Intertemporal Utility and Time Consistency*

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Abstract: We give a characterization of time-consistent preferences for the infinite time horizon case. Using Banach's fixed point theorem we show that there is a one-to-one relation between time consistent intertemporal utility functions and real-valued functions on the unit square. We investigate monotonicity as well as concavity properties of the time-consistent utility functions.

Keywords: Intertemporal utility, preferences, time consistency, optimal plans, fixed point theorems.

JEL classification: C73, D11, D91.

*We would like to thank Roger Hartley, Max Neunhöffer, John Proops and Alistair Ulph for enlightening discussions.

†The work has been supported in part by the HCM initiative of the Commission of the European Communities (Contract No. ERBCHBICT930815).

Contents

1 Introduction

It has been recognized that intertemporal planning may yield results that are inconsistent with a revised plan which is performed after a certain period of time (see, e.g., [Strotz 1955, Peleg and Yaari 1973, Blackorby et al. 1973, Kohlberg 1976], and [Thaler and Shefrin 1981]).

In particular, in models of altruistic generations, some generation's optimal future consumption path is inconsistent with the following generations' optimal plans. In the literature, two possibilities to deal with this problem are discussed.

First, each generation acts according to its optimal plan. Thus, it ignores that the plan will not be fulfilled by the following generations. Each generation revises the intertemporal plan. Only the first period of each plan is realized. This is known as the naive society [Blackorby et al. 1973]. As no plan is actually realized, each generation may suffer some loss of welfare. This aspect is analyzed by [Grout 1982].

Secondly, each generation takes into account the preferences and the behaviour of the following generation. Thus, the last (if it exists) generation determines its plan as a response to its initial endowments. Then the plans are recursively determined. It is obvious that this procedure yields time-consistent plans. In other words, the first generation's plan will be accepted to be optimal by all the following generations. The result of this procedure is a subgame-perfect Nash equilibrium in the game between the generations.[1] Note that this approach requires perfect information about future preferences. However, the problem of time-consistent planning under uncertainty has also been discussed in the literature (see, e.g., [Johnsen and Donaldson 1985]).

Time inconsistency in naive intertemporal planning is very likely. However, for the finite time horizon case, [Blackorby et al. 1973] characterize the sequences of intertemporal utility functions that yield time-consistent optima, even in the case of naive planning.

[1] For a discussion of this so-called Strotz-Pollak equilibrium see, e.g. [Strotz 1955, Pollak 1968, Peleg and Yaari 1973, Goldman 1980, Leininger 1986].

In this work we extend this characterization to the infinite time horizon case. Due to the lack of a starting point for the backward induction procedure and a different topological situation, several modifications are necessary. Relying on fixed point arguments, we classify the time-consistent sequences of intertemporal utility functions on the set of consumption streams (of infinite length).

We analyze the monotonicity and concavity properties of these time-consistent sequences of utility functions.

This article is organized as follows. Section 2 develops a criterion for time-consistent preferences in the infinite time horizon case. In Section 3 we deal with monotonicity and concavity of the intertemporal utility functions. Some examples illustrate our findings in Section 4. Section 5 summarizes the results. An appendix contains some technical details.

2 Time Consistency

We consider discrete time and denote consumption at time t by c_t. In our model we do not allow for unbounded consumption. That is, we consider the case $c_t \in [0,1]$ (after some rescaling, if necessary). The complete consumption sequence starting at time t is denoted by $C_t = (c_t, c_{t+1}, ...) \in [0,1]^{T-(t-1)}$ for a finite time horizon $T < \infty$ as well as infinite horizon $T = \infty$.

Generation t's preferences are described by an intertemporal utility function w_t which we normalize on the interval $[0,1]$:

$$\begin{aligned} w_t \;:\; [0,1]^{T-t+1} &\;\rightarrow\; [0,1] \\ C_t &\;\mapsto\; w_t(C_t). \end{aligned} \tag{1}$$

That is, only present and future consumption enter each generations' preferences (i.e. independence of history).

