
Quantum Field Theory 1 – Problem set 4

Lectures: Jan Pawłowski

J.Pawłowski@thphys.uni-heidelberg.de

Tutorials: Felipe Attanasio

F.Attanasio@thphys.uni-heidelberg.de

Institut für Theoretische Physik, Uni Heidelberg

tutorial date: week of 7.11.2022

Suggested reading before solving these problems: Chapter 3 in the script and/or Chapter 4.2 of *Peskin & Schroeder*.

Problem 1: Unitary evolution and T-product

Consider the decomposition of a Hamiltonian operator H in free and interaction parts, $H = H_0 + H_{\text{int}}$. In the interaction picture, operators evolve in time with the free Hamiltonian H_0 , while states $|f\rangle$ evolve with the interaction Hamiltonian,

$$i \partial_t |f\rangle = H_{\text{int}}(t) |f\rangle.$$

Show that this implies that we can write $|f(t)\rangle = U(t, t_0) |f(t_0)\rangle$, where the unitary operator $U(t, t_0)$ satisfies the differential equation (Schrödinger equation)

$$i \partial_t U(t, t_0) = H_{\text{int}}(t) U(t, t_0). \quad (1)$$

Show that the solution of equation (1) can be expressed as a power series, in which each term is an operator,

$$U(t, t_0) = \mathbf{1} - i \int_{t_0}^t dt_1 H_{\text{int}}(t_1) + (-i)^2 \int_{t_0}^t \int_{t_0}^{t_1} dt_1 dt_2 H_{\text{int}}(t_1) H_{\text{int}}(t_2) + \dots \quad (2)$$

Convince yourself that this series can be re-expressed as

$$U(t, t_0) = T \left\{ \exp \left[-i \int_{t_0}^t dt' H_{\text{int}}(t') \right] \right\}$$

where the T -product acts as $T A(t) B(t') = A(t) B(t') \Theta(t - t') + B(t') A(t) \Theta(t' - t)$. In particular, show that the expansion of the exponential up to second term provides Eq. (2), and try to generalise your argument for the higher order terms.

Problem 2: 2-to-2 scattering

In the lecture course you have learned how to describe the scattering of two particles in the interaction picture. Assume that the particles are characterised by momenta \mathbf{p}_1 and \mathbf{p}_2 in the initial state, and by momenta \mathbf{p}'_1 and \mathbf{p}'_2 in the final state. The interaction Hamiltonian is $H_I = \frac{\lambda}{4!} \phi^4$, where $\phi(t, \mathbf{x})$ is the time dependent operator associated with a real scalar field.

In particular, you have learned that the amplitude controlling the process can be obtained from the following quantity

$$i T_{fi} \simeq -i \langle \mathbf{p}_1' \mathbf{p}_2' | \left[\frac{\lambda}{4!} \int d^4x \phi^4(x) \right] | \mathbf{p}_1 \mathbf{p}_2 \rangle \quad (3)$$

by isolating the contributions proportional to $\delta^4(p_1 + p_2 - p_1' - p_2')$, and defining the scattering amplitude \mathcal{M}_{fi} as

$$i T_{fi} \equiv i \mathcal{M}_{fi} (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')$$

The amplitude can be extracted by expanding the scalar field $\phi(t, \mathbf{x})$ in terms of ladder operators $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$, and by plugging this expansion in eq. (3). Then, using the commutation relations of a and a^\dagger , one extracts the terms that are proportional to $\delta^{(4)}(p_1 + p_2 - p_1' - p_2')$.

- Identify the relevant terms in the expansion! How many are there? Do they all give the same contribution?
- Show that the final result is $\mathcal{M}_{fi} = -4! \frac{\lambda}{4!} = -\lambda$. Is there a connection between the coefficient in this result, and the number of terms in the expansion of eq. (3) that contribute to the scattering amplitude?
- Try to generalise the previous results to the $n - n$ scattering of a theory with interaction Hamiltonian $H_I = \frac{\lambda}{(2n)!} \phi^{2n}$, with n being a natural number.
- By inserting the scalar field expansion into eq. (3), do you also find terms that are *not* proportional to $\delta^{(4)}(p_1 + p_2 - p_1' - p_2')$? If so, what is their physical interpretation?