Baryonic matter in the lattice Gross-Neveu model

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Chiral phase transition in QCD:

- At \( T_c \) or \( \rho_c \), hadrons start to overlap, chiral symmetry is restored:
  \[
  T_c \simeq O(m_{\pi}) \\
  \rho_c \simeq 3 - 5 \rho_0, \quad \rho_0 \simeq 0.15 \text{fm}^{-1}
  \]
QCD Phase diagram

**Phase Diagram of QCD Matter**

- Early universe
- Temperature $T_c \sim 170$ MeV
- Quark-gluon plasma
- Hadron gas
- Nucleon gas
- Nuclei
- Color superconductor
- Neutron stars
- Vacuum

Understanding the properties of the transition is an intrinsically non-perturbative problem
⇒ methods like lattice simulations, effective theories, ... are necessary.

Standard lattice methods fail for finite density QCD
⇒ indirect methods have (large, unknown) systematic errors.

Study QCD related models where failure is absent or under control
⇒ Gross-Neveu model
**Definition of the model**

- **Euclidean lagrangian density in 2D** [Gross, Neveu '74]

\[ \mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \partial / \psi^\alpha(x) - \frac{g^2}{2} \left( \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \psi^\alpha(x) \right)^2 \]

where \( \psi^\alpha(x) \) are 2-component Dirac spinors and \( \alpha \) flavour index.

- Introduce a scalar field \( \sigma(x) \) conjugate to \( \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \psi^\alpha(x) \):

\[ \mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \partial / \psi^\alpha(x) + \frac{1}{2g^2} \sigma(x)^2 + \sigma(x) \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \psi^\alpha(x). \]
The Gross-Neveu model

The Gross-Neveu model is renormalisable and asymptotically free,

\[ \beta(g) = -\frac{N-1}{2\pi} g^3 + O(g^5), \]

has a \( O(2N) \times \Gamma \)-symmetry where \( \Gamma \) is the discrete chiral symmetry

\[ \Gamma : \quad \psi \rightarrow \gamma_5 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi} \gamma_5, \quad \sigma \rightarrow -\sigma, \]

exhibits spontaneous breaking of the discrete chiral symmetry

\[ \Rightarrow \text{fermions acquire non-vanishing mass } \sigma_0 = \langle \sigma \rangle \] (dimensional transmutation).

Note: there is no Goldstone boson due to \( \Gamma \) being a discrete symmetry.
In the large-$N$ limit with $\lambda = g^2 N$ fixed, the model can be solved analytically:

- Integrate out the fermions to obtain $Z = \int [d\sigma] \exp \left\{ -S_{\text{eff}} \right\}$,

\[
S_{\text{eff}} = N \left\{ \int [dx] \frac{\sigma(x)^2}{2\lambda} - \text{Tr} \log [\not{\partial} + \sigma] \right\}.
\]

- The minimum of the effective potential is given by

\[
\frac{\partial_{\sigma(x)} S_{\text{eff}}}{N} = \frac{\sigma(x)}{\lambda} - \partial_{\sigma(x)} \text{Tr} \log [\not{\partial} + \sigma] = 0, \ \forall x.
\]
Gap equation

- For constant $\sigma$ this reduces to a single equation

$$\frac{\sigma}{\lambda} = \partial_\sigma \text{Tr} \log [\Phi + \sigma],$$

or in momentum space

$$\sigma = 0 \quad \text{or} \quad \frac{1}{\lambda} = \int[dk] \frac{2}{k^2 + \sigma^2}.$$

⇒ Gap equation (self consistency equation)

- Equivalent equations via Hartree-Fock, Schwinger-Dyson, Bethe-Salpeter approaches.
To leading order in $1/N$ the spectrum consists of

\[ m_1 = \sigma_0 \sim \Lambda \exp \left\{-\frac{\pi}{\lambda}\right\}, \text{ single fermion,} \]

\[ m_n = \sigma_0 \cdot \frac{2N}{\pi} \sin \left(\frac{n\pi}{2N}\right), \text{ n-fermion bound state,} \]

\[ m_B = \sigma_0 \cdot \frac{2N}{\pi}, \text{ kink-antikink state ('baryon').} \]

