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Lectures on Supersymmetry[1]

I. Sachs
Dublin Institute for Advanced Studies,
10 Burlington Rd, Dublin 4, Ireland
and
Department of Mathematical Sciences
University of Durham
South Rd, Durham DH1 3LE, UK

Abstract

I first review some well known material such as Coleman-Mandula theorem, supersymmetry algebra, non-renormalisation theorems etc. Then, specialising to $N=2$ Yang-Mills theory, recent results are presented. In particular the superconformal Ward-identity for the effective action and the effective central charge are explained. Finally, the low energy effective action for $SU(2)$, $N=2$ YM theory is derived assuming that the superconformal anomaly parametrises the moduli space.

[1]Postgraduate lectures given at Trinity College Dublin and University of Durham, UK.

1 Introduction

A supersymmetric Lagrangian is invariant under a transformation which relates the Bosonic degrees of freedom, φ say to Fermionic degrees of freedom eg.

$$\delta\varphi = \epsilon^{\alpha}\psi_{\alpha}. \tag{1}$$

For consistency ϵ must have mass dimension $-\frac{1}{2}$ and must be Grassmann-valued, i.e. anti-commuting. The corresponding transformation of a fermion would then have to be of the form

$$\delta\psi_{\alpha} = (\epsilon\partial\!\!\!/\varphi)_{\alpha} + \quad \text{possibly other fields.} \tag{2}$$

Imposing invariance under such a transformation would then put conditions on the most general form of the Lagrangian. In a particular one would find that number of bosonic- and fermionic degrees of freedom would have to be the same and that Bosons and Fermions would have to have the same masses.

What is the physical motivation for imposing such a symmetry? One motivation for demanding supersymmetry is to improve the ultraviolet behaviour of the theory. For example if we consider the vacuum energy for free fields of mass m_j and spin j

$$\begin{aligned} E_j &\propto (-1)^{2j}(2j+1)\int \mathrm{d}^3k\sqrt{k^2+m_j^2} \\ &= (-1)^{2j}(2j+1)\int \mathrm{d}^3k\ |k|\Big(1+\frac{1}{2}\frac{m_j^2}{k^2}+O(k^{-4})\Big), \end{aligned} \tag{3}$$

We find that a necessary condition for the absence of power-like divergences is that both, integer and half-integer spin fields are present and that these fields have the same masses.

Supersymmetry may also be required to solve the *hierarchy problem*, which expresses our ignorance in explaining the value of the Higgs mass. Indeed, the Higgs mass is not protected by any symmetry. Therefore if m_H is different from zero its natural value would be of the order of the Plank scale. In a supersymmetric version of the Standard model however, the Higgs field is accompanied by its fermionic super partner which could acquire a mass through anomalous chiral symmetry breaking with condensate $\langle\bar{\psi}\psi\rangle \simeq \Lambda_W^3 \simeq m_H^3$ where the last identity follows from the fact that the Higgs field and its fermionic partner ψ are in the same supersymmetry multiplet. This explains the continued interest in supersymmetry despite of the fact that no experimental evidence in favour of supersymmetry has been found so far.

Another motivation is of conceptual nature. It originates in the attempt to unify in a non-trivial way internal symmetries with the Poincaré group. In the early 60's much effort was put into finding such a unified symmetry group. This came to an end after a series of no-go results culminating in the Coleman and Mandula no-go theorem [1] which excludes this possibility, at least within the context of Lie-groups.

More recently supersymmetry has become increasingly important as a tool analyse strongly coupled non-abelian Yang-Mills theories and string theories. Supersymmetrising ordinary YM-theories has lead to a number of new results on the strong-coupling, and therefore non-perturbative sector of these theories. In particular it lead to the discovery of explicit realisations of electromagnetic duality and generalisations of it to string theory. This electric-magnetic equivalence in turn provides evidence for the equivalence of a number of a priory different field theories. To what extend these properties rely on supersymmetry is still unclear.

The first part of these lectures, sections 1-4 reviews rather well known material, i.e. Coleman-Mandula theorem, Supersymmetry algebra, representation theory, superspace, action formulas and non-renormalisation theorems. The second part, sections 5 and 6 contains more recent results. In particular we obtain the quantum corrected central charge for $N=2$ Yang-Mills theory and the superconformal Ward-identities for the energy effective action. The low energy effective action is then derived by integrating the superconformal anomaly.

2 Coleman-Mandula Theorem

As this theorem applies to internal symmetries of the S-matrix we first recall some elements of *scattering theory:*
The Hilbert space of scattering theory, $\mathcal{H}$ is the infinite direct sum of n-particle subspaces

$$\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \cdots, \tag{4}$$

where $\mathcal{H}^{(n)}$ is the (symmetrised) subspace of the direct product of one-particle Hilbert spaces. The S-matrix is then a unitary operator on $\mathcal{H}$ describing all possible scattering processes in a given theory. It is usually written as

$$S = 1 - i(2\pi)^4 \delta^4(p_\mu - p'_\mu)T, \tag{5}$$

where the delta function ensures the conservation of energy and momentum during the scattering process. A unitary operator U on $\mathcal{H}$ is a symmetry transformation of the S-matrix if

- 1) It maps one-particle states into one-particle states;
- 2) U acts on many particle states by the tensor product representation of one-particle states;
- 3) U commutes with S.

In what follows we restrict ourself to theories for which all scattering states are in *positive mass* representations of the Poincaré group P. We shall further assume that for any finite mass M there are only a finite number of particles with mass smaller than M (*particle finiteness assumption*)
Theorem:
Let a Lie group G be a symmetry group of the S-matrix which contains the Poincaré group P and which connects a finite number of particles in a supermultiplet. Assume furthermore that

- 4) Elastic scattering amplitudes are analytic in the centre of mass energy $s = (p+q)^2$ and the momentum transfer $t = (p-p')^2$.

- 5) $T|p,q>\neq 0$ for almost all s

Then, G is (locally) isomorphic to a direct product of an internal symmetry group and the Poincaré group G.
Proof (sketch):
Let $\mathcal{D}$ be the subset of one-particle states whose momentum space wave-functions are test functions. For two two-particle states $x_1 \otimes x_2$ and $y_1 \otimes y_2$ in $\mathcal{D} \otimes \mathcal{D}$ the variation of its scalar product under infinitesimal G-transformations is then given by

$$(y_1, Ax_1) + (y_2, Ax_2) \tag{6}$$

where A is an infinitesimal generator of G. The G-invariance of the S-matrix is then equivalent to

$$(S\ y_1 \otimes y_2, A\ S\ x_1 \otimes x_2) = (y_1 \otimes y_2, A\ x_1 \otimes x_2) \tag{7}$$

For a given test function f with support in a region R not containing the origin (of momentum space) we then consider the distribution

$$f \cdot A = \int d^4a\, U^\dagger(1,a) A\, U(1,a) \tilde{f}(a), \tag{8}$$

where $\tilde{f}(a)$ is the Fourier transform of $f(p)$. Since

$$U(1,a)|p\rangle = e^{-ip\cdot a}|p\rangle \tag{9}$$

$f \cdot A$ has matrix elements

$$f \cdot A(p', p) = f(p - p')A(p', p). \tag{10}$$

By the finite particle assumption, if R is sufficiently small not containing the origin, there exist regions U_i in a given mass hyperboloid, such that for $p \in U_i$ and k in R, the sum $p + k$ is not in any mass hyperboloid and hence this state is annihilated by $f \cdot A$. We now choose p to be in the complement of these regions and $q, p', q' \in \cup_i U_i$ such that

$$p + q = p' + q'. \tag{11}$$

The centre of mass energy and invariant momentum transfer are defined as usual by

$$s = (p + q)^2 \quad \text{and} \quad t = (p - p')^2 \tag{12}$$

and are chosen to be below the threshold for pair production. By (7) The S-matrix element must be zero for such states. Using the analyticity the S matrix is then zero in the whole region of analyticity. Repeating the argument for multi particle scattering one obtains the result:
Lemma 1 *:* The support of $A(p, p')$ is restricted to $p{=}p'$.
Corollary: A cannot connect states on different mass hyperboloids (O'Raifeartaigh's theorem).

We make the technical assumption that A is a matrix valued distribution. Then, it follows that $A(p)$ is a polynomial in the differential operator on the mass hyperboloid,

$$\nabla_\mu = \frac{\partial}{\partial p_\mu} - \frac{p_\mu p_\nu}{m^2}\frac{\partial}{\partial p_\nu}. \tag{13}$$

In other words

$$A \;=\; \sum_{n=0}^{N} A^{(n)}(p)_{\mu_1,\cdots,\mu_n}\frac{\partial}{\partial p_{\mu_1}}\cdots\frac{\partial}{\partial p_{\mu_n}}, \tag{14}$$

with $[A, p^\mu p_\mu] = 0$, acting on any state on $\mathcal{D}$. To complete the proof we need another result which we state without proof

Lemma 2:

Let $\mathcal{B}$ be the subset of G-transformations which commute with space-time translations then for $B(p) \in \mathcal{B}$

$$B(p) = a_\mu p^\mu + b, \tag{15}$$

where a_μ is a constant four vector (i.e. no internal indices) and b is a constant Hermitian matrix which does not involve spin indices. Note hat lemma 2 assumes a Lie-multiplication [,] for generators of $\mathcal{B}$. In particular the result does not hold for

Now, taking the N-fold commutator of A with p_μ:

$$[p_{\mu_1}, [p_{\mu_2}\cdots, A]\cdots] = A^{(N)}_{\mu_1\cdots\mu_N}(p) \tag{16}$$

we obtain an object which is in $\mathcal{B}$, hence by lemma 2,

$$A^{(N)}_{\mu_1\cdots\mu_N}(p) = a_{\lambda\mu_1\cdots\mu_N}p^\lambda + b_{\mu_1\cdots\mu_N}. \tag{17}$$

However, (14) and the symmetry properties of N-fold commutators with p^μ imply

$$b_{\mu_1\cdots\mu_N} = 0, \tag{18}$$

unless $N=0$. Similarly

$$a_{\lambda\mu_1\cdots\mu_N} = 0, \tag{19}$$

unless $N=0$ or $N=1$. For $N=0$ we then showed that A is the sum of a translation and an internal symmetry transformation and for $N=1$ the antisymmetry of $a_{\lambda\mu}$ implies that A is just an infinitesimal Lorentz transformation. This then completes the proof of the theorem.