For $T < \infty$ we get from [Blackorby et al. 1973] that the sequence of utility functions w_t is time-consistent if and only if there are functions $F_t : [0,1]^2 \rightarrow [0,1]$ (for

$t \in \{1, 2, ..., T-1\}$) such that for all $C_t \in [0,1]^{T-(t-1)}$

$$w_t(C_t) = F_t[c_t, w_{t+1}(C_{t+1})]. \tag{2}$$

We now consider the infinite time horizon case. In an appendix we give a brief sketch of how to modify the proof of [BLACKORBY et al. 1973, theorem 3] for the infinite time horizon case. Again, a sequence of intertemporal utility functions w_t is time-consistent if there are functions F_t such that (2) holds.

We assume that the preferences do not depend on calendar time.[2] That is, the preferences for each generation are identical in the sense that

$$\forall t, t' \in \mathbb{N}\ \forall C \in [0,1]^\infty : \ w_t(C) = w_{t'}(C). \tag{3}$$

Therefore, we may skip the time index of the utility function and define the universal intertemporal utility function $w := w_t$. The consistency criterion is then

$$w(C_t) = F_t[c_t, w(C_{t+1})]. \tag{4}$$

We will now prove that the functions F_t also have to be identical.

The utility functions of periods t and $t+i$ are given by

$$\begin{aligned} w(C_t) &= F_t[c_t, w(C_{t+1})] \\ w(C'_{t+i}) &= F_{t+i}[c'_{t+i}, w(C'_{t+i+1})]. \end{aligned} \tag{5}$$

with arbitrary consumption paths $C_t, C'_{t+i} \in [0,1]^\infty$. Now we choose

$$c'_{t+j+i} = c_{t+j} \quad \text{for } j \geq 0. \tag{6}$$

With $T = \infty$ we get

$$\begin{aligned} C'_{t+i} &= C_t \\ w(C'_{t+i}) &= w(C_t) = F_t[c_t, w(C_{t+1})]. \end{aligned} \tag{7}$$

On the other hand, we have

$$w(C'_{t+i}) = F_{t+i}[c'_{t+i}, w(C'_{t+i+1})] = F_{t+i}[c_t, w(C_{t+1})] \quad \text{(from (6))}. \tag{8}$$

[2] Note that this assumption is not possible in the finite time horizon case.

As (7) and (8) hold for arbitrary paths C_t we get

$$F_t[x,y] \quad = \quad F_{t+i}[x,y] \qquad \text{for } x \in [0,1],\ y \in w([0,1]^\infty). \tag{9}$$

For $w([0,1]^\infty) = [0,1]$ we get $F_t = F_{t+i}$. In the case $w([0,1]^\infty) \neq [0,1]$, however, F_t, F_{t+i} might differ on the set $[0,1] \times ([0,1] \setminus w([0,1]^\infty))$. The values of F_t, F_{t+i} on this set do not matter; so we can simply set $F_t := F_0$ for all t on this set.

This enables us now to define $F := F_t$ and write the criterion of time consistency in the form

$$w(C_t) \quad = \quad F[c_t, w(C_{t+1})]. \tag{10}$$

So far we have shown that for a time-consistent sequence of preferences characterized by the intertemporal utility function w there is a function F with the property (10). We may ask now whether for a given $F: [0,1]^2 \to [0,1]$ there is a solution w.

The utility functions we are interested in should give more weight to present rather than future consumption. To be more precise let's consider $C_t^1 \neq C_t^2$ with $c_t^1 = c_t^2 = c_t$. The difference in the consumption paths gives rise to a difference in the level of welfare[3] Generation $t+1$ faces the difference in consumption earlier then generation t. Generation t discounts the value of the utility difference due to the difference in future consumption. We will reflect this by assuming

$$\begin{aligned} |w(C_t^1) - w(C_t^2)| \quad &\leq \quad L\, |w(C_{t+1}^1) - w(C_{t+1}^2)| \\ |F[c_t, w(C_{t+1}^1)] - F[c_t, w(C_{t+1}^2)]| \quad &\leq \quad L\, |w(C_{t+1}^1) - w(C_{t+1}^2)| \end{aligned} \tag{11}$$

with a constant $L \in (0,1)$.[4]

In the following we confine the functions F to the set

$$\begin{aligned} \mathcal{S} \quad = \quad & \{F \in \mathcal{C}([0,1]^2,[0,1]) \mid |F[x,y^1] - F[x,y^2]| \ \leq \ L \cdot |y^1 - y^2|, \\ & \forall x, y^1, y^2 \in [0,1] \ \text{ with } L \in (0,1)\} \end{aligned} \tag{12}$$

[3]Here we use cardinal utility in order to speak about utility levels and differences between them.