For chirally twisted spatial boundary conditions the single kink state

\[ \sigma(x) = \sigma_0 \tanh (\sigma_0 x) \]

is topologically stable,

interpolates between the two vacua related by the discrete $\gamma_5$-symmetry.
The GN model possesses a rich $\mu$-$T$ phase structure:

[Dashen, Ma, Rajaraman '75; Wolff '85; Karsch, Kogut, Wyld '87]

- Mermin-Wagner-Coleman theorems forbid spontaneous breaking of
  - continuous symmetry at $T = 0$,
  - discrete symmetry at $T \neq 0$.
- Fluctuations are expected to destroy any long range order
  \[ \Rightarrow \] free massless boson propagator is logarithmic in 2D,
- However, fluctuations are suppressed at large $N$:

\[
\langle \bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y) \rangle \sim 1 + \frac{1}{N} \ln |x - y| + O(1/N^2)
\]

becomes constant as $N \to \infty$.

\[ \Rightarrow \] take large-$N$ limit before thermodynamic limit!
From the homogeneous mean field approximation [Wolff '85]:

- AB $\rightarrow$ 2$^{\text{nd}}$ order
- BD $\rightarrow$ 1$^{\text{st}}$ order
- B $\rightarrow$ tricritical point
- BCE $\rightarrow$ metastability region
- $T_c^A = e^{C/\pi} = 0.5669$
- $\mu_c^D = 1/\sqrt{2} = 0.7071$
- $T_c^B = 0.3183, \mu_c^B = 0.6082$
On general grounds one expects from widely separated baryons

$$-\frac{\partial}{\partial \rho} \ln Z \bigg|_{\rho=0, T=0} \equiv \mu_c = m_B. $$

- mean field approximation is in conflict with this,
- ad-hoc reconciliation via a droplet model of baryons, yielding a modified baryon mass $m_B = 1/\sqrt{2}$.
- Something wrong with the mean field approach? No, but…

⇒ Assumption of translational invariance of $\sigma$ is invalid.
Thies et al. recently clarified the structure of cold baryonic matter in the GN model:

[Schön, Thies '00; Thies, Brzoska '02; Thies, Urlichs '03; Thies '03; Schnetz, Thies, Urlichs '05]

- they use a Hartree-Fock approach with a spatially varying scalar potential,
- the gap equation becomes a set of non-linear self-consistency equations,
- potential ansatz inspired by the scalar potential for a single baryon:

\[ \sigma(x) = 1 + y \left[ \tanh(yx - c_0) - \tanh(yx + c_0) \right], \]

where \( c_0 = \frac{1}{2} \text{arctanh}(y) \) and \( y = y(\sigma_0) \).
The revised phase diagram II

Scalar potential ansatz:

- motivated by matter at low density forming isolated baryons,
- Pöschl-Teller potential wells can be periodically extended,
- leads to a general ansatz satisfying self-consistency equation.
In addition to the massive and massless Fermi gas, there is a new baryonic crystal phase at low temperature:

\[ \mu_c = \frac{2}{\pi} \] now consistent with \( m_B \), no first order transition at \( \mu \neq 0 \).
The revised phase diagram IV

\[(T, \rho)\text{-phase diagram:}\]

- \(T_{\text{crit}}\) unchanged,
- tricritical point turns into multi-critical point at the same location.

