Quantum Field Theory 1 – Problem set 9

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Suggested reading before solving these problems: Chapter 4.2 and 4.3 in the script and/or Chapter 3.3 and 4.7 of *Peskin & Schroeder*.

Problem 1: Properties of the Dirac field

Since a Dirac field satisfies a Klein-Gordon equation, it is natural to expand it in plane waves. In the lectures you have learned the properties of the quantities u_s and v_s, associated with positive and negative frequency waves respectively.

a) Prove the following relations,

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) \,=\, \not{p} + m\,, \qquad \sum_{s=1}^{2} v_s(p)\bar{v}_s(p) \,=\, \not{p} - m\,.$$

b) Show that

$$\bar{u}_r(p)\,\gamma^0\,u_s(p) \,=\, 2\,p^0\,\delta_{rs}\,, \qquad \bar{v}_r(p)\,\gamma^0\,v_s(p) \,=\, 2\,p^0\,\delta_{rs}\,.$$

c) Using a) and the expansion of ψ and $\bar{\psi}$ in terms of creation and annihilation operators, compute

$$\langle 0|\,\psi(x)\bar{\psi}(y)\,|0\rangle$$

d) Prove by using c) that the Feynman propagator is

$$\langle 0|\,T\,\psi(x)\bar{\psi}(y)\,|0\rangle \,=\, \int \frac{d^4p}{(2\pi)^4}\,\frac{\not{p}+m}{p^2-m^2+i\epsilon}\,e^{-i\,p(x-y)}$$

e) Consider the conserved quantity $j^\mu = \bar{\psi}\gamma^\mu\psi$. Expanding ψ and $\bar{\psi}$, compute the corresponding charge

$$Q \,=\, \int d^3x\, j^0\,.$$

Problem 2: Vacuum polarization

Consider the following Lagrangian, describing a theory of interacting scalar and spinor fields:

$$\mathcal{L} \,=\, \frac{1}{2}\,(\partial\phi)^2 - \frac{m_\phi^2}{2}\,\phi^2 + \bar{\psi}\,\left(i\partial\!\!\!/ - m_\psi\right)\psi - \,h\,\phi\,\bar{\psi}\psi\,.$$

a) Write in detail the Feynman rules associated with this theory.

b) Convince yourself that, due to the interaction with the fermion, the scalar propagator acquires a one-loop contribution.

c) Write the amplitude $i\,\mathcal{M}(p^2)$ associated to this loop contribution. Show that it reads

$$i\,\mathcal{M}(p^2) \;=\; -\,(-ih)^2 \int \frac{d^4q}{(2\pi)^4}\,\mathrm{tr}\left[\frac{i\left(q\!\!\!/ + p\!\!\!/ + m_\psi\right)}{(q+p)^2 - m_\psi^2}\,\frac{i\left(q\!\!\!/ + m_\psi\right)}{q^2 - m_\psi^2}\right]$$

where p is the momentum of the incoming scalar field. Where does the overall minus sign on the right hand side come from? What is the role of the trace in the integrand?

d) Perform the trace by using $\mathrm{tr}\,\mathbb{1} \,=\, 4$, $\mathrm{tr}\,\gamma_\mu \,=\, 0$ and $\mathrm{tr}\,\gamma_\mu\gamma_\nu \,=\, 4\eta_{\mu\nu}$. Convince yourself that the latter identity follows directly from the Clifford algebra. Plug the result in the integral, and show that

$$i\,\mathcal{M}(p^2) \;=\; -\,4h^2 \int \frac{d^4q}{(2\pi)^4}\,\frac{q\,(p+q) + m_\psi^2}{\left[(q+p)^2 - m_\psi^2\right]\left[q^2 - m_\psi^2\right]}$$

e) Consider the so called Feynman trick:

$$\frac{1}{AB} \;=\; \int_0^1 dx dy\,\frac{\delta(1-x-y)}{(xA+yB)^2}$$

Use it to show that, after making an appropriate shift in q to remove the cross terms containing qp, our integral becomes

$$i\,\mathcal{M}(p^2) \;=\; -\,4h^2 \int \frac{d^4q}{(2\pi)^4}\int_0^1 dx\,\frac{q^2 - x(1-x)p^2 + m_\psi^2}{\left[q^2 + x(1-x)p^2 - m_\psi^2\right]^2}$$

f) Perform a Wick rotation and show that, calling $\Delta \,\equiv\, x(1-x)p^2 - m_\psi^2$, one can write

$$i\,\mathcal{M}(p^2) = \;\frac{ih^2}{2\pi^2}\left[\int_0^{+\infty} dq\,\int_0^1 dx\,\frac{q^5}{(q^2-\Delta)^2} + \int_0^{+\infty} dq\,\int_0^1 dx\,\frac{\Delta\,q^3}{(q^2-\Delta)^2}\right]\,.$$

The last two integrals in q are divergent: use an ultraviolet cut-off Λ in order to regularize them. How does each integral diverge in the ultraviolet for large values of Λ?

On the next sheet, we will discuss how one can actually deal with such cut-off dependent amplitudes.