[4] To illustrate the impact of our assumption (11) we consider two consumption paths $C_t^1 = (1,0,0,0,...)$ and $C_t^2 = (0,1,0,0,...)$. If generation t discounts future consumption we get $w_t(C_t^1) > w_t(C_t^2)$. On the other hand, the constraint (11) leads to $w_t(1,0,0,0,...) - w_t(0,0,0,0,...) > L \cdot [w_t(1,0,0,0,...) - w_t(0,0,0,0,...)] \geq w_t(0,1,0,0,...) - w_t(0,0,0,0,...)$. That gives $w_t(1,0,0,0,...) > w_t(0,1,0,0,...)$, i.e. discounting $w_t(C_t^1) > w_t(C_t^2)$.

where we use the notation $\mathcal{C}(\mathcal{A},\mathcal{B})$ for the set of all continuous mappings $f:\ \mathcal{A}\to\mathcal{B}$. We consider the product topology of $[0,1]^\infty$ and define the mapping

$$\begin{aligned} \Phi_F\ :\ \mathcal{C}([0,1]^\infty,[0,1]) &\ \to\ \mathcal{C}([0,1]^\infty,[0,1]) \\ w &\ \mapsto\ \Phi_F[w] \end{aligned} \tag{13}$$

with

$$\Phi_F[w](C_t)\ \ :=\ \ F[c_t, w(C_{t+1})]. \tag{14}$$

Hence, the criterion for time consistency (10) turns out to be a fixed point equation, i.e.,

$$\Phi_F[w]\ \ =\ \ w. \tag{15}$$

Thus, the question whether a utility function w exists for a given $F\in\mathcal{S}$, is now the question of the existence of a fixed point in (15).

We approach the problem by use of Banach's fixed point theorem. Therefore we need the following

Definition 1 *For a non-empty subset $\mathcal{A}$ of a normed space, a mapping $f:\mathcal{A}\to\mathcal{A}$ is called contracting if there is a $L\in(0,1)$ such that $\|f(x)-f(y)\|\ \leq\ L\ \|x-y\|$ for all $x,y\in\mathcal{A}$.*

Of course, a contracting mapping is continuous.

Now we can state the following fixed point theorem (see, e.g., [Heuser 1981]).

Theorem 1 (Banach) *Let $\mathcal{X}$ be a non-empty closed subset of a Banach space. If there is a contracting mapping $f:\mathcal{X}\to\mathcal{X}$, then the equation $f[x]\ =\ x$ has one and only one solution (i.e., the fixed point).*

In order to apply Banach's Theorem to our problem, we have to prove

Proposition 1 *For all $F\in\mathcal{S}$ the mapping Φ_F is contracting.*

Proof: Let $u, w \in \mathcal{C}([0,1]^\infty, [0,1])$ and $F \in \mathcal{S}$. From (12) we know that there is an $L \in (0,1)$ such that

$$\begin{aligned} \|\Phi_F[u] - \Phi_F[w]\|_\infty &= \sup_{C_1 \in [0,1]^\infty} |\Phi_F[u](C_1) - \Phi_F[w](C_1)| \\ &= \sup_{C_1 \in [0,1]^\infty} |F[c_1, u(C_2)] - F[c_1, w(C_2)]| \\ &\leq L \cdot \sup_{C_2 \in [0,1]^\infty} |u(C_2) - w(C_2)| \\ &= L \cdot \|u - w\|_\infty. \end{aligned} \tag{16}$$

Thus, Φ_F is contracting. ■

Now we can formulate

Proposition 2 *For all $F \in \mathcal{S}$ there is exactly one function w such that for all $C_t \in [0,1]^\infty$ the equation $w(C_t) = F[c_t, w(C_{t+1})]$ is fulfilled.*

Proof: Let $F \in \mathcal{S}$. Then Φ_F is contracting (Proposition 1). Thus, we may apply Banach's theorem and get a unique solution to $\Phi_F[w] = w$. This proves the Proposition. ■

3 Quasiconcavity and Monoticity

The utility functions w generated by $F \in \mathcal{S}$ should be strictly quasiconcave and non-decreasing. In order to achieve this we define the set

$$\mathcal{S}^* \quad := \quad \{F \in \mathcal{S} |\ F \text{ strictly monotonically increasing and strictly concave}\}. \tag{17}$$

We formulate the following Proposition.