These findings motivate to look for the new phase in lattice models.
Consider the staggered GN action:

\[ S = N \sum_x \frac{\sigma(x)^2}{2\lambda} + \sum_x \sum_{\alpha=1}^N \overline{\chi}^\alpha(x) \left[ D_{xy} + \Sigma_{xy} \right] \chi^\alpha(y) \]

where the Dirac operator

\[ D_{xy} = \frac{1}{2} \left[ \delta_{x,y+\hat{1}} - \delta_{x,y-\hat{1}} \right] + \frac{1}{2} (-1)^{x_1} \left[ \delta_{x,y+\hat{2}} - \delta_{x,y-\hat{2}} \right] \]

describes 2 flavours and

\[ \Sigma_{xy} = \frac{1}{4} \delta_{xy} \left( \sigma(x) + \sigma(x - \hat{1}) + \sigma(x - \hat{2}) + \sigma(x - \hat{1} - \hat{2}) \right). \]

Modification \( \sigma \rightarrow \Sigma \) is necessary to ensure correct continuum limit [Cohen, Elitzur, Rabinovici '83].
Discrete chiral symmetry is preserved:

\[ \chi(x) \rightarrow (-1)^{x_1+x_2} \chi(x), \quad \bar{\chi}(x) \rightarrow -(-1)^{x_1+x_2} \bar{\chi}(x), \quad \sigma(x) \rightarrow -\sigma(x). \]

A finite chemical potential \(\Leftrightarrow\) time component of an imaginary external constant Abelian vector potential [Hasenfratz, Karsch '83]:

\[ \Rightarrow \] weighting the temporal derivatives with factors \(\exp(\pm \mu)\).

In momentum space this amounts to the replacement

\[ k_t \Rightarrow k_t - i\mu. \]

Imaginary chemical potential corresponds to a non-trivial magnetic flux [Huang, Schreiber '94].
Consider the massless overlap Dirac operator

\[ D = m \left\{ 1 + D_W(-m) \left[ D_W(-m) D_W(-m) \right]^{-1/2} \right\} \]

satisfying the Ginsparg-Wilson relation \( D^\dagger + D = \frac{1}{m} D^\dagger D \).

The coupling to the scalar field is introduced like

\[ \mathcal{L} = \bar{\psi}(x) \left[ \left( D_{x,y} - \frac{\sigma(x)}{4m} D_{x,y} - D_{x,y} \frac{\sigma(y)}{4m} \right) + \sigma(x) \delta_{x,y} \right] \psi(y) \]

consistent with a covariant scalar density.

For \( \sigma \to \text{const.} \) it is just the usual mass term

\[ \left( 1 - \frac{\sigma}{2m} \right) D + \sigma. \]
For constant $\sigma$ we can work in momentum space; we have (for $m = 1$):

$$D = \left\{ 1 + \left( i\gamma_\mu \hat{p}_\mu + \frac{1}{2} \hat{p}_\mu^2 - 1 \right) \left[ \left( \frac{1}{2} \hat{p}_\mu^2 - 1 \right)^2 + \hat{p}_\mu^2 \right] \right\}^{-1/2}$$

where $\hat{p}_\mu = \sin(k_\mu)$, $\hat{p}_\mu = 2 \sin(k_\mu/2)$ with appropriate b.c.

Chemical potential as before, replacing everywhere

$$k_t \Rightarrow k_t - i\mu.$$
We need to calculate the (real) fermion determinant

\[ \ln \det D = 2 \sum_{t=0}^{L_t/2-1} \sum_{k=0}^{L_x/2-1} \ln \left[ \hat{p}_t^2 + \hat{p}_k^2 + \sigma^2 \right] \]

where \( \hat{p}_t, \hat{p}_k \) are lattice momenta, \( L_t, L_x \) lattice extensions.

