Quantum Field Theory 1 – Tutorial 3

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Problem 1: Lorentz invariant quantities

In the lecture we have used the following important identity for the spatial three-momentum integral

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} f(p) \;=\; \int \frac{d^4p}{(2\pi)^4}\, 2\pi\, \delta(p^2 - m^2)\, \Theta(p_0) f(p)\,. \tag{1}$$

Use the delta function identity

$$\delta(g(x) - g(a)) \;=\; \frac{1}{|g'(a)|}\, \delta(x-a)\,.$$

to derive Eq. (1). Further, convince yourself that this implies that the integral in Eq. (1) is invariant under Lorentz transformations Λ if $f(\Lambda p) = f(p)$.

Remark: We consider proper, orthochronous Lorentz transformations $\Lambda \in SO(1,3)$.

Optional: In the lecture we have also introduced conveniently normalised one-particle states with momentum $\boldsymbol{p}$ as (see Eq. (2.104) on p. 23 in the script)

$$|\boldsymbol{p}\rangle = \sqrt{2\omega_{\boldsymbol{p}}}\, a^\dagger(\boldsymbol{p})\, |0\rangle\,,$$

with $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. Consider a Lorentz boost in the p_3 direction:

$$p_3' = \gamma(p_3 + \beta\, E)\,, \quad \text{and} \quad E' = \gamma(E + \beta\, p_3)\,.$$

Use the delta function identity from above to show that the boost gives

$$\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) = \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{q}') \frac{E'}{E}\,.$$

This non-trivial transformation property is related to the fact that volumes are not invariant under boosts due to Lorentz contraction ($V \to V/\gamma$). Instead, the quantity $E\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$ is invariant under boosts. More generally, $E\, \delta^{(3)}\delta(\boldsymbol{p}-\boldsymbol{q})$ is Lorentz invariant which also makes $\langle \boldsymbol{p}|\boldsymbol{q}\rangle = 2\omega_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})$ a Lorentz invariant quantity.