Proposition 3 *If $F \in \mathcal{S}^*$ and $w = \Phi_F[w]$, then w is strictly monotonically increasing and strictly concave.*

Proof: By means of $w = \Phi_F[w]$ we write w in the iterative form

$$w(C_1) \quad = \quad F[c_1, F[c_2, F[c_3, ...F[c_n, w(C_{n+1})]...]]]. \tag{18}$$

We define functions F^n for $n \in \mathbb{N}$ to write (18) in a compact form:

$$\begin{aligned} F^1[C_1, x] &:= F[c_1, x] \\ F^n[C_1, x] &:= F[c_1, F^{n-1}[C_2, x]]. \end{aligned} \tag{19}$$

As F is strictly increasing the functions F^n are strictly increasing with respect to the first n components of the vector $C_1 \in [0,1]^\infty$ and also with respect to the last argument $x \in [0,1]$.[5] Now equation (18) takes the form

$$w(C_1) = F^n[C_1, w(C_{n+1})]. \tag{20}$$

Thus, we get from (20) and $0 \le w[C_{n+1}] \le 1$:

$$F^n[C_1, 0] \le w[C_1] \le F^n[C_1, 1]. \tag{21}$$

As $F \in \mathcal{S}^*$ has a Lipschitz constant $L \in (0,1)$ we can show that F^n has the Lipschitz constant L^n with respect to the second entry. We show this by induction. For any $C_1 \in [0,1]^\infty$ and $x, x' \in [0,1]$ we have

$$\begin{aligned} |F^1[C_1, x] - F^1[C_1, x']| &= |F[c_1, x] - F[c_1, x']| \\ &\le L \cdot |x - x'|. \end{aligned} \tag{22}$$

Assume now that F^{n-1} has the Lipschitz constant L^{n-1}. Then for any $x, x' \in [0,1]$ we get

$$\begin{aligned} |F^n[C_1, x] - F^n[C_1, x']| &= |F[c_1, F^{n-1}[C_2, x]] - F[c_1, F^{n-1}[C_2, x']]| \\ &\le L \cdot |F^{n-1}[C_2, x] - F^{n-1}[C_2, x']| \\ &\le L^n \cdot |x - x'|. \end{aligned} \tag{23}$$

Thus we get $F^n[C_1, 1] = F^n[C_1, 0] + |F^n[C_1, 1] - F^n[C_1, 0]| \le F^n[C_1, 0] + L^n$. From (21) we get

$$F^n[C_1, 0] \le w[C_1] \le F^n[C_1, 0] + L^n \tag{24}$$

for all $n \in \mathbb{N}$. As $L \in (0,1)$ the term L^n vanishes in the limit $n \to \infty$. Thus, we get

$$F^\infty[C_1, 0] \le w[C_1] \le F^\infty[C_1, 0] \tag{25}$$

[5] Therefore we split the support $\mathcal{D}$ of F^n as follows: $\mathcal{D} = [0,1]^n \times [0,1]^\infty \times [0,1]$.