For a homogeneous condensate one can perform the thermodynamic limit analytically,

\[ \lambda = \frac{L_t}{2} \left( \sum_{t=0}^{L_t/2-1} \frac{1}{\left( \sigma^2 + \hat{p}_t^2 \right) \sqrt{1 + \frac{1}{\sigma^2 + \hat{p}_t^2}}} \right)^{-1} \]
For an inhomogeneous condensate we have

\[
\det D = \prod_{t=0}^{L_t-1} 2^{L_x} \det \left( P_t - \left( \frac{1}{2} \right)^{L_x} \right)
\]

with the reduced matrices [Gibbs '86; Hasenfratz, Toussaint '91]

\[
P_t = \prod_{x=0}^{L_x/2-1} \left( \Omega_t(2x)\Omega_{L_t/2+t}(2x+1) \right)
\]

and

\[
\Omega_t(x) = \begin{pmatrix}
\hat{p}_t + \sigma(x) & \frac{1}{2} \\
\frac{1}{2} & 0
\end{pmatrix}.
\]

Can be interpreted as a transfer matrix in space at each \( t \).
If the condensate is invariant under translation by $l_x$ and $L_x = nl_x$, 
\[ \det D_t = 2^{nl_x} \det \left( P_t^n - 2^{-nl_x} \right) . \]

In the thermodynamic limit we then simply have
\[ \lim_{n \to \infty} \frac{1}{n} \ln \det D_t = \sum_{t=0}^{L_t-1} \ln \lambda_t^{(1)} \]

where $\lambda_t^{(1)}$ is the larger of the two eigenvalue of $P_t$.

Length scale $L_x$ of the box size is replaced by $l_x$, the wave length of the condensate.
Homogeneous mean field results at $\mu = 0, T = 0$

- Gap equation yields $\sigma$ as a function of $\lambda$:

  $\Rightarrow$ non-perturbative $\beta$-function vs asymptotic scaling

- Staggered operator, asymptotic scaling $2^{3/2} e^{-\pi/2\lambda}$
- Overlap operator, asymptotic scaling $1.5539\ldots e^{-\pi/\lambda}$
Homogeneous mean field results at $\mu = 0$

- second order transition at $T_c, \mu = 0$:
  (overlap Dirac operator, $\lambda = 1.0$, $L = 200$)

![Graph showing second order transition at $T_c, \mu = 0$.](image)
Homogeneous mean field results at $\mu = 0$

- Scaling of $\frac{T_c}{\sigma_0}$ vs $(a\sigma_0)^2$:

```
<table>
<thead>
<tr>
<th>L</th>
<th>Tc/\sigma_0</th>
<th>(a\sigma_0)^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>48</td>
<td>0.55</td>
<td>0.0002</td>
</tr>
<tr>
<td>100</td>
<td>0.56</td>
<td>0.0004</td>
</tr>
<tr>
<td>200</td>
<td>0.57</td>
<td>0.0006</td>
</tr>
<tr>
<td>400</td>
<td>0.58</td>
<td>0.0008</td>
</tr>
<tr>
<td>800</td>
<td>0.59</td>
<td>0.0010</td>
</tr>
</tbody>
</table>
```

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Baryonic matter in the GN model
second order transition at $T_c, \mu = 0$ vs first order at $T = 0, \mu_c$:
(overlap Dirac operator, $\lambda = 1.0, L = 200$)
Homogeneous mean field results

- Normalised fermion density vs chemical potential at $T \approx 0$:

\[ \rho/(N\sigma_0) \]

\[ \mu/\sigma_0 \]

\[ L=200 \]
Homogeneous mean field results

- Normalised fermion density vs chemical potential at $T \approx 0$:
Normalised fermion density vs chemical potential at $T \approx 0$:

$$\rho/(N\sigma_0)$$ vs $\mu/\sigma_0$ for $L=200$ and $L=400$.

