
Quantum Field Theory 1 – Problem set 3

Lectures: Jan Pawłowski

J.Pawłowski@thphys.uni-heidelberg.de

Tutorials: Felipe Attanasio

F.Attanasio@thphys.uni-heidelberg.de

Institut für Theoretische Physik, Uni Heidelberg

tutorial date: week of 31.10.2022

Suggested reading before solving these problems: Chapter 2.3 in the script and/or Chapters 2.2, 2.3 and 2.4 of *Peskin & Schroeder*.

Problem 1: A commutation relation for the three-momentum operator

Consider a real scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

and conjugate momentum $\pi = \partial \mathcal{L} / \partial \dot{\phi}$. You have learned how to quantise the system by promoting ϕ and π to operators with the commutation relations

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (1)$$

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0. \quad (2)$$

The associated Hamiltonian and the three-momentum operators read

$$H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right]$$

$$\mathbf{P} = \int d^3x (\pi \nabla \phi)$$

Show that these operators commute, $[H, \mathbf{P}] = 0$, by *only* using the commutation relations (1) and (2).

Problem 2: Charge of a complex scalar field

Consider the Lagrangian density for a free complex scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi.$$

and define the associated conjugate momenta π and π^* . The Noether theorem leads to a conserved charge, given in terms of 0-component of the Noether current,

$$Q \equiv \int d^3x j^0.$$

For a complex scalar field, the four-vector associated with the current j reads

$$j^\mu = i [(\partial^\mu \phi)^* \phi - \phi^* (\partial^\mu \phi)],$$

from which an expression for the corresponding charge Q can be easily obtained.

The theory is quantised by promoting ϕ , ϕ^* and their conjugate momenta to operators. To that end it is convenient to introduce the creation and annihilation operators

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \{a(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + b^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}}\}, \quad (3)$$

$$\pi(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \{b(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} - a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}}\}. \quad (4)$$

with commutation relations

$$\begin{aligned} [a(\mathbf{p}), a^\dagger(\mathbf{q})] &= [b(\mathbf{p}), b^\dagger(\mathbf{q})] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \\ [a(\mathbf{p}), a(\mathbf{q})] &= [b(\mathbf{p}), b(\mathbf{q})] = 0, \\ [a(\mathbf{p}), b(\mathbf{q})] &= [a(\mathbf{p}), b^\dagger(\mathbf{q})] = 0. \end{aligned}$$

Express the charge Q in terms of the operators a , a^\dagger and b , b^\dagger , carrying out all the details of the calculation.