Hence,

$$w[C_1] \quad = \quad F^{\infty}[C_1, 0]. \tag{26}$$

Now it remains to be shown that F^{∞} has the desired monotonicity and concavity properties. We show this by intermediate results on the properties of F^n for $n \in \mathbb{N}$. We know already that F^n is strictly increasing in the first n components of C_1, as F is strictly increasing. Next we show that the F^n are strictly concave in the first n components by complete induction. F^1 is concave on $[0,1]^{\infty}$ and strictly concave in $c_1 \in [0,1]$ as F is strictly concave. Now we assume F^{n-1} to be strictly concave in the first $n-1$ components of C_1. For any $C_1, C_1' \in [0,1]^{\infty}$ (with $c_i \neq c_i'$ for some $i \in \{1, ..., n\}$) , $x, x' \in [0,1]$ and $\lambda \in (0,1)$ we get

$$\begin{aligned}
& F^n[\lambda C_1 + (1-\lambda)C_1',\ \lambda x + (1-\lambda)x'] \\
= \;& F[\lambda c_1 + (1-\lambda)c_1',\ F^{n-1}[\lambda C_2 + (1-\lambda)C_2',\ \lambda x + (1-\lambda)x']] \\
\geq \;& F[\lambda c_1 + (1-\lambda)c_1',\ \lambda F^{n-1}[C_2, x] + (1-\lambda)F^{n-1}[C_2', x']] \\
\geq \;& \lambda F[c_1, F^{n-1}[C_2, x]] + (1-\lambda)F[c_1', F^{n-1}[C_2', x']] \\
= \;& \lambda F^n[C_1, x] + (1-\lambda)F^n[C_1', x'].
\end{aligned} \tag{27}$$

Thus, F^n is concave. Moreover, as F is strictly concave, the second inequality in (27) becomes strict, if $c_1 \neq c_1'$. If $c_1 = c_1'$, there is an $i \in \{2, ..., n\}$ such that for $c_i \neq c_i'$. Then, as F is strictly increasing and F^{n-1} strictly concave, the first inequality becomes strict.

This proves the strict concavity of F^n in the first n arguments for all $n \in \mathbb{N}$.

This shows the strict concavity of F^{∞} in $C_1 \in [0,1]^{\infty}$.

We conclude (see (26)) that w is strictly concave and strictly increasing, which proves Proposition 3. ∎

Any strictly concave function is strictly quasiconcave. So each fixed point of Φ_F for $F \in \mathcal{S}^*$ is an intertemporal utility functions with the usual properties.

In the following we confine ourselves to differentiable functions, for the sake of simplicity. We use the notation: $F_i[x_1, x_2] := \partial F/\partial x_i$ for $i = 1, 2$ and $\partial_i w[C_1] := \partial w[C_1]/\partial c_i$ for the partial derivatives of F and w.

We showed already that w with $w = \Phi_F[w]$ is strictly increasing for all $F \in \mathcal{S}^*$. In the case of differentiable functions F we formulate this as follows.

Proposition 4 *Let* $w = \Phi_F[w]$. *If* $F_1, F_2 > 0,\ \forall x, y \in [0,1]$, *then* $\forall n \in \mathbb{N} : \partial_n w > 0$.

Proof: Proposition 3 ■

However, we can also show the other direction, that non-decreasing w can only be the solution to the fixed point equation $w = \Phi_F[w]$ if F is non-decreasing.

Proposition 5 *Let* $w = \Phi_F[w]$. *If* $\partial_1 w, \partial_2 w > 0$, *then for all* $n \in \mathbb{N}$ *we have* $\partial_n w > 0$ *and* $F_1, F_2 > 0\ \forall x, y \in w([0,1])$.

Proof: Let $C_1 \in [0,1]^\infty$. Then $\partial_2 w(C_1) = F_2[c_1, w(C_2)] \cdot \partial_1 w(C_2)$. As $\partial_1 w, \partial_2 w > 0$ we get also $F_2 > 0$. From $F_1[c_1, w(C_2)] = \partial_1 w(C_1) > 0$ we get $F_1 > 0$. Thus, the requirements of Proposition 4 are met which proves $w_n \geq 0\ \forall n \in \mathbb{N}$. ■

Remark: Proposition 5 says that altruism for the next generation (i.e. $\partial_2 w > 0$) implies altruism for all generations (i.e. $\partial_n w > 0$ for $n \geq 2$) if the criterion for time consistency holds. This is a generalization of the result known from the discussion of maximin-optimal plans in [Dasgupta 1974, Calvo 1978, Rodriguez 1981, Leininger 1985, Leininger 1986, Rodriguez 1990]. If each generation cares for its direct successor but not for other future generations, then the behaviour is time-inconsistent.