Slope $\sim 1/\pi$. 
Normalised fermion density vs chemical potential at $T \approx 0$:

The graph shows the normalised fermion density $\rho/(N\sigma_0)$ plotted against the chemical potential $\mu/\sigma_0$. The slope of the graph is approximately $1/\pi$. At $T=0$, the graph indicates the onset of fermion condensation.
Homogeneous mean field results

Normalised fermion density vs chemical potential at $T \approx 0$:

![Graph showing normalised fermion density vs chemical potential at $T \approx 0$]
Homogeneous mean field results

Scaling of $\mu_c/\sigma_0$ vs $(a\sigma_0)^2$:

- Staggered operator
- Overlap operator
Scaling of the entry into the metastable region at $T \approx 0$:

Scaling of start of metastability region, staggered fermions
Phase diagram from homogeneous mean field, in the thermodynamic limit:
Homogeneous mean field results

- Phase diagram from homogeneous mean field, in the thermodynamic limit:
- Phase diagram from homogeneous mean field, in the thermodynamic limit:
Phase diagram from homogeneous mean field, in the thermodynamic limit:

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Baryonic matter in the GN model
Phase diagram from homogeneous mean field, in the thermodynamic limit:
Homogeneous mean field results

Phase diagram from homogeneous mean field, in the thermodynamic limit:

\[ \frac{T}{\sigma_0} \]

\[ \frac{\mu}{\sigma_0} \]

\[ L_t = 8 \]
Phase diagram from homogeneous mean field, in the thermodynamic limit:
Phase diagram from homogeneous mean field, in the thermodynamic limit:
Phase diagram from imaginary chemical potential [Huang, Schreiber '94]:

\[ \text{Phase diagram from imaginary chemical potential } [\text{Huang, Schreiber '94}]: \]
Phase diagram from imaginary chemical potential [Huang, Schreiber '94]:

tricritical point is missed
Crystal phase results

- Free energy density for kink-antikink solutions, variational calculation at $\mu = 0.45$:

![Graph showing free energy density for kink-antikink solutions at $\mu = 0.45$.]
Crystal phase results

- behaviour suggests second order transition:

\[ \mu=0.55, \lambda=1.4, L=24, T=0.0 \]

\[ \sigma \]

Free energy density vs. \( \sigma \) for different values of \( Q \):
- \( Q=4 \)
- \( Q=3 \)
- \( Q=2 \)
- \( Q=1 \)
- \( Q=0 \)

\( Q=4 \) shows a peak at a higher value of \( \sigma \) compared to the other values of \( Q \), indicating a different transition behavior.
Crystal phase results

**Free energy density vs. different kink-antikink solutions, \( \lambda = 0.8 \):**

\[ \lambda = 0.8, L = 48, \text{staggered operator} \]

![Graph showing free energy density vs. \( \mu/\sigma_0 \) for different values of \( B \).]
Crystal phase results

Crystal phase towards strong coupling, $\lambda = 1.15$:

![Graph showing crystal phase results](image)
Crystal phase results

Crystal phase towards strong coupling, $\lambda = 1.25$:

- $L=24$, $\lambda=1.25$, staggered Dirac operator

Graph showing the free energy versus $\mu/\sigma_0$ with lines for different values of $\sigma$.
Crystal phase towards strong coupling, $\lambda = 1.35$:
Crystal phase results

Crystal phase towards strong coupling, $\lambda = 1.50$:

- $L=24$, $\lambda=1.50$, staggered Dirac operator

![Graph showing free energy vs. $\mu/\sigma_0$]

- Free energy scale
- $\sigma=\text{const.}$
- $\sigma=0$
- Single kink
Crystal phase results

Crystal phase towards strong coupling, $\lambda = 1.65$:

L=24, $\lambda=1.65$, staggered Dirac operator

-28
-29
-30

-30 0.69 0.7 0.71 0.72 0.73

\( \mu/\sigma_0 \)

\( \sigma=\text{const.} \)

\( \sigma=0 \)

single kink
Single kink solutions towards strong coupling:

Kink profiles towards strong coupling

- $\lambda=1.00$
- $\lambda=1.25$
- $\lambda=1.50$
- $\lambda=2.00$
Crystal phase results

