Wilsonian Flows in Non-Abelian Gauge Theories

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We review an approach to non-Abelian gauge theories with Wilsonian (or Exact) flow equations. The renormalisation aspects of such an approach are detailed. In particular the consequences for gauge invariance in the presence of an infrared regularisation and background fields are examined for general linear gauges.

The presence of background fields allows the application of analytic heat kernel methods. On the basis of these investigations we outline a feasible way of attacking problems related to the low energy sector of QCD. We discuss the relation between gauges and the validity of particular approximations. The conceptual results presented here are used for a particularly simple calculation of universal properties. The inclusion of topological effects is discussed with the example of instanton-induced effects and chiral symmetry breaking. Finally we propose a gauge invariant thermal renormalisation group on the basis of the flow equation studied here.

Contents

I. INTRODUCTION

Quantum chromodynamics (QCD) is widely accepted as the fundamental theory of strong interaction. Its predictions have been verified very impressively by experiments in a large energy range, for momenta ranging from about 1GeV up to several 100GeV [163]. Theoretical predictions in this momentum/energy region can be obtained in an expansion about vanishing coupling: perturbation theory. The reliability of such an expansion is guaranteed by asymptotic freedom [82,149]. This property entails that the coupling strength decreases with increasing momenta, thus making an expansion about vanishing coupling at high momenta, reliable. In turn, in the low momentum region, at large distances, QCD is confining. The coupling becomes strong and perturbation theory ceases to work. The fundamental degrees of freedom in the asymptotically free region, quarks and gluons, are not the appropriate degrees of freedom for low momenta. In this regime, mesons and baryons are observable particles. Their quark-gluon content is quite complicated.

By itself a strong coupling is not the end of the story. We know of many cases where a power series in a parameter reaches its convergence radius but can be continued uniquely to larger values of the parameter. Such resummation techniques are widely used in quantum field theory, e.g. at finite temperature they play a pivotal rôle. Confinement, however, requires the dynamical generation of a physical mass-scale Λ at low energies. It follows from renormalisation group invariance of physics that (in massless QCD or pure Yang-Mills theory) this scale is exponentially suppressed in the limit where the coupling tends to zero, the ultra-violet fixed point. In turn, all dependence on Λ is projected out in an expansion about vanishing coupling. This implies that within such an expansion or resummations upon it we never see the formation of bound states with quark and gluon constituents.

To sum up, from the point of view of theoretical predictions, the situation is rather unsatisfactory. The standard method of quantum field theory, perturbation theory, is inapplicable and we have to deal with the situation, that physics and the relevant degrees of freedom change qualitatively, when going from the ultraviolet UV (large momenta, small distances) to the infrared IR (small momenta, large distances). This makes it hard to get analytical access to the physics under investigation.

Hence, for the reasons outlined above, one of the most interesting questions in the theory of strong interaction is, so far, quantitatively unanswered: the physics of the confinement mechanism. However, a quantitative understanding of the physics is essentially out of reach. It is clear that in order to get more insight into this question, one has to devise truly non-perturbative techniques. It would be most desirable to develop a method which works in both, the confining IR sector of the theory and the perturbative UV sector. This would allow us to study the phase transition or cross-over phenomena taking place in the interesting intermediate region ($\sim 1\,\mathrm{GeV}- \sim 200\,\mathrm{MeV}$). As the ultraviolet physics is well understood such an evolution would start in the perturbative high energy sector with well-understood perturbative QCD. In a Gedankenexperiment this is achieved by squeezing our physical system into a square box of size smaller than $(1\,\mathrm{GeV})^{-1}$. Then, the infra-red regime is included by extending the size towards $(200\,\mathrm{MeV})^{-1}$. At each infinitesimal step we have good control about the new physics entering our system.

Practically, such an infra-red suppression is better achieved in momentum space. There, we introduce an IR cut-off k into the theory. The propagation of momentum modes with momenta p^2 smaller than k^2 is suppressed and the theory is well under control. When lowering the cut-off scale k we introduce more and more infra-red physics into our theory. In quantum field theory physics is encoded in effective actions, analogues to the classical action. The evolution of these objects with the cut-off scale k is described by the so-called Exact Renormalisation Group (ERG), or Wilsonian flow equation, or shortly 'flow equation' [88,174,179]. They have been employed by Polchinski [148] for simplifying proofs of perturbative renormalisability in the ϕ^4-model. The crucial improvement is that the book-keeping of (sub-) divergences of diagrams is greatly simplified, see also [91–94,143–146]. Its formulation used in the present review was developed in [178], see also [22,51,124].

