Quantum Field Theory 1 – Problem set 13

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Suggested reading before solving these problems: Chapter 7.1 in the script.

Problem 1: Ultraviolet Landau pole in four-dimensional ϕ^4 theory

In the lecture, you have calculated the β function (Gell-Mann & Low, 1954) for the coupling in ϕ^4 theory in four spacetime dimensions to be

$$\beta(\mu) = \mu \frac{d}{d\mu} \lambda = \frac{3}{16\pi^2} \lambda^2 \,. \tag{1}$$

where the renormalization condition fixes $\lambda = \lambda_{\text{phys}}$ at the momentum scale μ.

a) Solve the differential equation (1) and explicitly give the coupling function $\lambda(\mu)$ as a function of the momentum scale μ.

b) Suppose the value of the coupling λ is known at an infrared momentum scale μ_{IR} to be $\lambda(\mu_{\text{IR}}) = \lambda_{\text{IR}} > 0$. Show that for large μ the coupling exhibits a singularity at a finite μ_{L}. Calculate μ_{L}.

c) What is the value of the coupling if we demand $\lambda < \infty \; \forall \mu$?

This is the *Landau pole* or *triviality problem* and it indicates that the theory becomes strongly coupled, i.e. perturbation theory predicts its own failure. A similar behavior is observed in QED, where $\beta = e^3/(12\pi^2) + \ldots$. Here, the Landau-pole is predicted to appear on energy scales much larger than the Planck scale, $m_{\text{P}} \sim 10^{19}$GeV. The LHC operates at energies of the order of $\sim 10^4$GeV.

Problem 2: Infrared Landau pole in six-dimensional ϕ^3 theory

Another perturbatively renormalizable theory is given by a scalar field with a cubic interaction term $g\phi^3$ in six-dimensional spacetime. This should rather be considered as a toy model, however, it turns out that it shares a fundamental similarity with the theory of the strong interaction, i.e. Quantum Chromodynamics (QCD): it has a negative sign in the β function. Here, we will explore the consequences:

The β function for the coupling g in ϕ^3 theory in six spacetime dimensions is

$$\beta(\mu) = \mu \frac{d}{d\mu} g = -\frac{3}{256\pi^3} g^3 \,. \tag{2}$$

a) Solve the differential equation (2) and explicitly give the coupling function $g(\mu)$ as a function of the momentum scale μ.

b) Suppose the value of the coupling g is known at an infrared momentum scale μ_{IR} to be $g(\mu_{\mathrm{IR}}) = g_{\mathrm{IR}} > 0$. Show that for large μ the theory becomes weakly coupled.

c) What happens at small momentum scales $\mu < \mu_{\mathrm{IR}}$?

This behaviour is called *asymptotic freedom*. In QCD, we have $\beta = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$ on one-loop level, where the number of colors is $N_c = 3$ and asymptotic freedom can accordingly be observed for a number of fermion flavors $N_f < 33/2$.