4 Examples

In order to give some intuition for the problem of time-(in)consistent utility functions, we give some simple examples:

1. $w[C_t] = u(c_t) + \lambda u(c_{t+1})$ with a discount factor $\lambda \in (0,1)$ and some instantaneous utility function u (with $u' > 0$) is time-inconsistent. We get this by use of the contraposition of Proposition 5 ($\partial_n w = 0$ for $n \geq 2$).

   On the other hand, we can see directly that w is time-inconsistent, as there is no function F which fulfills (10).

2. We consider the separable function $F[x, y] := u(x) + \lambda y$ with $\lambda \in (0,1)$. The corresponding utility function is defined by

$$w[C_t] \quad = \quad u(c_t) + \lambda w[C_{t+1}]. \tag{28}$$

   This is solved by the present value of future utility

$$w[C_t] \quad = \quad \sum_{\tau=0}^{\infty} \lambda^{\tau} \cdot u(c_{t+\tau}). \tag{29}$$

   The convergence of the sum is guaranteed if we assume u to be bounded above. This intertemporal utility function w represents the simple present value of future utility, with a constant discount factor.

3. The case $F[x, y] = u(x) \cdot y^{\lambda}$ gives $w[C_t] = u(c_t) \cdot w[C_{t+1}]^{\lambda}$. For $\lambda \in (0,1)$ and bounded u, the solution is $w[C_t] = \prod_{\tau=0}^{\infty} u(c_{t+\tau})^{\lambda^{\tau}}$. Taking the logarithm yields a form as given by (29).

5 Conclusions

In this article we investigated the problem of time consistency in intertemporal decision making. In particular, we considered an infinite sequence of intertemporal utility functions, and looked for criteria for time consistency.

The article by [Blackorby et al. 1973] provides a criterion for the finite time horizon case. Under the assumption that calendar time does not matter, we extended the criterion to the case of an infinite time horizon. The homogeneity of time allowed us to characterize the set of time-consistent utility functions by the solutions of a fixed point equation in the space of functions $\mathcal{C}([0,1]^\infty,[0,1])$. The assumption of bounded consumption enabled us to apply Banach's fixed point theorem to the problem.

Under the assumptions above, we proved the following statement:

For each strictly increasing and strictly concave function $F : [0,1]^2 \to [0,1]$ that is contracting in the second argument, there is one and only one time-consistent utility function w. Furthermore, every bounded utility function w is the fixed point of a mapping $\Phi_F : \mathcal{C}([0,1]^\infty,[0,1]) \to \mathcal{C}([0,1]^\infty,[0,1])$, where F is contracting in the second component. We emphasize that the assumptions made on F do not restrict us essentially. Our classification contains all cases of interest.

Furthermore, we analyzed the impact of properties of the functions F on the utility functions w. Both monotonicity and concavity properties are inherited in a very intuitive manner. This close link between F and w restricts considerably the set of economically meaningful functions F which characterize time-consistent preferences.

6 Appendix

We want to answer the question, whether the proof of theorem 3[6] can be extended to the case of an infinite time horizon.

The prior maximization problem is

$$\max_{C_t} w_t[C_t] \qquad \text{s.t.} \qquad \sum_{s=0}^{\infty} p_{s+t} \cdot c_{s+t} \ \leq \ m_t \tag{30}$$

where $m_t > 0$ is the wealth of generation t. The infinite sum in (30) cannot diverge to infinity as the wealth m_t is finite.

Depending on the exogenous price path P_t and the intertemporal utility function w_t there might be both interior and boundary solutions to the maximization program (30).

[6] All references to theorems, equations and pages in this appendix refer to [Blackorby et al. 1973].

We can exclude the possibility of boundary solutions with zero consumption in a period by assuming that w_t goes to $-\infty$ for any of its arguments approaching 0. Upper boundary solutions (i.e. consumption in some period equals unity) can be excluded by assuming that marginal utility goes to 0 for consumption level approaching 1. In the following we consider interior solutions only.

The dual problem of (30) is

$$h = \min_{C_t} \sum_{s=0}^{\infty} p_{s+t} \cdot c_{s+t} \qquad \text{s.t.} \qquad w_t[C_t] \geq \bar{w}_t \tag{31}$$

where $\bar{w}_t$ is a certain level of utility.