In the present formulation one studies the Legendre effective action $\Gamma[\phi]$ of the theory at hand, the generating functional of 'one particle irreducible Green functions'. This object encodes the full physical information about the theory in its vertex functions. The scale dependent effective action $\Gamma_k[\phi]$ agrees with the full effective action $\Gamma[\phi]$ for momentum modes with momenta much larger than the cut-off scale k, whereas it is trivial for momenta much smaller than k. As we shall derive later, its derivative w.r.t. $t = \ln k$ is given by

$$\partial_t \Gamma_k[\phi] = \int d^d p \left(\frac{\delta^2 \Gamma_k[\phi]}{(\delta\phi)^2} + R_k \right)^{-1} (p,p)\, \partial_t R_k(p), \tag{1.1}$$

where we work in the Euclidean formulation of quantum field theory. Eq. (1.1) constitutes an equation for the evolution of the effective action Γ_k with the scale. Here, $R_k(p)$ is the regulator which is as a modification of the propagator. To act as an infrared cut-off it has to behave like a momentum dependent mass: for momenta much smaller than the cut-off scale $p^2 \leq k^2$ the regulator tends to a mass or even diverges with inverse momentum. For momenta much larger than the cut-off scale $p^2 \geq k^2$ the regulator should tend to zero, as the ultraviolet region should remain unchanged. As a result, for $k \to 0$, we end up with the full effective action Γ of the theory at hand, as no propagation is suppressed. For $k \to \infty$ all propagation is suppressed. Thus no quantum fluctuations contribute in this limit and we approach the classical action. Consequently, the flow equation interpolates between the classical action and the full quantum effective action. Below, we include a typical plot of R_k and its scale derivative.

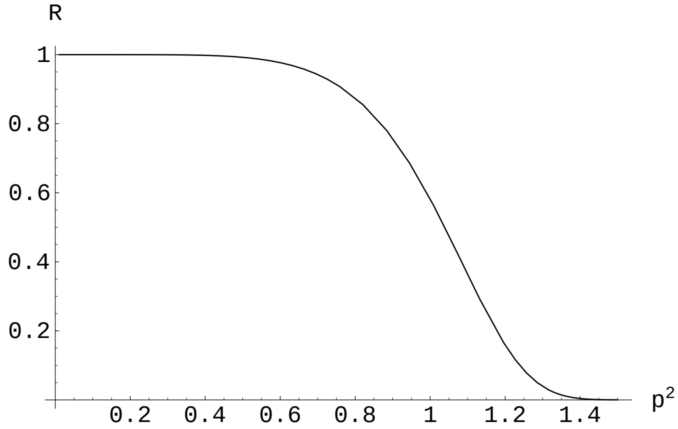

Figure 1: Plot of a typical regulator R/k^2 as a function of p^2/k^2

Its derivative is a smeared out delta function peaked at about k^2.

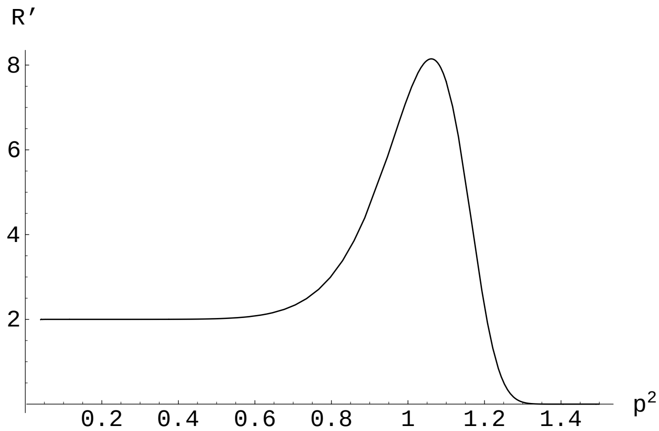

Figure 2: Plot of the t-derivative $R' = (\partial_t R)/k^2$ as a function of p^2/k^2

Eq. (1.1) with a regulator R_k enjoying the properties plotted in Figs. 1,2 leads to a well-defined flow equation (1.1). As the regulator is well localised in momentum space, the right hand side of the flow equation is very stable within numerical applications.