3 Supersymmetry

A way out of this dilemma was discovered by Golfand and Likhtman [2] in 1971. They avoided the conclusions of Coleman and Mandula by introducing a graded Lie-algebra, that is, to the standard generators of a Lie-algebra, satisfying commutation relations they added (fermionic) generators Q^i_α with the corresponding anti-commutation relations[2]. We have

$$[A, B\} \equiv AB - (-1)^{ab} BA, \tag{20}$$

where a, b is the number of fermionic generators in A and B respectively. There is a corresponding generalised Jacobi identity (exercise)

$$[A, [B, C\}\} + (-)^{a(b+c)}[B, [C, A\}\} + (-)^{c(a+b)}[C, [A, B\}\} = 0. \tag{21}$$

3.1 SUSY-algebra

Let us now determine the structure constants of the general super algebra. Because of $[odd, even\} = odd$ we must have

$$[Q^i_\alpha, B_a] = -(h_a)_{\alpha j}^{\beta i} Q^j_\beta. \tag{22}$$

The (Q, B_a, B_b)-Jacobi identity then requires that the matrices h_a represent the bosonic symmetry algebra i.e.

$$[h_a, h_b] = i f_{ab}{}^c h_c. \tag{23}$$

This implies in particular the commutation relations

$$\left[Q^i_\alpha, P_\mu\right] = 0, \quad \text{and} \quad \left[Q^i_\alpha, J_{\mu\nu}\right] = (b_{\mu\nu})_\alpha{}^\beta Q_\beta{}^i, \tag{24}$$

with

$$[b_{\mu\nu}, b_{\rho\lambda}] \;\; = \;\; \eta_{\mu\lambda} b_{\nu\rho} + \eta_{\nu\rho} b_{\mu\lambda} - \eta_{\mu\rho} b_{\nu\lambda} - \eta_{\nu\lambda} b_{\mu\lambda}, \tag{25}$$

showing that $b_{\mu\nu}$ forms a representation of the of the Lorentz transformations. We shall assume that Q^i_α be in the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation of the Lorentz group and hence $b_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$. Similarly for the generators of the internal symmetry group T^a

$$[Q_\alpha{}^i, T^a] = (l^a)^i{}_j Q^j{}_\alpha + i(t^a)^i{}_j (\gamma_5)_\alpha{}^\beta Q^j{}_\beta, \tag{26}$$

where $l^a + it^a\gamma_5$ represents the the Lie algebra of the internal symmetry group. Here we have used that $\delta_\alpha{}^\beta$ and γ_5 are the only (pseudo-) scalar invariant tensors.

Let us now consider the $odd - odd$ commutator $\{Q^i_\alpha, Q^j_\alpha\}$. The result must be composed of even generators and must be symmetric under simultaneous exchange of $i \leftrightarrow j$ and $\alpha \leftrightarrow \beta$. The most general possibility, compatible with the Coleman-Mandula theorem is then

[2]Without restricting the generality we can assume the Q^i_α to be Majorana fermions

$$
\begin{aligned}
\{Q^i_\alpha, Q^j_\beta\} &= r\delta^{ij}(\gamma^\mu C)_{\alpha\beta}P_\mu + s\delta^{ij}(\gamma^{\mu\nu}C)_{\alpha\beta}J_{\mu\nu} \\
&\quad + C_{\alpha\beta}U^{ij} + (\gamma_5 C)_{\alpha\beta}V^{ij}.
\end{aligned} \tag{27}
$$

The (Q, Q, Q) Jacobi-identity together with (Q, Q, B) Jacobi-identity then imply that U and V commute with everything and are therefore *central charges*. The latter identity also implies that $s=0$. The remaining undetermined constant r can be absorbed in a redefinition of P_μ. Here we take $r=-1$.

We end this section by noting that the above supersymmetry algebra is the most general superalgebra consistent with an interacting quantum field theory [3]. Note also that in the presence of fermionic charges the first part of the proof of the no-go theorem (O'Raifeartaigh's theorem) goes still through. In particular, supersymmetry transformations cannot connect different mass hyperboloids.

3.2 Irreducible Representations

Before constructing explicit irreducible representations of the algebra explained in the last section we recall some representation independent consequences of the algebra.
i) The Hamiltonian of a supersymmetric theory is positive definite.
Proof:
From the Majorana property $\bar{Q}^i = Q^{Ti}C$ we have with (27,154)

$$\{Q^i_\alpha, \bar{Q}^j_\beta\} = \delta^{ij} \not{P}_{\alpha\beta} - \delta_{\alpha\beta}U^{ij} - (\gamma_5)_{\alpha\beta}V^{ij}. \tag{28}$$

Multiplying with γ_0 and taking the trace we then obtain

$$4NP_0 = Q^i_\alpha(Q^i_\alpha)^\dagger + h.c. \geq 0. \tag{29}$$

ii) The number of fermionic and bosonic states in a supersymmetric system is the same.

Proof: (so lala)
Define the fermion number operator $(-1)^{N_F}$. Then, using $(-1)^{N_F}Q^i_\alpha = -Q^i_\alpha(-1)^{N_F}$,

$$0 = \mathrm{tr}\left[(-1)^{N_F}\{Q^i_\alpha, \bar{Q}^{i\beta}\}\right] = 2(\gamma^\mu)_\alpha{}^\beta \mathrm{tr}\left[(-1)^{N_F}P_\mu\right], \tag{30}$$

where the trace is over a single supermultiplet. Noting that supersymmetry transformations leave the four momentum invariant, we can pull p_μ out of the trace obtaining in this way the desired result.

The irreducible representations of the super-Poincaré group can be found using the Wigner method for constructing irreducible representations of the Poincaré group. This method consists of finding a representation of the little group leaving a given four-momentum invariant and boosting it up to a representation of the full Poincaré group. In this way one obtains irreducible representations labelled the values of the Casimir operators $P_\mu P^\mu = -m^2$ and $W_\mu W^\mu = m^2 s(s+1)$, where $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}J^{\nu\rho}P^\lambda$ is the Pauli-Lubanski vector. The two Casimir operators of the super Poincaré group are p^2 and $\hat{W}^2$, with $\hat{W}_\mu$ a linear combination of W_μ and $\bar{Q}\gamma_\mu\gamma_5 Q$.

3.2.1 Massless Supermultiplet

We choose $p_\mu = (E, 0, 0, E)$. The little group (in the Lorentz group) of that vector is generated by

$$J_1 = J_{10} + J_{13}, \quad J_2 = J_{20} + J_{23} \quad \text{and} \quad J = J_{12}. \tag{31}$$

They satisfy the commutation relations of E_2 i.e.

$$[J_1, J] = -iJ_2, \qquad [J_2, J] = iJ_1, \qquad [J_1, J_2] = 0. \tag{32}$$

As in any finite dimensional unitary representation, J_1 and J_2 are trivially represented. The remaining operator $J = J_{12}$ has the helicity-eigenstates

$$J|\lambda\rangle = \lambda|\lambda\rangle, \tag{33}$$

with $\lambda \in \frac{1}{2}\mathbf{Z}$.

On the other hand for vanishing central charge the *odd-odd* commutator (27) simplifies to

$$\{Q^i_\alpha, Q^{\dagger j}_\beta\} = E\delta^{ij}(\gamma^0 + \gamma^3)_{\alpha\beta}. \tag{34}$$

Making use of the Majorana condition one then easily shows that (34) contains only one independent non-vanishing commutator i.e.

$$\{Q^i_1, Q^{\dagger j}_1\} = 2E\delta^i_j. \tag{35}$$

But from (24) and (25)

$$\left[J, Q^\dagger_1\right] = -\frac{1}{2}Q^\dagger_1 \quad \text{and} \quad [J, Q_1] = \frac{1}{2}Q_1, \tag{36}$$

and therefore Q^i_1 and $Q^{\dagger i}_1$ act as creation and annihilation operators on the Hilbert space spanned by the eigenvectors of J (33). Positivity then requires that that the remaining supercharges are trivially realised on the physical states (eg. [4]). Starting from the highest weight vector $|\Omega\rangle$ defined by $Q^i_1|\Omega\rangle = 0$ we then obtain the full Hilbert space upon acting with $Q^{\dagger i}_1$, $i = 1, \cdots, N$. i.e.

$$|\Omega\rangle, \quad Q^{\dagger i}_1|\Omega\rangle, \quad \cdots\cdots, \quad Q^{\dagger i_1}_1 \cdots Q^{\dagger i_N}_1|\Omega\rangle \tag{37}$$

with helicities

$$\lambda, \quad \lambda - \frac{1}{2}, \quad \cdots\cdots, \quad \lambda - \frac{N}{2} \tag{38}$$

and multiplicities

$$1, \quad \binom{N}{1}, \quad \cdots\cdots, \quad \binom{N}{N}. \tag{39}$$

The total number of states is then $\sum\limits_{k=0}^{N} \binom{N}{k} = 2^N$. Because of the CPT-theorem, a physical massless state contains always the two helicities λ and $-\lambda$ and therefore a single supermultiplet can contain all massless states for $\lambda = \frac{N}{4}$. Otherwise the states must be doubled starting with a second vacuum $\lambda' = \frac{N}{2} - \lambda$.

