Quantum Field Theory 1 – Problem set 6

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Suggested reading before solving these problems: Chapters 3.3 to 3.4 in the script and/or Chapters 4.4 to 4.6 of *Peskin* & *Schroeder*.

Problem 1: Key figures of LHC

The protons circulating in the LHC accelerator (the proton beam) are divided into 2808 bunches. Find out

a) How many particles will be in one bunch?

b) What is the maximal energy per proton and what is therefore the energy of the whole proton beam? How fast could one accelerate a ICE (400 000 kg) with that energy?

c) What is the width of a bunch at the collision point?

d) The Luminosity is the event rate for a process divided by the corresponding cross section $L = \dot{N}/\sigma$. What follows from the numbers obtained in parts a) and c) for L? (The accelerator ring is 27 km long, the velocity of a particle is $0.9999c$.)

Problem 2: Scattering matrix and cross-section

On sheet 5 you considered the following theory involving two real scalar fields:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - h\Phi\phi\phi.$$

a) Write down the Feynman rules in momentum space for this theory.

b) Consider a scattering experiment where two ϕ-particles with initial momenta $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are scattered to final momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Show that at order h^2 there are 3 tree-level[1] contributions to the scattering matrix:

$$i\mathcal{M} = \frac{-i(2h)^2}{(k_1+k_2)^2 - M^2} + \frac{-i(2h)^2}{(k_1-p_1)^2 - M^2} + \frac{-i(2h)^2}{(k_1-p_2)^2 - M^2}.$$

[1]A tree-level process has a Feynman diagram without loops.

c) Derive an expression for the differential cross section $d\sigma/d\Omega$ in the center-of-mass frame as a function of the angle ϑ between $\boldsymbol{k}_1$ and $\boldsymbol{p}_1$.

d) Consider the limit $h^2 \to \infty, M^2 \to \infty$ with fixed ratio

$$\frac{1}{4!}\lambda = -\frac{1}{2}\frac{h^2}{M^2}.$$

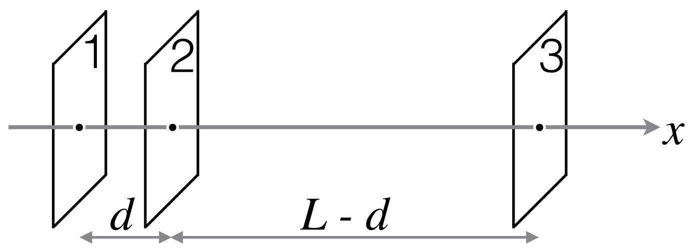
Calculate the total cross section σ in this limit. Do not forget to take into account that the final state has two identical particles.

e) Show that the result for σ is identical to the cross-section of ϕ^4 theory calculated at tree-level and to first order in the interaction parameter λ.

Problem 3: Casimir effect

For this exercise, we go back a little bit in the script (chap. 2.3) and recall that the quantization of the free scalar field led to a divergent vacuum energy density $\epsilon = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\hbar\omega_{\boldsymbol{p}}$, cf. p. 24, remark (i). The corresponding energy cannot be directly measured as only energy differences matter. An interesting measurable consequence, however, follows if we disturb the vacuum.

To imagine a real physical setup think of the following: Take the electromagnetic field with vacuum energy density $\epsilon = 2\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\hbar\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{p}} = c|\boldsymbol{p}|$. Now, introduce three parallel and perfectly conducting plates with distances d and $L-d$ into the electromagnetic vacuum, see figure. The plates of infinite size and zero thickness introduce boundary conditions for the electromagnetic field which comes with two possible polarizations.



a) To simplify the analysis, we won't do the full calculation for the electromagnetic field which is, e.g., complicated by considerations about polarization. Instead, we consider a massless scalar field in one-dimensional space. Therefore, the 'plates' define the positions x where the field vanishes and introduce discrete field modes. Show that the energy between 'plate' 1 and 'plate' 2 is given as

$$E(d) = \frac{\pi\hbar c}{2d}\sum_{n=1}^{\infty} n\,.$$

What is the energy between plates 2 and 3? What is the full energy $E_f(d)$ between plates 1 and 3?

b) Real physical plates do not set boundary conditions for arbitrarily high frequencies but become transparent, i.e., high frequency modes don't contribute to the sum. This can be accounted for by the replacement $n \to n\, e^{-a\,\omega_n}$ in $E(d)$ with small but arbitrary constant a which controls when high frequencies ω_n cease to contribute to the sum. Show that with this *regularization* the sum can be evaluated to be

$$E(d) = \frac{\pi \hbar c}{2d} \frac{e^{a\pi c/d}}{(e^{a\pi c/d} - 1)^2}\,.$$

In the limit of small a, determine the first two terms in an expansion of $E(d)$.

c) The two outer "plates" shall be fixed and only the inner "plate" can be moved. Calculate the Casimir force $-\frac{\partial E_f(d)}{\partial d}$ in the limit of $L \gg d$. Is the force attractive or repulsive? What happens to the $1/a^2$ divergence?

Remarks:

(i) For the electromagnetic field in (3+1) spacetime dimensions one finds:
$F_c(d) = -\frac{\pi^2 \hbar c}{240\, d^4}$.

(ii) You can calculate the result for the full EM field yourself, for help see:
https://arxiv.org/pdf/hep-th/9908149.pdf

(iii) The Casimir force between two parallel plates has been measured in 2002:
https://arxiv.org/pdf/quant-ph/0203002v1.pdf.