However, the main strength of this formalism is its flexibility, when it comes to approximations to the full problem at hand. This flexibility allows for the straightforward implementation of systematic non-perturbative approximations. This is a key advantage in situations where one wants to test possible physical mechanisms as it is the case for confinement in QCD. We only have to expand the effective action in the degrees of freedom which seem relevant to us. Then with the integration of the flow equation we are able to test our physical picture. In case one has reached some qualitative understanding about the relevant degrees of freedom the flow equation can also be used to get quantitative numerical results.

These properties have been successfully used in a number of applications in scalar as well as fermionic theories. Abelian gauge invariance poses no additional problem due to the linear realisation (in the field) of the gauge symmetry. For an exhaustive discussion of results obtained in scalar field theories theories with fermions as well as references on Abelian gauge theories we refer the reader to the reviews [11,16]

These successes have fueled hopes that a suitable formulation for non-Abelian gauge theories provides new insight into non-perturbative effects in QCD. However, progress in this direction has been hampered by the problem of maintaining gauge invariance in this approach. The introduction of a cut-off term as in (1.1) quadratic in the fields leads to gauge *variant* flow trajectories for the effective action Γ_k. Instead of satisfying the standard Ward-Takahashi identities (WI) or BRST (BRST) identities, Γ_k satisfies modified Ward-Takahashi (mWI) or BRST identities (mBRST) [24,52,69,84,110,151,158]. These identities are evolving according to the flow and approach the usual Ward-Takahashi identities as $k \to 0$. Consequently the full effective action Γ satisfies the usual Ward-Takahashi identities. In other words, gauge invariance of the full theory is preserved if the effective action Γ_{k_0} satisfies the mWI at the initial scale k_0 (see e.g. [44,52,109]).

As a first step towards a non-perturbative implementation of mWI/mBRST, much work has been devoted to a perturbative investigation of gauge theories, including anomalies and chiral gauge theories, in the presence of an infra-red cut-off [22–37]. The formulation of supersymmetric theories has been studied in [34,63]. This perturbative analysis corresponds to the analysis of standard renormalisation where the counter terms break the symmetry.

So far, non-perturbative approximations in non-Abelian gauge theories have been studied in quite different approaches. In [76,151,158] the flow equation in the background field approach was expanded in orders of the classical action density, and dropping all higher (covariant) derivative terms. This approximation is the analogue of the lowest order of a derivative expansion in scalar theories. The observables determined in this approximation were the running gauge coupling and the gluon condensate. Even though no quantitative statement can be drawn within these rough approximations, the estimate on the gluon condensate $\langle F^2 \rangle \sim [3\Lambda_{\rm QCD}]^4$ is promising. In [151], the running coupling was given as a resummed expression based on a one-loop approximation. It had a remarkable agreement with two loop perturbation theory. In [76] the agreement was improved up to 99% by further improving the approximation, also relying on a particular regulator.

In [15,53–55] an alternative approach was taken. Pure non-Abelian gauge theories were studied in covariant gauges. For the numerics the Landau gauge has been chosen. The approximation allowed for a general momentum dependence of the gluon and ghost propagators which were used to construct the heavy quark potential. As a first non-trivial consequence the mBRST was employed to fix the mass term for the gluon. Apart from some technical differences this is quite similar to approximation schemes used for Schwinger-Dyson equations in Landau gauge QCD [169,3] which also reviews the relevant lattice results. The results of [53,54] match the expected behaviour for the momentum dependence in the validity regime of the approximation. However, from studies of Dyson-Schwinger equations we expect an infra-red enhancement of the ghost propagator in combination with an infra-red well-behaved Gluon propagator. To get access to these interesting properties, the approximations in [53,54] have to be improved. Including more terms in the effective action necessitates a further investigation of the mWI/mBRST.

In [56–62] (see also [71,72]) an approach has been developed, based on the confinement scenario of the dual Meissner effect. There, confinement originates in the condensation of magnetic monopoles. This scenario is taken care of by introducing collective fields for the related (Abelian) components of the field strength. These collective fields carry the monopole degrees of freedom. The results indicate the existence of an infra-red attractive fixed point. In turn, this supports the scenario behind the approach, as the collective degrees of freedom seem to carry the relevant physics. Still, more work is required in order to improve on the truncations. So far only the classical terms with multiplicative scale factors for the original fields and the collective degrees of freedom have been considered.