Examples ($N\!=\!1$):

chiral multiplet $\lambda = \frac{1}{2}$:

$$
\begin{array}{rcl}
\text{helicity} \quad \frac{1}{2} & : & \text{Majorana Fermion} \\
0 & : & \text{complex scalar} \\
-\frac{1}{2} & : & \text{Majorana Fermion}
\end{array}
\tag{40}
$$

vector multiplet $\lambda = 1$:

$$
\begin{array}{rcl}
\text{helicity} \quad 1 & : & \text{gauge field} \\
\frac{1}{2} & : & \text{Majorana Fermion} \\
-\frac{1}{2} & : & \text{Majorana Fermion} \\
-1 & : & \text{gauge field} \quad .
\end{array}
\tag{41}
$$

Exercise: What is the field content of a minimal $N\!=\!2$ YM-theory?

3.2.2 Massive Hypermultiplets

All fields of a super multiplet will have the same mass M. We choose the rest frame $p_\mu\!=\!(M,0,0,0)$. Using again the Majorana property we have in the absence of central charges

$$\{Q^i_\alpha, Q^{j\dagger}_\beta\} = M\delta^{ij}\delta_{\alpha\beta} \tag{42}$$

that is

$$\{Q^i_1, Q^{j\dagger}_1\} = \{Q^i_2, Q^{j\dagger}_2\} = M\delta^{ij}, \tag{43}$$

with all other commutators vanishing. One then proceeds as in the massless case with the difference however that the number of generators is twice that of the massless representations. Correspondingly the dimension of a supersymmetry multiplet is now 2^{2N}. Note that in the presence of central charges there is a possibility of constructing massive representations that are in 'small' i.e. 2^N-dimensional representations of the SUSY algebra.

Exercise: What is the condition between mass and central charge for a massive representation of a $N\!=\!2$ SUSY algebra to be 2^N, rather than 2^{2N}-dimensional.

Solution: Making use of the Majorana property and the γ_5-invariance (27) can be written as

$$\{Q_1^i, Q_1^{j\dagger}\} = \{Q_2^i, Q_2^{j\dagger}\} = M\delta^{ij} + \epsilon^{ij}\gamma^0_{\alpha\beta}Z, \tag{44}$$

where Z is a real number. The second term on the right of (44) has eigenvalues $\pm Z$. In the eigenbasis the it is therefore proportional to $M \pm Z$. In particular the combinations $Q_1^1 \pm Q_2^{2\dagger}$ diagonalise the r.h.s. For $Z = \pm M$ there is then only one linearly independent, non-vanishing commutator, reducing the representation to a 2^N-dimensional one. The little group of $p_\mu = (M, 0, 0, 0)$ is generated by J_{12}, J_{23}, J_{13}, i.e. the $SO(3) \sim SU(2)$ algebra. The states are labelled by the spin which is an $SU(2)$-Casimir. The combinations $Q_1^1 \pm Q_2^{2\dagger}$ and $Q_1^{1\dagger} \pm Q_2^2$ form a doublet with spin $\frac{1}{2}$. This follows from (27) and its hermitian conjugate. For $N > 2$, Z^{ij} is a totally antisymmetric $N \times N$ matrix. For N even there exists a unitary transformation U that brings Z into the form $\epsilon^{ij} \otimes$ diag$(\lambda_1, \cdots, \lambda_{\frac{N}{2}})$.

4 Superspace

In order formulate manifestly SUSY-invariant field theories we first generalise ordinary Minkowski space to super-Minkowski space (or short super space) $\mathbb{R}^{4|2N}$. It can be realised as the coset SP/L of the super-Poincaré group and the Lorentz group. Indeed, any element of an $N=1$ SP can be written in the form[3]

$$g = \exp\{a^\mu P_\mu + \epsilon^{\dot\alpha} Q_{\dot\alpha} + \epsilon^\alpha Q_\alpha\} \exp\{\frac{1}{2}\omega^{\mu\nu} J_{\mu\nu}\}, \tag{45}$$

where we have switched to Weyl notation ($Q_\beta = (Q_\alpha, Q_{\dot\alpha})$; $\alpha, \dot\alpha = 1, 2$) for the supercharges. The coset space is then parametrised by coordinates $(x^\mu, \theta^\alpha, \theta^{\dot\alpha}) \equiv z^A$ corresponding to the group element

$$\exp\{x^\mu P_\mu + \theta^\alpha Q_\alpha + \theta^{\dot\alpha} Q_{\dot\alpha}\}. \tag{46}$$

Superspace is therefore the tensor product of Minkowski space with the Grassmann algebra of anticommuting variables

$$\Lambda = \oplus_{l=0}^n (\wedge V)^l \tag{47}$$

where V is a vector space over $\mathbf{C}$. The left action by a group element g_0 is equivalent to the coordinate transformation

$$\begin{aligned} x^\mu &\to x^\mu + i\epsilon^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\theta^{\dot\alpha} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\epsilon^{\dot\alpha} + \omega^{\mu\nu}x_\nu \\ \theta^\alpha &\to \theta^\alpha + \epsilon^\alpha + \frac{1}{4}(\sigma^{\mu\nu})^\alpha_\beta\theta^\beta\omega_{\mu\nu} \\ \bar\theta^{\dot\alpha} &\to \bar\theta^{\dot\alpha} + \epsilon^{\dot\alpha} + \frac{1}{4}\bar\theta^{\dot\beta}(\bar\sigma^{\mu\nu})_{\dot\beta}{}^{\dot\alpha}\omega_{\mu\nu}. \end{aligned} \tag{48}$$

[3]For simplicity we restrict ourself here to $N=1$ supersymmetry

Exercise: How does $z^{\mathcal{A}}$ transform under the product of two left multiplications?

We define left- and right differentiation on the algebra by

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta, \tag{49}$$

or, in two component notation[4]

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta \quad \text{and} \quad \frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}\bar{\theta}^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}, \tag{50}$$

with a sign change for left differentiation. Similarly we can define an integration on the algebra as the functional

$$\int d\theta\, f(\theta) = \frac{\partial}{\partial\theta} f(\theta)|_{\theta=0}. \tag{51}$$

In analogy with usual x-space integrals we may define δ-functions by

$$\int d\theta\delta(\theta-\theta')f(\theta) = f(\theta'), \tag{52}$$

and correspondingly for more that one Grassmann variable. For example, the delta function for $N=1$-superspace, $\delta^4(x-x')\delta^4(\theta-\theta')$ can be represented by

$$4\delta^4(x-x')(\theta-\theta')^2(\bar\theta-\bar\theta')^2. \tag{53}$$

From the above it follows in particular that

$$\int d^8z\, f(z) = \int d^4x \mathrm{d}^2\theta d^2\bar\theta f(x,\theta,\bar\theta), \tag{54}$$

is an invariant integration over the coset space SP/L. Functional differentiation in superspace is then simply

$$\frac{\delta f(x',\theta',\bar\theta')}{\delta f(x,\theta,\bar\theta)} = \delta^4(x-x')\delta^4(\theta-\theta'). \tag{55}$$

4.1 Superfields

We now consider functions (superfields) on the superspace introduced above. A scalar superfield is defined by its transformation property:

$$\phi'(z') = \phi(z), \tag{56}$$

or, under an infinitesimal transformation

[4] The lowering of the the internal index j for the antichiral fermions is purely conventional. The dotted indices indicate that $\bar{Q}_j$ transforms according to the $(0,\frac{1}{2})$-representation

$$\delta\phi(z) = \phi(z + \delta z) - \phi(z) = \zeta^A X_A \phi(z), \tag{57}$$

where

$$\begin{aligned}
X_\mu &= \partial_\mu \\
X_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \\
X_{\dot\alpha} &= \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\sigma^\mu)_{\alpha\dot\alpha}\theta^\alpha\partial_\mu \\
X_{\mu\nu} &= -(x_\mu\partial_\nu - x_\nu\partial_\mu) - \frac{1}{2}\theta^\beta(\sigma^{\mu\nu})_\beta{}^\alpha\frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\bar\theta^{\dot\beta}(\bar\sigma^{\mu\nu})_{\dot\beta}{}^{\dot\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}},
\end{aligned} \tag{58}$$

are the Killing vector fields and $\zeta^A = (a^\mu, \epsilon^\alpha, \bar\epsilon^{\dot\alpha}, \omega^{\mu\nu})$ are the infinitesimal parameters of the transformation.

A general superfield will be of the form ϕ^i_A where i denotes the collective index for the internal indices and A for the space-time indices. The obvious generalisation of (56) is then

$$\phi'^i_A(z^g) = [D(g)]^{iB}_{Aj}\ \phi^j_B(z). \tag{59}$$

Next we introduce the covariant derivatives, that is derivatives that (anti-) commute with supersymmetry transformations. It is easy to see (exercise) that these are obtained from the first three Killing vectors in (58) by simply changing the sign of the second term, i.e.

$$\begin{aligned}
D_\mu &= \partial_\mu \\
D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \\
D_{\dot\alpha} &= \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\sigma^\mu)_{\alpha\dot\alpha}\theta^\alpha\partial_\mu
\end{aligned} \tag{60}$$

Note that the connection appearing in the covariant derivatives (60) is not torsion free. Indeed computing the (anti-) commutators of the different covariant derivatives in (60) we obtain

$$[D_A, D_B\} = T_{AB}{}^C D_C, \tag{61}$$

where

$$T_{\alpha\dot\alpha}{}^\mu = -2i(\sigma^\mu)_{\alpha\dot\alpha}, \tag{62}$$

and all other elements of the torsion vanishing.

