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# Quantum Field Theory 1 – Problem set 9

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Suggested reading before solving these problems: Chapter 4.2 and 4.3 in the script and/or Chapter 3.3 and 4.7 of *Peskin & Schroeder*.

## Problem 1: Properties of the Dirac field

Since a Dirac field satisfies a Klein-Gordon equation, it is natural to expand it in plane waves. In the lectures you have learned the properties of the quantities  $u_s$  and  $v_s$ , associated with positive and negative frequency waves respectively.

a) Prove the following relations,

$$\sum_{s=1}^2 u_s(p) \bar{u}_s(p) = \not{p} + m, \quad \sum_{s=1}^2 v_s(p) \bar{v}_s(p) = \not{p} - m.$$

b) Show that

$$\bar{u}_r(p) \gamma^0 u_s(p) = 2p^0 \delta_{rs}, \quad \bar{v}_r(p) \gamma^0 v_s(p) = 2p^0 \delta_{rs}.$$

c) Using a) and the expansion of  $\psi$  and  $\bar{\psi}$  in terms of creation and annihilation operators, compute

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle$$

d) Prove by using c) that the Feynman propagator is

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

e) Consider the conserved quantity  $j^\mu = \bar{\psi} \gamma^\mu \psi$ . Expanding  $\psi$  and  $\bar{\psi}$ , compute the corresponding charge

$$Q = \int d^3 x j^0.$$

## Problem 2: Vacuum polarization

Consider the following Lagrangian, describing a theory of interacting scalar and spinor fields:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m_\phi^2}{2}\phi^2 + \bar{\psi}(i\not{\partial} - m_\psi)\psi - h\phi\bar{\psi}\psi.$$

- Write in detail the Feynman rules associated with this theory.
- Convince yourself that, due to the interaction with the fermion, the scalar propagator acquires a one-loop contribution.
- Write the amplitude  $i\mathcal{M}(p^2)$  associated to this loop contribution. Show that it reads

$$i\mathcal{M}(p^2) = -(-ih)^2 \int \frac{d^4q}{(2\pi)^4} \text{tr} \left[ \frac{i(\not{q} + \not{p} + m_\psi)}{(q+p)^2 - m_\psi^2} \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2} \right]$$

where  $p$  is the momentum of the incoming scalar field. Why there is an overall minus sign in the right hand side? And which is the role of the trace in the integrand?

- Perform the trace by using  $\text{tr} \mathbf{1} = 4$ ,  $\text{tr} \gamma_\mu = 0$  and  $\text{tr} \gamma_\mu \gamma_\nu = 4\eta_{\mu\nu}$ . Convince yourself that the latter identity follows directly from the Clifford algebra. Plug the result in the integral, and show that

$$i\mathcal{M}(p^2) = -4h^2 \int \frac{d^4q}{(2\pi)^4} \frac{q(p+q) + m_\psi^2}{[(q+p)^2 - m_\psi^2][q^2 - m_\psi^2]}$$

- Consider the so called Feynman trick:

$$\frac{1}{AB} = \int_0^1 dx dy \frac{\delta(1-x-y)}{(xA+yB)^2}$$

Use it to show that, after making an appropriate shift in  $q$  to remove the cross terms containing  $qp$ , our integral becomes

$$i\mathcal{M}(p^2) = -4h^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{q^2 - x(1-x)p^2 + m_\psi^2}{[q^2 + x(1-x)p^2 - m_\psi^2]^2}$$

- Perform a Wick rotation and show that, calling  $\Delta \equiv x(1-x)p^2 - m_\psi^2$ , one can write

$$i\mathcal{M}(p^2) = \frac{ih^2}{2\pi^2} \left[ \int_0^{+\infty} dq \int_0^1 dx \frac{q^5}{(q^2 - \Delta)^2} + \int_0^{+\infty} dq \int_0^1 dx \frac{\Delta q^3}{(q^2 - \Delta)^2} \right].$$

The last two integrals in  $q$  are divergent: use an ultraviolet cut-off  $\Lambda$  in order to regularize them. How does each integral diverge in the ultraviolet for large values of  $\Lambda$ ?

On the next sheet, we will discuss how one can actually deal with such cut-off dependent amplitudes.