- Instability of the $\sigma = 0$ free energy density wrt spatial variations $\Leftrightarrow$ end of the crystal phase

$\lambda_{\text{min}}$ from Hessian

$L_t=80$, unit cell $l_x=80$

$\lambda=0.4815118031$
Instability of the $\sigma = 0$ free energy density wrt spatial variations $\Leftrightarrow$ end of the crystal phase

$\lambda_{\text{min}}$ from Hessian

$\lambda = 0.4815118031$

$L_t = 80$, unit cell $l_x = 80$

incommensurability effects
Crystal phase results

Phase diagram:

staggered fermions, weak coupling $L_t=80$

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Phase diagram with crystal phase, unit cell $l_x = 80$:

staggered fermions, weak coupling $L_t=80$
Crystal phase results

Phase diagram with crystal phase, thermodynamic limit:

staggered fermions, weak coupling $L_t = 80$
Baryonic matter in the lattice GN model

Crystal phase results

Phase diagram with crystal phase:

staggered fermions, weak coupling $L_t=80$
The breakdown of translational invariance of the ground state requires a revision of the GN model phase diagram.

Besides the massive and massless Fermi gas phase, a new phase of baryonic matter emerges:

⇒ it forms a baryon crystal

The transition to the new phase is always second order.

We are investigating the new phase on the lattice:

- crystal phase disappears at strong coupling, topological excitations fall through the lattice,
- large volumes are necessary,
- poses potential obstacle for simulations at finite density.
Crystal phase is caused by topological excitations
⇒ look for effect in other models
  - 't Hooft model in 1+1 dimensions (chiral spiral)
  - NJL model in 2+1 dimensions (with continuous chiral symmetry)
  - QCD with $N_f = 2$ in 3+1 dimensions:
    ⇒ $SU(N_f = 2)$ symmetry allows topological Skyrmion solutions

Other related work:
- large-$N_c$ QCD in 3+1 dimension [Deryagin, Grigoriev, Rubakov '92]
Gross-Neveu model at finite $N$

Most natural formulation in terms of **Majorana fermions**.

For the Wilson lattice discretisation:

$$\mathcal{L} = \frac{1}{2} \xi^T C (\gamma_\mu \tilde{\partial}_\mu + m - \frac{1}{2} \partial^* \partial) \xi - \frac{g^2}{4} \left( \xi^T C \xi \right)^2.$$

For even $N$ each pair of Majorana fermions may be considered as on Dirac fermion

$$\psi = \frac{1}{\sqrt{2}} (\xi_1 + i \xi_2), \quad \bar{\psi} = \frac{1}{\sqrt{2}} (\xi_1^T - i \xi_2^T) C.$$

Integrating the fermions yields the Pfaffian

$$Z = \text{Pf}[C (\gamma_\mu \tilde{\partial}_\mu + m - \frac{1}{2} \partial^* \partial)].$$
Expanding the Grassmannian Boltzmann factor one obtains a loop representation in terms of **monomers and dimers**.

Partition function sum is over all non-oriented, self-avoiding loops

\[ Z = \sum_{\{k(x,\mu)\} \in \mathcal{L}} \rho[k], \quad \mathcal{L} \in \{\mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{01}, \mathcal{L}_{11}\}. \]

This is equivalent to a special case of the 8-vertex model

\[ Z_{8-\text{vertex}} = \sum_{l \in \mathcal{L}} \prod_{x \in l} w(x). \]
Examples of 8-vertex models

- Ising model on the dual lattice,
- Ising model in high-temperature expansion,
- Close packed dimer problem,
- QED$_2$ at $\beta = 0$ with Wilson fermions,
- GN model with Majorana Wilson fermions.
Very powerful ‘Worm’-type algorithms can be applied:
- amounts to enlarging the configuration space by open loops,
- corresponds to sampling directly the correlation function.

Critical slowing-down much suppressed.

I see no objections to adapt this to $d > 2$. 