4.2 Superfield Actions

We are now ready to construct supersymmetric field theories choosing different superfield functional that is, superspace integrals of suitable combinations of superfields. The simplest example, is

$$\int d^4x \mathrm{d}^2\theta d^2\bar{\theta}\ \phi(x,\theta,\bar{\theta}). \tag{63}$$

However if we want irreducible representations of the supersymmetry algebra, (63) is not a valid action. Indeed, the constraint

$$D_{\dot{\alpha}}\phi = 0 \tag{64}$$

is supersymmetry invariant and hence ϕ as it stands is not an irreducible multiplet. We can remedy this by considering only chiral scalar fields ϕ which satisfy (64). we shall see that chiral super fields form an irreducible representation of the $N=1$ SUSY-algebra. Substitution of a chiral super field into (65) however leads to a zero result. The next simplest possibility is then

$$\int d^4x \mathrm{d}^2\theta d^2\bar{\theta}\ \bar{\phi}(x,\theta,\bar{\theta})\phi(x,\theta,\bar{\theta}), \tag{65}$$

which defines a valid supersymmetric field theory. To see what it corresponds to in terms of Bosons and fermions we need to disentangle the component field content of the chiral superfield ϕ. Due to the constraint (64) the only independent degrees of freedom in ϕ are

$$\begin{aligned} \varphi &\equiv \phi|_{\theta=0}, \\ \chi_\alpha &\equiv D_\alpha\phi|_{\theta=0}, \\ d &\equiv -\frac{1}{2}D^2\phi|_{\theta=0}, \end{aligned} \tag{66}$$

where $D^2 \equiv D^\alpha D_\alpha$. Higher derivatives of the superfield vanish identically. Assuming that the fields fall off fast enough at infinity we then use $\mathrm{d}\theta^\alpha = D_\alpha$ and hence (65) becomes

$$\begin{aligned} \frac{1}{4}\int d^4x\ \bar{D}^2\ D^2\bar{\phi}\phi|_{\theta,\bar{\theta}=0} &= \int d^4x\ \bar{D}^2(\bar{\phi}D^2\phi)|_{\theta,\bar{\theta}=0} \\ &= \int d^4x\ \left(|d|^2 + \frac{1}{4}\left[\bar{\chi}^{\dot{\alpha}}[\bar{D}_{\dot{\alpha}}, D^2]\phi - \{D^{\dot{\alpha}}, [\bar{D}_{\dot{\alpha}}, D^2]\}\phi\right]\Big|_{\theta,\bar{\theta}=0}\right) \\ &= \int d^4x\ \left(|d|^2 - i\bar{\chi}\partial\!\!\!/\chi + 2\bar{\varphi}\partial^2\varphi\right), \end{aligned} \tag{67}$$

where we have used (see Appendix)

$$\{D_\alpha, D_{\dot{\alpha}}\} = 2i(\sigma^\mu)^{\alpha\dot{\alpha}}\partial_\mu \tag{68}$$

and

$$(\sigma^\mu)^{\alpha\dot{\alpha}}(\sigma^\nu)_{\alpha\dot{\alpha}} = (\sigma^\mu)^{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)_{\dot{\alpha}\alpha} = \eta^\mu_\nu. \tag{69}$$

The component field action is therefore just that of a free Weyl fermion and a complex scalar field. Note the presence of the so-called *auxiliary field d*. It has two purposes: i) to match the counting between fermionic and bosonic degrees of freedom of shell and ii) to linearise the supersymmetry transformations. We can read off the supersymmetry transformations of the component fields repeating (66) after making an infinitesimal super translation (57). Alternatively, using that the covariant derivatives commute with the Killing fields we have

$$\begin{aligned} \delta\varphi = (\epsilon^{\alpha} D_{\alpha} + \bar{\epsilon}^{\dot{\alpha}} D_{\dot{\alpha}})\phi|_{\theta,\bar{\theta}=0} &= \epsilon^{\alpha}\chi_{\alpha} \\ \delta\chi_{\alpha} = (\epsilon^{\alpha} D_{\alpha} + \bar{\epsilon}^{\dot{\alpha}} D_{\dot{\alpha}})D_{\alpha}\phi|_{\theta,\bar{\theta}=0} &= \epsilon^{\alpha} d - 2i(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\partial_{\mu}\varphi \\ \delta d = -\frac{1}{2}(\epsilon^{\alpha} D_{\alpha} + \bar{\epsilon}^{\dot{\alpha}} D_{\dot{\alpha}})D^{2}\phi|_{\theta,\bar{\theta}=0} &= 2i(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\partial_{\mu}\chi^{\alpha}. \end{aligned} \tag{70}$$

Let us now consider a Yang-Mills multiplet. As the lowest component is a spinor we expect the corresponding superfield to be a spinorial one. Indeed the chiral superfield W_{α} with $D_{\dot{\alpha}} W_{\alpha} = 0$ has the components

$$\lambda_{\alpha}, F_{\alpha\beta} = F_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta} \quad \text{and} \quad D \tag{71}$$

respectively. In order to identify $F_{\mu\nu}$ with the field strength we impose the Bianchi constraint $\partial_{[\mu} F_{\nu\lambda]} = 0$ which, in superspace becomes (exercise)

$$D^{\alpha} W_{\alpha} = D^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}. \tag{72}$$

This constraint can be solved introducing a real superfield V as

$$W_{\alpha} = \bar{D}^{2}(e^{-V} D_{\alpha} e^{V}). \tag{73}$$

Under gauge transformations the *prepotential* V transforms like

$$e^{V} \to e^{i\Lambda^{\dagger}} e^{V} e^{-i\Lambda}, \tag{74}$$

where Λ is a chiral super field[5]. The Yang-Mills action is simply

$$\mathrm{Tr} \int d^{4}x \, d^{2}\theta \, \left(W^{\alpha} W_{\alpha} \right). \tag{75}$$

Next we consider an $N = 2$ multiplet. As explained in section 3, a $N = 2$ supermultiplet containing the gauge field contains furthermore two Weyl fermions and a complex scalar. In terms of the $N = 1$-superfields, ϕ introduced earlier the $N = 2$-superfield $\mathcal{A}$ can be written as

$$\mathcal{A} = \phi + \theta^{2\alpha} W_{\alpha} + \theta^{2\alpha} \theta^{2}_{\alpha} D^{2} \phi, \tag{76}$$

or equivalently

$$\phi = \mathcal{A}|_{\theta^{2}=0}, \quad W_{\alpha} = D^{2}_{\alpha} \mathcal{A}|_{\theta^{2}=0}. \tag{77}$$

In order to satisfy the chirality and Bianchi condition for the $N = 1$ superfields $\mathcal{A}$ must satisfy the $N = 2$-constraints

[5]The chirality of Λ is required in order to preserve the chirality of ϕ.

$$D^i_{\dot\alpha}\mathcal{A}=0, \quad i=1,2 \quad \text{and} \quad D^{ij}\mathcal{A}=\bar{D}^{ij}\bar{\mathcal{A}}, \tag{78}$$

where $D^{ij}=D^{i\alpha}D^j_\alpha$. Unlike the $N=1$ constraints, a closed solution for the $N=2$-constraints is not available. The action of the $N=2$-YM-theory then is given by

$$\begin{aligned} S &= \frac{1}{8\pi}\mathrm{Im}\mathrm{Tr}\int d^4x d^2\theta_1 d^2\theta_2\, \tau\mathcal{A}^2 \qquad (79)\\ &= \frac{1}{16\pi}\mathrm{Im}\mathrm{Tr}\int d^4x\int d^2\theta\, \tau W^\alpha W_\alpha + \frac{\mathrm{Im}(\tau)}{16\pi}\mathrm{Tr}\int d^4x\int d^2\theta\, d^2\bar\theta\, \phi^\dagger\phi \\ &= -\frac{1}{g^2}\mathrm{Tr}\int\Big\{\frac{1}{4}(F_{\mu\nu})^2-\frac{1}{2}\varphi^\dagger\Delta\varphi+\frac{i}{2}\bar\chi^i\bar\sigma^\mu D_\mu\chi_i+\frac{i}{2}\chi^i\sigma^\mu D_\mu\bar\chi_i \\ &\quad -\frac{i}{2}\varphi^\dagger\{\chi^i,\chi_i\}+\frac{i}{2}\varphi\{\bar\chi^i,\bar\chi_i\}-\frac{1}{4}[\varphi^\dagger,\varphi]^2-X^2\Big\}, \\ &\quad +\frac{i\theta}{8\pi^2}\mathrm{Tr}\int\frac{1}{4}F_{\mu\nu}\tilde F^{\mu\nu}, \end{aligned}$$

where $\tau=\frac{i4\pi}{g^2}+\frac{\theta}{2\pi}$ and X is an auxiliary field. The prepotential V has been absorbed in the definition of the chiral super field ϕ which is therefore *covariantly chiral* i.e. $[\nabla_{\dot\alpha},\phi]=0$ where $\nabla_{\dot\alpha}=e^{-V}D_\alpha e^V$. Note that the chiral fields are in the adjoint representation. An important consequence of this fact is the presence of flat directions in the potential for the scalar component φ of the $N=2$-multiplet, $V(\varphi)=\frac{1}{g^2}\mathrm{Tr}[\varphi,\varphi^\dagger]^2$, where g is the coupling constant. The potential vanishes for constant φ taking its value in the Cartan subalgebra of the gauge group. For $\varphi\neq 0$, the Higgs mechanism breaks the gauge symmetry spontaneously down to $U(1)^l$, where l is the rank of the Cartan subalgebra. In what follows we take the gauge group to be $SU(2)$. The classical theory is then parameterised by two real parameters, g^2 and $|a|$, were $a=\mathrm{Tr}(\langle\varphi\rangle\sigma_3)$ is the expectation value of the scalar component of the $N=2$ superfield[6]. The mass of the charged fields (with respect to the remaining $U(1)$) is proportional to a. In addition, this model allows for solitonic solutions with magnetic- or dyonic charge. The simplest of them is the so-called t'Hooft-Polyakov monopole. It is obtained starting with a radially symmetric ansatz:

$$\varphi_a=\frac{x^a}{gr^2}H(agr) \qquad A^0_a=0 \qquad A^i_a=-\epsilon_{aij}\frac{x^j}{gr^2}\left[1-K(agr)\right] \tag{80}$$

Insertion of (80) in the equations of motions yields the system of coupled differential equations

$$\begin{aligned} r^2\frac{d^2K}{dr^2} &= KH^2+K(K^2-1) \\ r^2\frac{d^2H}{dr^2} &= 2K^2H. \end{aligned} \tag{81}$$