Now, we can follow the analysis given in [Blackorby et al. 1973]. We do this in a slightly different notation.

By solving the dual problem (31) for the tth generation we get the compensated demand functions

$$c_s^{(t)} = \psi_s^{(t)}[P_t, \bar{w}_t] \text{ for } s \geq t \tag{32}$$

or, in vector notation,

$$C_t^{(t)} = \Psi_t^{(t)}[P_t, \bar{w}_t]. \tag{33}$$

The expenditure function is

$$e_t[P_t, \bar{w}_t] = \sum_{s=t}^{\infty} p_s \cdot \psi_s^{(t)}[P_t, \bar{w}_t]. \tag{34}$$

The compensated demand functions can be expressed by the partial derivatives of the expenditure function:

$$c_s^{(t)} = \frac{\partial e_t[P_t, \bar{w}_t]}{\partial p_s}. \tag{35}$$

Intertemporal consistency is defined as

$$c_s^{(t)} = c_s^{(t+1)} \text{ for } s \geq t+1 \tag{36}$$

or, in vector notation,

$$C_{t+1}^{(t)} \;=\; C_{t+1}^{(t+1)}. \tag{37}$$

Thus, in terms of the partial derivatives of the expenditure function we get

$$\frac{\partial e_t[P_t, \bar{w}_t]}{\partial p_s} \;=\; \frac{\partial e_{t+1}[P_{t+1}, \bar{w}_{t+1}]}{\partial p_s} \text{ for } s \geq t+1. \tag{38}$$

We can decompose the expenditure function for the tth generation into present and future expenditure:

$$e_t[P_t, \bar{w}_t] \;=\; p_t \cdot \psi_t^{(t)}[P_t, \bar{w}_t] + e_{t+1}[P_{t+1}, \bar{w}_{t+1}]. \tag{39}$$

We take the total derivative of the above:

$$\begin{aligned} & \frac{\partial e_t}{\partial p_t}\mathrm{d}p_t + \sum_{s=t+1}^{\infty} \frac{\partial e_t}{\partial p_s}\mathrm{d}p_s + \frac{\partial e_t}{\partial w_t}\frac{\partial w_t}{\partial c_t}\mathrm{d}c_t + \frac{\partial e_t}{\partial w_t}\sum_{s=t+1}^{\infty}\frac{\partial w_t}{\partial c_s}\mathrm{d}c_s \\ = \;& c_t\mathrm{d}p_t + p_t\mathrm{d}c_t + \sum_{s=t+1}^{\infty}\frac{\partial e_{t+1}}{\partial p_s}\mathrm{d}p_s + \frac{\partial e_{t+1}}{\partial w_{t+1}}\sum_{s=t+1}^{\infty}\frac{\partial w_{t+1}}{\partial c_s}\mathrm{d}c_s. \end{aligned} \tag{40}$$

From (35) the first terms on both sides are equal. The third term on the left-hand side cancels with the second term on the right-hand side because of the first-order conditions of the dual problem:

$$p_t \;=\; \frac{\partial e_t}{\partial w_t}\frac{\partial w_t}{\partial c_t} \tag{41}$$

where $\partial e_t / \partial w_t = \lambda_t$ is the solution for the Lagrange multiplier. Intertemporal consistency (38) requires that the second term on the left and the third term on the right are equal. After these cancellations, we get

$$\frac{\partial w_t}{\partial c_s} \;=\; \frac{\lambda_{t+1}}{\lambda_t}\frac{\partial w_{t+1}}{\partial c_s} \text{ for } s \geq t+1. \tag{42}$$

These conditions entail that the intertemporal utility function w_t is Leontief separable [LEONTIEF 1947]. From [GORMAN 1959] we get that there are functions F_t such that (2)

$$w_t(C_t) \;=\; F_t[c_t, w_{t+1}(C_{t+1})]. \tag{43}$$

The generalization to infinite dimension of this result does not cause any problems as for each $t \in \mathbb{N}$ we we might set an artificial time horizon $T > t$ and can use the finite time-horizon result.

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