The boundary conditions are determined by imposing finite energy and $\varphi^2\to a^2$ for $r\to\infty$. The solutions of (81) are given by

$$K(\xi)=\frac{\xi}{\sinh\xi} \qquad H(\xi)=\frac{\xi}{\tanh\xi}-1 \tag{82}$$

[6]The phase of a is irrelevant because of the R-symmetry $\mathcal{A}\to e^{i\alpha}\mathcal{A}$

The mass of these states can be obtained by computing the their energy. Substituting (80,82) into the energy functional yields

$$E = \int d^3x \partial^i \left[B^i_a \varphi_a \right] \equiv a g_m \tag{83}$$

where $\mathbf{B^a}$ is the non-abelian magnetic field and $g_m = 4\pi/g$ is the magnetic charge of the monopole solution (82). Corrigan et al. [5] have shown that the most general solution of the previous equations is of the form

$$A^\mu_a = \frac{1}{a^2 e} \epsilon_{abc} \varphi_b \partial^\mu \varphi_c + \frac{1}{a} \varphi_a B^\mu \tag{84}$$

where B^μ is arbitrary. The corresponding non-abelian magnetic field

$$B^i_a = -\frac{1}{2} \epsilon_{ijk} F_{a\ jk} = -\frac{1}{2a^4 e} \varphi_a \epsilon_{ijk} \epsilon^{bcd} \varphi^b \partial^j \varphi^c \partial^k \varphi^d \tag{85}$$

is aligned with the Higgs field. The magnetic charge is therefore

$$g_m = \frac{4\pi}{g} \epsilon_{abc} \int \mathrm{d}\varphi^a \wedge \mathrm{d}\varphi^b \wedge \mathrm{d}\varphi^c. \tag{86}$$

It is clear from (86) that the magnetic charge is topological since it counts the times that the two-sphere, defined by $\varphi^2 = a^2$ is covered when the two-sphere at infinity in space is covered once. To summarise we get

$$M = ag \quad \text{and} \quad M = a g_m, \tag{87}$$

for the masses of the electrically and magnetically charged states respectively. This structure generalises to states with both electric and magnetic charge (dyons) [6]. In the quantum theory the monopoles and dyons correspond to a new particles of the spectrum which are extended objects (with size $\sim 1/a$) located in the region where the energy density is appreciably different from zero. The quantum Hamiltonian is normally taken to be the Laplace operator on the moduli space of the classical solutions [7].

According our discussion of the massless and massive representations of the supersymmetry algebra in section 3, assuming continuity in a, the central extension for these states must be equal to their mass. To see that this is indeed the case we determine the central extension in these charge sectors. It is convenient to combine the two Majorana supercharges Q_1 and Q_2 into a Dirac supercharge $Q = Q_1 - iQ_2$. For the classical action (79) Q is then found to be

$$Q = \frac{1}{2g^2} \mathrm{Tr} \int \mathrm{d}^3x \ \mathcal{S} \psi, \tag{88}$$

where $\mathcal{S} = X + iY$ and

$$\begin{aligned} X &= -i\gamma \cdot (g^2 \mathbf{\Pi} + i\gamma_5 \mathbf{B}) - i\gamma^0 \gamma_5 [A, B], \\ Y &= g^2 (\pi_A - i\gamma_5 \pi_B) + \gamma \cdot \mathbf{D}(A + i\gamma_5 B)\gamma^0. \end{aligned} \tag{89}$$

Here A and B are the real- and imaginary part of the scalar field φ respectively and

ψ is a Dirac Fermion. The central charges U and V in turn are related to the Dirac super charges by

$$U = \frac{1}{4i}\mathrm{Tr}\{Q, \bar{Q}\} \quad \text{and} \quad V = \frac{1}{4i}\mathrm{Tr}(\{Q, \bar{Q}\}\gamma_5), \tag{90}$$

respectively. Evaluating the Poison brackets and taking the trace one one finds

$$U + V = \mathrm{Tr}\int \mathrm{d}^3x[\mathbf{\Pi}\cdot\mathbf{D}\varphi + 2[A, B]\pi_\varphi^\dagger + i\{\psi, \pi_\psi\}], \tag{91}$$

where $\pi_\varphi^\dagger = \frac{1}{2}(\pi_A + i\pi_B)$. Now, using the Bianchi identity $\mathbf{D}\cdot\mathbf{B}=0$ and Gauss's law

$$\mathbf{D}\cdot\mathbf{\Pi} = i[\varphi, \pi_\varphi] + i[\varphi^\dagger, \pi_\varphi^\dagger] + i\{\psi, \pi_\psi\} \tag{92}$$

the centre (91) can be written as a pure boundary term

$$U + V = \mathrm{Tr}\int \mathrm{d}\Omega\varphi\left(\mathbf{\Pi} + \frac{\tau}{4\pi}\mathbf{B}\right). \tag{93}$$

Using the asymptotic behaviour of the fields appearing in (93) the centre becomes

$$U + V = a(n_e + \frac{i}{g^2}n_m), \tag{94}$$

where n_e and n_m are the electric and magnetic charges respectively. This last formula shows that the classical spectrum of $N=2$ YM-theory indeed coincides with the BPS-states.

5 Quantisation

In this lecture we consider the quantisation of supersymmetric Yang-Mills theory. The quantum theory is encoded in the set of all Green functions that is the Schwinger functional $\mathcal{W}[J]$, or, equivalently in the 1PI effective action $\Gamma[\varphi, \mathbf{W}_\alpha]$, which is the Legendre transform of $\mathcal{W}$. Evaluated at zero momentum the effective action reduces to the effective potential. It characterises the ground state of the quantum theory and, in particular determines the value of the different order parameters. What distinguishes supersymmetric theories is that the effective (super[7]) potential is an integral over only half of the superspace, in other words a holomorphic function (or more precisely, a section) of the superfield. This strongly restricts the general form of the super potential and often allows to determine it exactly.

5.1 Background Field Effective Action

The background field effective action is best adapted to supersymmetric theories. In contrast to the standard definition via Legendre transform it is (formally) defined by

[7]Because a superfield also contains derivatives of the component fields, the super potential typically contains some derivative terms as well.

$$e^{\Gamma[\phi]} \equiv \int [D\Phi] e^{S[\Phi+\phi] - \int d^4x \, \frac{\delta\Gamma}{\delta\phi}\Phi + S_{gf}[\Phi,\phi]}, \tag{95}$$

where ϕ is a general background field and Φ is the fluctuation field and is integrated out. In a loopwise ($\hbar$) expansion the second term in the exponent of (95) is is of one order lower in $\hbar$ and therefore the definition makes sense. The last term in (95) is the gauge-fixing term which is necessary when ϕ is a gauge field. It can be shown on general grounds [8] that the two definition agree provided the Schwinger functional is evaluated in a particular gauge. In a supersymmetric theory the functional integral which corresponds to a sum over all possible configurations therefore includes also all configurations which are related to a given configuration by supersymmetry transformations. On the other hand, because the action is invariant under such transformations this this is equivalent to multiplication with the supervolume $V_s = \int \mathrm{d}^4x d^2\theta d^2\bar{\theta} = 0$ unless the background field absorbs the θ-integration. This qualitative argument suggests the following structure

$$\begin{array}{rrcl} \text{Partition function:} & Z & = & e^{\Gamma[0]} = \sum\limits_{\{c\,:\,Qc=\bar{Q}c=0\}} e^{S[c]} \\ \text{Super potential:} & e^{\Gamma[\phi]_\theta} & = & \sum\limits_{\{c\,:\,\bar{Q}c=0\}} e^{S[c+\phi]} \\ \text{D-term:} & e^{\Gamma[\phi]_{\theta,\bar{\theta}}} & = & \sum\limits_{\{c\}} e^{S[c+\phi]}, \end{array} \tag{96}$$

where $\{c\}$ denotes the set of configurations corresponding to the functional integral. This simple picture is only approximately correct. This is because the relation between functional integrals and the above sum over configurations is formal. In particular, regularisation and gauge fixing lead to additional complications. The purpose of the following subsections is to make these properties more precise.

5.2 Non-Renormalisation Theorems

Here we discuss two different but related approaches to the non-renormalisation properties. The first uses the holomorphicity [10] and the global symmetries of the classical action. The second is based on simple properties of the super Feynman rules [4].

5.2.1 Holomorphicity

The starting point is the assumption that superpotential is a holomorphic section of the superfield (and the coupling constants), excluding essential singularities. If the classical theory has furthermore continuous global symmetries and if the fate of these upon quantisation is under control, one can make use of these symmetries to derive *selection rules*. Let us illustrate this approach with two examples.

1.) *Wess-Zumino Model:*
The Wess-Zumino model is defined by the classical super-field Lagrangian

$$S=\int d^4x\mathrm{d}^2\theta d^2\bar{\theta}\;\bar{\phi}\phi+\int d^4x\mathrm{d}^2\theta\;\left(m\phi^2+g\phi^3\right). \tag{97}$$

This Lagrangian is invariant under $U(1)\times U(1)_R$-transformations, provided the 'fields' ϕ, m, g are given the charges

$$\begin{array}{cclcl} U(1) & : & Q_\phi=1 \quad , \quad Q_m=-2 \quad , \quad Q_g=-3 \\ U(1)_R & : & Q_\phi=1 \quad , \quad Q_m=0 \quad , \quad Q_g=-1. \end{array} \tag{98}$$

Provided there are no holomorphic anomalies the effective super potential must then be of the form

$$\Gamma[\phi]_\theta=\int d^4x\mathrm{d}^2\theta\; m\phi^2\; f\left(\frac{g\phi}{m}\right), \tag{99}$$

where $f(z)$ is some holomorphic section. For small g perturbation theory is valid and therefore $\Gamma[\phi]$ has can be expanded as

$$\Gamma[\phi]=\sum_{n=0}^{\infty}c_n\frac{g^n\phi^{n+2}}{m^{n-1}}, \tag{100}$$

corresponding to a tree-diagram with $n+2$ external legs, n-vertices and $n-1$ propagators. For $n>1$ such a diagram would however not be 1-particle irreducible and does therefore not contribute to the effective action. Loop corrections are not compatible with the structure of (100) and must therefore be absent. We therefore conclude that in the Wess-Zumino model the classical potential is not modified by quantum corrections.

2.) *SUSY QED*:
Using (74) we find that the minimal gauge invariant combination of chiral fields and YM-fields is given by

$$S=\frac{1}{4\pi}\mathrm{Im}\int\; d^4x\; d^2\theta\tau W^\alpha W_\alpha+\int\; d^4x\; d^2\theta\; d^2\bar{\theta}\left(\phi^\dagger e^V\phi+\tilde{\phi}^\dagger e^{-V}\tilde{\phi}\right), \tag{101}$$

where $\tau=\frac{i4\pi}{g^2}+\frac{\theta}{2\pi}$ and we have used that

$$\mathrm{Im}\int\; d^4x\; d^2\theta\; W^\alpha W_\alpha=\int\; d^4x\; F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{102}$$

The matter fields ϕ and $\tilde{\phi}$ have electric charge 1 and -1 respectively. The effective coupling τ_{eff} can only depend ϕ and $\tilde{\phi}$. The classical theory is invariant under $U(1)_R$-transformations. The R-charges of W_α, ϕ and $\tilde{\phi}$ are 1, 2 and 2 respectively. Due to the axial anomaly the continuous $U(1)_R$ is broken down to the discrete subgroup $\mathbf{Z}_2$. In order to reproduce this anomaly τ_{eff} must be a linear combination of $\log(\phi)$ $\log(\tilde{\phi})$ and $\log(\phi+\tilde{\phi})$. Higher log's are incompatible with the Adler-Bardeen theorem and therefore absent. This in turn implies that the β-function for the gauge coupling gets no contribution above 1-loop. On the other hand an explicit computation shows that the β-function gets contributions at all orders. This discrepancy is due to the presence of massless charged fields which typically lead to holomorphic anomalies [9]. This can be cured by introducing an explicit infrared cut-off. The corresponding effective action is the *Wilsonian effective action* and is in general different from the 1PI-effective action. This shows the limitations of the holomorphic approach.

5.2.2 Super Feynman rules

The derivation of the superfield Feynman rules proceeds in close analogy to the non-supersymmetric case by expanding the Schwinger functional about the free field action. That is, for a general action in $N=1$ superspace

$$S = \int d^4x\ d^2\theta\ d^2\bar{\theta}\ \mathcal{L}(V,\phi,\bar{\phi}) + \int d^4x\ d^2\theta\ V(\phi) + h.c., \tag{103}$$

we define the Schwinger functional $\mathbf{W}[J,j]$ by

$$\begin{aligned} e^{\frac{i}{\hbar}\mathbf{W}[J,j]} &\equiv \int [DV\ D\phi\ D\bar{\phi}]\ e^{\frac{i}{\hbar}S+\frac{i}{\hbar}\int\ d^4x\ d^2\theta\ d^2\bar{\theta} VJ+\frac{i}{\hbar}\int\ d^4x\ d^2\theta\phi j+h.c} \\ &= \exp\left[iS_{int}\left(\frac{1}{i}\frac{\delta}{\delta j},\frac{1}{i}\frac{\delta}{\delta J}\right)e^{\frac{i}{\hbar}\mathbf{W}_0[J,j]}\right], \end{aligned} \tag{104}$$

where $\mathbf{W}_0[J,j]$ is the corresponding Schwinger functional for the free theory. It is important to note that functional integrals are defined only for unconstrained fields[8]. Functional differentiation of both expressions in (104) with respect to the currents then yields the Feynman rules in the usual way:

- i) the propagators are

$$\begin{aligned} \langle\phi(1)\bar{\phi}(2)\rangle &= \frac{-i}{p^2-m^2}\delta_{12} \\ \langle\phi(1)\phi(2)\rangle &= \frac{imD^2}{4p^2(p^2-m^2)}\delta_{12} \\ \langle V(1)V(2)\rangle &= (\Pi_{\frac{1}{2}}+\alpha\Pi_0)\frac{i}{p^2}\delta_{12}, \end{aligned} \tag{105}$$

  where $\Pi_{\frac{1}{2}} = \frac{D^\beta\bar{D}^2D_\beta}{8p^2}$, $\Pi_0 = -\frac{\bar{D}^2D^2+D^2\bar{D}^2}{16p^2}$ are projection operators [4] and α is the gauge-fixing parameter.

- ii) The vertices have a factor $-\frac{1}{4}\bar{D}^2$ $(-\frac{1}{4}D^2)$ for each internal line as well as a $\int \mathrm{d}^2\theta_{vert}\ d^2\bar{\theta}_{vert}$ associated with it.

- iii) Each independent loop receives a $\int\frac{d^4k}{(2\pi)^4}$.

- iv) $\prod\limits_{p_{ext}}\int\frac{d^4p_{ext}}{(2\pi)^4}(2\pi)^4\delta\left(\sum\limits_{p_{ext}}p_{ext}\right)$ for the external momenta.

[8] In the case of the chiral field ϕ both, the constrained and the unconstrained integral lead to the same result. This is not so for the YM-multiplet.

We are now ready to formulate the perturbative version of the non-renormalisation theorem:
Any counter term in the effective action must be a full superspace integral of local functionals of the fields corresponding to the external lines.
If we restrict ourself to graphs with no external lines, then these functionals depend only on the background fields. This then implies that counter terms, holomorphic in the background fields cannot arise in perturbation theory and we recover in particular the previous result for the Wess-Zumino model. What about SUSY QED? First note that the above result does not apply at one loop in super YM-theories. This is due to the gauge fixing where ghost terms must be included. In a supersymmetric theory however the ghosts themselves have a gauge-invariance, and in this way one obtains ghosts for ghosts. This can only be avoided by introducing prepotential for the background fields which invalidates the above results. However, provided there are no massless charged fields in the theory, one can show [9, 11] that the situation is unchanged beyond the the one-loop corrections. Let me illustrate the above remarks for the two-point function on $N=1$ YM-theory. Using the Feynman rules listed above one obtains that the vacuum polarisation contributes to the effective action a term

$$\begin{aligned}
&\propto \; g^2 \int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} d^2\theta_1 d^2\bar{\theta}_1 d^2\theta_2 d^2\bar{\theta}_2 V(-p,\theta_1)\frac{\bar{D}_1^2 D_1^2}{16}\frac{\delta_{12}}{(p^2+k^2)}\frac{D_1^2\bar{D}_1^2}{16}\frac{\delta_{12}}{k^2}V(p,\theta_2)\\
&= \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (106)\\
&= \left(\frac{1}{\epsilon}+O(\epsilon)\right)\int \frac{d^{4-\epsilon}p}{(2\pi)^4} d^2\theta d^2\bar{\theta} V(-p,\theta) D^\alpha \bar{D}^2 D_\alpha V(p,\theta),
\end{aligned}$$

where we have used dimensional regularisation to regulate the divergent momentum integral. Now, using (73) the integral can be written as

$$\int \frac{d^{4-\epsilon}p}{(2\pi)^4} d^2\theta \; W^\alpha(-p,\theta) W_\alpha(p,\theta), \tag{107}$$

and therefore this is indeed a counter term for the super potential leading to the 1-loop running of the gauge coupling. Similarly, contributions of the form

$$\int \frac{d^4p}{(2\pi)^4} d^2\theta d^2\bar{\theta} \; W^\alpha(-p,\theta)\frac{D^\beta D_\beta}{p^2} W_\alpha(p,\theta) \tag{108}$$

arise at higher orders. They contribute to the superpotential unless an infrared cut-off μ is introduced in the momentum integral. This is the so-called Wilsonian effective potential [9]. The problem of construction of such a cut-off without destroying gauge-invariance is to date unsolved.

5.3 Currents in Supersymmetric Theories

The conserved Noether currents for a Poincaré invariant theory are given by

$$T_{\mu\nu} \quad \text{and} \quad L_{\mu\nu\rho} = -x_\nu T_{\mu\rho} + x_\mu T_{\nu\rho} \tag{109}$$

respectively. Apart from from general relativity all fundamental theories are in fact invariant under a bigger group of transformations, namely the conformal transformations, at least if the bare masses are taken to be zero. This group contains in addition to the Poincaré transformations the dilatation D and the special conformal transformations K_ν. The corresponding conserved currents are

$$x^\nu T_{\mu\nu} \quad \text{and} \quad 2x_\nu x^\lambda T_{\lambda\mu} - x^2 T_{\mu\nu}. \tag{110}$$

These currents are conserved provided the stress tensor is traceless i.e. $T^\mu_\mu = 0$. In a supersymmetric theory the set of conserved currents is supplemented by the supersymmetry current $j_{\mu\alpha}$ and the superconformal current $(\not{x} j_\mu)_\alpha$. The corresponding conservation equations are

$$\partial_\mu j^\mu_\alpha = \gamma_\mu j^\mu_\alpha = 0. \tag{111}$$

The idea is that in a supersymmetric theory the different currents form a super multiplet. Indeed the counting of degrees of freedom, taking into account the constraints

$$T_{\mu\nu} \quad : \quad 5, \qquad j_{\mu\alpha} \quad : \quad 8, \qquad j^{(5)}_\mu \quad : \quad 3, \tag{112}$$

as well as the dimension are such that we can collect them into the super current $j_{\alpha\dot\alpha}$ which has $j^{(5)}_\mu$ as its lowest component.

5.4 Superconformal Ward Identities

In a non-supersymmetric theory the the conformal transformations are generated by $\int d^3x J_0$, where $J_\mu = f^\nu T_{\mu\nu}$ and $f^\nu(x)$ satisfies the Killing equation

$$f_{(\mu,\nu)} = \frac{d}{2}\eta_{\mu\nu}(f^\lambda_{,\lambda}). \tag{113}$$

Similarly the superconformal transformations are generated by $f^{\alpha\dot\alpha} j_{\alpha\dot\alpha}$ where $f^{\alpha\dot\alpha}$ satisfies the Killing equation

$$D_{(\alpha} f_{\beta)\dot\beta} = 0. \tag{114}$$

It is solved by writing $\frac{1}{2} f_{\alpha\dot\alpha} = D_\alpha L_{\dot\alpha}$ where $L_{\dot\alpha}$ is an unconstrained superfield. Generically the scale invariance and therefore the superconformal invariance is broken by quantum corrections. We can nevertheless use the classical symmetry to derive Ward-identities for the effective action. For this we couple the theory to supergravity and use the fact that any conformal transformation is equivalent to a Weyl transformation of the metric. Using the invariance of the classical action the Schwinger functional satisfies the formal identity for an arbitrary superconformal transformation

$$\int [DV\ D\Phi\ D\bar\Phi]\ e^{S[H_{\alpha\dot\alpha},\Phi,\bar\Phi,V]+(J,\Phi+\delta\Phi)} = \int [DV\ D\Phi\ D\bar\Phi]\ e^{S[H_{\alpha\dot\alpha}+\delta H_{\alpha\dot\alpha},\Phi,\bar\Phi,V]+(J,\Phi)} \tag{115}$$

where Φ, V are the unconstrained $N=1$ integration variables and $H_{\alpha\dot\alpha}$ is the supergravity prepotential coupling to the supercurrent $j_{\alpha\dot\alpha}$. Expanding both sides of the equality to the first order and using

$$J = \frac{\delta\Gamma[\phi,\bar{\phi}]}{\delta\phi}, \tag{116}$$

where $\Gamma[\phi,\bar{\phi}]$ is the effective action for the chiral superfield ϕ, we obtain

$$\int d^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \frac{\delta\Gamma[\phi,\bar{\phi}]}{\delta\phi}\langle\delta\phi\rangle + h.c. = \int d^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \delta H^{\alpha\dot{\alpha}}\langle j_{\alpha\dot{\alpha}}\rangle. \tag{117}$$

The $N\!=\!1$ fields and the supergravity prepotential transform under an arbitrary transformation as

$$\delta\phi = \bar{D}^2 L^\alpha D_\alpha\phi - q(\bar{D}^2 D_\alpha L^\alpha)\phi \quad \text{and} \quad \delta H^{\alpha\dot{\alpha}} = (D^\alpha L^{\dot{\alpha}} - D^{\dot{\alpha}} L^\alpha), \tag{118}$$

respectively, where L^α is the parameter superfield [13] and q is the R-weight of ϕ. After partially integrating the r.h.s. of (118) we then obtain

$$\begin{aligned} \delta_\Delta\Gamma[\phi,\bar{\phi}] &= \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, (L^\alpha D^{\dot{\alpha}} - L^{\dot{\alpha}} D^{\dot{\alpha}}) J_{\alpha\dot{\alpha}} \\ &= -2\Delta(\int \mathrm{d}^2\theta\, S - \mathrm{h.c.}), \end{aligned} \tag{119}$$

where the last equality uses the anomaly equation [12] and $\Delta\!=\!\bar{D}^2 D^\alpha L_\alpha$.

$$D^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = D_\alpha S, \tag{120}$$

with S a chiral super field ($D_{\dot{\alpha}}S\!=\!0$). This is the super conformal Ward-identity for the chiral $N\!=\!1$ effective action. We will make extensive use of it in the discussion of $N\!=\!2$ YM-theory in the next section.

6 N=2 Effective Action

In this we compute the super potential for $N\!=\!2$ YM theory with gauge group $SU(2)$ by integrating the superconformal anomaly. When expressed in terms of the massless $N\!=\!2$-superfield[9] $\mathcal{A} = \varphi + \theta\chi + \cdots$, the most general local $N = 2$ supersymmetric low energy superpotential must be of the form

$$\Gamma[\mathcal{A}] = \frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta_1 d^2\theta_2 \mathcal{F}(\mathcal{A}), \tag{121}$$

where the prepotential $\mathcal{F}$, to be determined, is the result of integrating out the massive (i.e. charged, root-valued) fields. Using the formalism developed in section 4, it is straight forward to show that in $N\!=\!1$ notation (121) becomes

$$\Gamma[A,W_\alpha] = \frac{1}{8\pi}\mathrm{Im}\,\mathrm{Tr}\int \mathrm{d}^4x\Big\{\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}(\phi_D\bar{\phi} - \bar{\phi}_D\phi) + \int \mathrm{d}^2\theta\tau(a)\, W^\alpha W_\alpha\Big\}, \tag{122}$$

[9]As we neglect derivative terms, it is not meaningful to retain the heavy (root-valued fields) in the super potential.

where ϕ and W_α are the (covariantly) chiral- and vector $N=1$ superfields respectively. Furthermore

$$\phi_D = \mathcal{F}'(\phi) \quad \text{and} \quad \tau(a) = \mathcal{F}''(a) = a'_D(a). \tag{123}$$

The spectrum of the quantum theory is contained in the set of BPS-states of the supersymmetry algebra. The quantum corrections to the centre is determined by the Poison brackets of the effective supercharges i.e.

$$\{Q^{\text{eff}}, \bar{Q}^{\text{eff}}\} = i(\not{P}^{\text{ eff}} - Z^{\text{eff}}), \tag{124}$$

where Q^{eff}_α are the supercharges obtained from the low energy effective Lagrangian $\mathcal{F}(\sqrt{\mathcal{A}^a\mathcal{A}^a})$ and $Z = U + V\gamma_5$. Apart from the fermionic contributions, the effective Lagrangian (122) has the same structure in the classical theory with

$$\begin{aligned} \frac{4\pi}{g^2} &\rightarrow \mathcal{I}_{ab} \equiv \text{Im}\frac{\partial^2\mathcal{F}}{\partial\mathcal{A}^a\partial\mathcal{A}^b} \\ \frac{\theta}{2\pi} &\rightarrow \mathcal{R}_{ab} \equiv \text{Re}\frac{\partial^2\mathcal{F}}{\partial\mathcal{A}^a\partial\mathcal{A}^b}. \end{aligned} \tag{125}$$

Accordingly one has

$$Q = \frac{1}{8\pi}\int \mathrm{d}^3x\ \mathcal{S}^a\mathcal{I}_{ab}\psi^b, \tag{126}$$

where $\mathcal{S} = X + iY$ and

$$\begin{aligned} X^a &= -i\gamma\cdot(4\pi\mathcal{I}^{ab}\tilde{\mathbf{\Pi}}_b + i\gamma_5\mathbf{B}^a) - i\gamma^0\gamma_5[A,B]^a + O(\psi^2), \\ Y^a &= 4\pi\mathcal{I}^{ab}(\tilde{\pi}_A - i\gamma_5\tilde{\pi}_B)_b + \gamma\cdot\mathbf{D}(A + i\gamma_5 B)^a\gamma^0 + O(\psi^2), \end{aligned} \tag{127}$$

with $\mathcal{I}^{ab} = \mathcal{I}^{-1}_{ab}$, $\tilde{\mathbf{\Pi}}^{\mathbf{a}} = \frac{\mathbf{1}}{\mathbf{4\pi}}\mathcal{I}_{\mathbf{ab}}\mathbf{E}^{\mathbf{b}}$ and $\tilde{\pi}_{Aa} = \frac{1}{4\pi}\mathcal{I}_{ab}(D_0\mathbf{A})^b$. Repeating the steps outlined for the classical case one ends up with the simple result [14]

$$U + V = \int \mathrm{d}\Omega\left(\varphi^a\mathbf{\Pi}_a + \frac{1}{4\pi}\varphi_{Da}\mathbf{B}^a\right) = an_e + a_D n_m, \tag{128}$$

where $a_D = \mathcal{F}'(a)$. Hence, independent, of the actual form of the effective potential$\mathcal{F}$, the spectrum of the quantum theory is entirely determined by the quantities a_D and a! The effective coupling $\tau = \frac{\mathrm{d}a_D}{\mathrm{d}a}$ is the coefficient of the kinetic term in (122) its imaginary part must be positive. On the other hand its real part plays the role of an effective θ-angle: $\text{Re}\,\tau = \frac{\theta}{2\pi}$. Thus a shift $\tau \mapsto T(\tau) = \tau + 1$ is a symmetry transformation of the effective potential. The invariance of the chiral part in (122) together with (123) requires then that this transformation induces on (a, a_D) a linear transformation in $U(1) \times SL(2, \mathbf{Z})$. In general the mass spectrum and hence the low energy effective theory my have other invariances which are not symmetries of the effective action. These play an important role for the solution of the model. Let us now compute the effective potential in turn. Writing the superconformal Ward Identity (119) in $N=2$-superspace it becomes

$$\mathcal{F}(\mathcal{A}) - \frac{1}{2}\mathcal{F}'(\mathcal{A})\mathcal{A} = \frac{i}{\pi}\mathcal{U}, \tag{129}$$

where $\mathcal{U}$ is the $N=2$ anomaly multiplet. In $N=2$-Yang-Mills one has $\mathcal{U} = \frac{i}{2\pi}\mathcal{A}^2$ to all orders in perturbation theory as a consequence of the non-renormalisation theorems in the last section. Indeed, because there are no massless charged fields present in the Coulomb phase of this model, the only perturbative contribution to the holomorphic part of the effective action arises from the vacuum polarisation, where the effective field must be expressed in terms of the prepotential in order to avoid an infinite tower of ghosts. The higher loop contributions are absent because for non-zero a the theory has a natural infrared cut-off. Integrating eqn (129) we obtain

$$\mathcal{F}(\mathcal{A}) = \frac{i}{2\pi}\mathcal{A}^2 \log\left(\frac{\mathcal{A}^2}{\Lambda^2}\right). \tag{130}$$

If non-perturbative effects are included the relation between $\mathcal{U}$ and $\mathcal{A}$ becomes more complicated. Integrating the corresponding equation will occupy the rest of these section. In the low energy *effective* potential the phase of a plays a role because the R-symmetry is anomalous at the quantum level. The moduli space of inequivalent vacua is therefore 1-complex-dimensional. The equivalence is here with respect to the mass spectrum. On the other hand a cannot be a global parameter on the moduli space since in that case the monodromy group of the couple (a_D, a) would be abelian, contradicting the positivity of $\mathrm{Im}(\tau)$. We therefore need to choose some other (uniformising) variable to parametrise the theory. The only other complex parameter available is the superconformal anomaly $u = \mathcal{U}|_{\theta=0}$. Let us assume that this is indeed a uniformising variable[10] on the moduli space and think of the couple (a, a_D) and τ as sections of $u \in \mathbf{C} - \{u_i\}$ where $\{u_i\}$ is the set of singular points of these functions. One such singular point is at infinity, where $\tau(u)$ diverges logarithmically and the functions a_D and τ are defined only up to a discrete transformation

$$a_D \to a_D + 2na, \; n \in \mathbf{Z} \quad \text{and} \quad \tau \to \tau + 2n. \tag{131}$$

Note that the latter identification corresponds to a shift of the θ angle by 4π and is therefore a symmetry of the effective action. Given this form at infinity there must be at least two more singular points u_i to avoid that the imaginary part of τ becomes negative. On the other hand, if $u \in \mathbf{C}$ indeed parametrises the space of inequivalent vacua, the mass spectrum must be single-valued everywhere and hence the ambiguity in a_D and a at these points must be by a $SL(2,\mathbf{Z}) \times U(1)$-transformation. From this we conclude in particular that the system a_D, a is equivalent to the solutions of a second order differential equation (Riemann-Hilbert Problem). In fact we can say more. Differentiating the anomaly equation (119) with respect to u we obtain

$$\mathcal{W}(a_D, a) = const. \tag{132}$$

where $\mathcal{W}(a_D, a)$ is the Wronskian of a_D and a. From this we conclude that there is no

[10] This assumption can in fact be derived from more fundamental properties of the theory [15]

first derivative term in the differential equation for $\mathbf{a}=(a_D, a)$ i.e.

$$a'' + Va = 0, \tag{133}$$

for some potential V. Differentiating this equation once more we get for $b=a'$

$$b'' - \frac{V'}{V}b' + Vb = 0. \tag{134}$$

On the other hand it is a well known fact in the theory of conformal mappings that any $SL(2, \mathbf{Z})$-section can be written as

$$\begin{aligned} \tau(u) &= \frac{y_1}{y_2} \quad \text{with} \\ y'' + Qy &= 0 \end{aligned} \tag{135}$$

where Q is of the form

$$Q = \frac{1}{2}\sum_i \frac{1}{2}\frac{1-\alpha_i^2}{(u-u_i)^2} + \frac{\beta_i}{u-u_i}, \tag{136}$$

with $0 \le \alpha_i < 1$. Compatibility of the two differential equations and $\tau = \frac{a'_D}{a'}$ then requires

$$\mathbf{b} = V^{\frac{1}{2}}\mathbf{y} \quad \text{and} \quad \frac{V''}{V} - \frac{3}{2}\left(\frac{V'}{V}\right)^2 + 2V = 2Q, \tag{137}$$

supplemented by the boundary conditions $Q \to 1/4u^2$ and $V \to 1/4u^2$ for $|u| \to \infty$ in order to recover the logarithm at infinity. Near a singular point u_0 of V we have $V \simeq V_0(u-u_0)^\gamma$ with $V_0, \gamma \neq 0$. Eqn (137) then implies

$$-\frac{1}{2}\gamma(\gamma+2)\frac{1}{(u-u_0)^2} + 2V_0(u-u_0)^\gamma \simeq 2Q(u). \tag{138}$$

Consistency with (136) requires that u_0 coincides with one of the singularities u_i of Q and

$$\begin{aligned} V_0 &= \frac{1}{4}(1-\alpha_i^2) \quad \text{for} \quad \gamma = -2 \quad \text{and} \\ \gamma &= -1 \pm \alpha_i \quad \text{for} \quad \gamma > -2. \end{aligned} \tag{139}$$

Consequently V must be of the form

$$V(u) = V\prod_r (u-u_r)^{-2}\prod_s (u-u_s)^{\pm\alpha_i - 1}, \tag{140}$$

where $u_{r,s}$ are the singular points of $Q(u)$. On the other hand, substitution of (140) into (137) implies that V is rational function with no zeros. The boundary condition then implies that V has only simple and double poles. But because it must have at least two poles it can have only simple poles and hence it has precisely two simple poles. More precisely

$$V(u)=\frac{1}{4(u-a)(u-b)}. \tag{141}$$

We are free to choose $a{=}-b{=}1$. Furthermore Q takes the form

$$Q=\frac{1}{2}\left(\frac{1}{(u-1)^2}+\frac{1}{(u+1)^2}\right)-\frac{1/2}{(u-1)(u+1)}. \tag{142}$$

The equation for $a_D(u)$ and $a(u)$ therefore becomes a standard hypergemoetric differential equation with solutions [16]

$$\begin{aligned} a_D(u) &= i\frac{u-1}{2}F\left(\frac{1}{2},\frac{1}{2},2;\frac{1-u}{2}\right), \\ a(u) &= \sqrt{2}\sqrt{u+1}F\left(-\frac{1}{2},\frac{1}{2},1;\frac{2}{u+1}\right). \end{aligned} \tag{143}$$

The prepotential can then, at least in principle be obtained by solving for $a_D(a)$. Instead of pursuing this route we note that the monodromy group associated with (142) is

$$\Gamma_2=\{M\in SL(2,\mathbf{Z})\mid M-\mathbf{1}_2=0\,\mathrm{mod}2\}. \tag{144}$$

This group is generated by

$$T^{-2}=\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}\quad\text{and}\quad (T^2)^T=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}. \tag{145}$$

The first generator corresponds to the the singularity at infinity whereas the second to $u{=}1$. Hence we conclude that the involution

$$u\to 1-\frac{4}{1-u}, \tag{146}$$

is equivalent to exchanging a_D and a i.e.

$$\begin{pmatrix}a_D\\ a\end{pmatrix}\to\begin{pmatrix}-a\\ a_D\end{pmatrix}, \tag{147}$$

or, equivalently

$$\tau\to\frac{-1}{\tau}\equiv\tau_D. \tag{148}$$

But this is precisely the electric-magnetic duality transformation $g\to\frac{4\pi}{g}$ and $n_e\leftrightarrow n_m$. This is the manifestation of electric-magnetic duality in this model. It tells us that for $u\simeq 1$ the system is can equivalently be described by strongly coupled electrically charged fields or, by weakly coupled monopole fields. The only $N{=}2$ supersymmetric theory with the same number of degrees of freedom and which is weakly coupled in the infrared $u\simeq 1$ is $N{=}2$ QED where we replace the electric charges by magnetic charges. The action for this theory is given by the usual kinetic terms plus a potential (in $N=1$ formulation)

$$\int d^4x\ d^2\theta\ \tilde{M}\phi_D M, \tag{149}$$

where M and $\tilde{M}$ are the $n=1$ monopole fields in the fundamental representation. We can now break $N=2$ SUSY to $N=1$ by adding a mass term to the chiral multiplet in the classical action (79). That is

$$S \to S + m^2 \int d^4x\ d^2\theta\ \mathrm{Tr}(\phi^2). \tag{150}$$

The dual effective potential then becomes

$$\mathrm{Re}\int d^4x\ d^2\theta\ \left(\tilde{M}\phi_D M + m\langle \mathrm{Tr}\phi^2\rangle\right) = \mathrm{Re}\int d^4x\ \left(\tilde{\mathcal{M}}a_D\mathcal{M} + m^2\langle \mathrm{Tr}\varphi^\dagger\varphi\rangle\right), \tag{151}$$

where $\mathcal{M}$ is the scalar component of the chiral monopole multiplet. Now, it can be shown [17] that $\langle \mathrm{Tr}\varphi^2\rangle = u$. Using this we then obtain upon minimising the dual effective potential

$$\begin{aligned} a_D &= 0 \quad \text{and} \\ \tilde{M} = M &= \left(-\frac{1}{2}m\frac{\bar{u}}{u}\frac{\mathrm{d}u}{\mathrm{d}a_D}|_{a_D=0}\right)^{\frac{1}{2}} \neq 0. \end{aligned} \tag{152}$$

Hence the monopoles condense! In other words the vacuum of softly broken $N=2$ YM-theory is similar to that of a dual superconductor. In particular the electric flux lines are confined to a tube, which is precisely what is necessary to create a linear potential for the quarks (here gluinos). We therfore have proved that this theory confines! This is the main result of Seiberg and Witten's seminal paper [18] on $N = 2$ Yang-Mills theory.

The introductory part of these notes is based partly on P. West's book [4] and on A. Chamsedine's lecture notes [19] on Supersymmetry. The material of section 6 relies on unpublished results obtained in collaboration with M. Magro and L. O'Raifeartaigh.

7 Appendix: Conventions

We choose the "relativist's" signature for the metric

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \tag{153}$$

The charge conjugation matrix in 4-dimension has the representation independent properties

$$C^T = C^\dagger = -C \quad \text{and} \quad CC^\dagger = C^\dagger C = -C^2 = 1 \tag{154}$$

In the chiral representation of the gamma-matrices i.e.

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu \\ i\bar\sigma^\mu & 0 \end{pmatrix} \quad \text{where} \quad (\sigma^\mu)^{\alpha\dot\beta} = (1, \vec\sigma) \quad \text{and} \quad (\bar\sigma^\mu)_{\dot\beta\alpha} = (1, -\vec\sigma) \tag{155}$$

the charge conjugation matrix is then given by $C = i\gamma^0\gamma^2$.

Majorana spinors satisfy $\bar\psi \equiv \psi^\dagger\gamma^0 = \psi^T C$, or equivalently, in two-component notation

$$\psi \equiv \begin{pmatrix} \chi \\ -\sigma^2\bar\chi \end{pmatrix}. \tag{156}$$

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