Topological consideration also play a pivotal rôle for chiral symmetry breaking. Instantons have been successfully used for a quantitative explanation of the anomalously big η'-mass within the instanton-liquid model, see [162]. Within flow equations, the evolution of the corresponding instanton-induced couplings can be studied , hence adding insight into the chiral phase transition within the fundamental theory. Such a task necessitates the computation of instanton-induced terms in the effective action at high scales. A priori, it is not clear whether topological properties survive the introduction of a cut-off term which modifies the infra-red behaviour of the fields. In [137,138] the 't Hooft determinant [171] was derived in the presence of the cut-off term. These terms also depend on the θ-angle, and are related to the strong CP-problem, i.e. the extremely small value for θ, (non-) observed in nature. The flow of θ has been investigated in [154] in a first approximation. Unfortunately, conceptual problems, related to the space-time dependence of θ in the approach, remain open. Apart from these intricacies, the results showed the potential for a non-trivial flow of θ, which could solve the strong CP-problem. Related work on Chern-Simons theories is found in [152,153].

Finally we would like to discuss the application of ERG methods to gravity in an approach similar to the background field method presented above [18–21,97–100,155,159,160], see also ERG flows in Liouville theory, [156,157]. The flow equation is used in a background field approach, in the same spirit as discussed above for Yang-Mills theories. Quantum gravity, however, is not renormalisable.

Interestingly enough, within the flow equation, an ultraviolet non-Gaußian fixed point is found. This is taken as an indication that quantum gravity might be non-perturbatively renormalisable in the spirit of Weinberg.

To summarise, the lesson to learn from the results obtained in the applications discussed above is twofold. Firstly, it confirms the hopes to gain non-trivial insight into the non-perturbative regime of QCD by this method, even more so, as promising results already have been obtained in rough approximations, leaving much space for improvement. Secondly, it is mandatory to more carefully investigate the impact of the modification of gauge invariance in the present approach, before embarking on more elaborated applications.

More explicitly, the following questions have to be investigated in more detail: First of all, the modification of gauge invariance during the flow, encoded in the mWI, makes life harder when it comes to consistent approximations. To understand this point, let us have a closer look at standard perturbation theory. There, physical gauge invariance is encoded in non-trivial Ward-Takahashi identities for the effective action. Much work has been devoted in order to make these non-trivial constraints algebraically accessible as these WI typically involve loop terms. This led to BRST invariance, which encodes physical gauge invariance on the level of an algebraic identity for the effective action, at the cost of auxiliary fields, the ghosts.

In the flow equation approach, this algebraic identity is lost again due to the presence of the cut-off term. Even when using the BRST formulation, the mBRST involves loop terms, leading to a non-trivial task when devising gauge consistent approximations. Consequently it is of chief importance to minimise the technical effort connected with this situation. There are several options to overcome this obstacle:

Firstly one can use gauges which minimise the technical problems related to the solution of the mWI/mBRST. Such an approach using general axial gauges was developed in [107,109,110,119] (see also [74]). In axial gauges we have the additional benefit that the subtleties of the non-trivial ghost sector are circumvented as the ghosts decouple. The only terms in the mWI or mBRST are the loop terms introduced via the cut-off term. Indeed, in perturbation theory the WI or BRST identities are trivial. There, however, this triviality exacts a high price, because the propagators of the gauge field involves additional so-called spurious poles, which have to be treated separately. This also applies to DS-equations [3,169]. Given also the more general tensor structure in the presence of an additional Lorentz vector, no simplification is achieved in comparison to covariant gauges. Even more so, numerically, the treatment of the spurious singularities is cumbersome. It has been shown in [107] that the spurious singularities are absent in the ERG equation for axial gauges. This opens a path for using the advantages of axial gauges. Additionally, inherent approximations made in the background field approach to flow equations are more justified here than in the background gauge.

Moreover, the additional tensor structures pose no additional complication for computations at finite temperature. Here, the heat-bath singles out a rest frame anyway. This approach is discussed here in Chapter III and Chapter V.

Secondly, one can try to reinstall gauge invariance in the gauge fixed formalism as much as possible. In perturbation theory such an approach was introduced via the background field formalism. It allows the definition of a gauge invariant effective action. Nevertheless, this formalism has additional non-trivial WI or BRST identities, which are the usual identities of the formulation without background field. The advantage of this approach is that even though this is so, the physical Green functions can be extracted more easily due to gauge invariance. This stays so in the flow equation approach, but non-trivial mWI or mBRST are still present. This makes it inevitable to resort to inherent approximations in practical computations. In Chapter III and Chapter IV we detail flow equations in the background gauge with particular emphasis on symmetry aspects and the impact of inherent approximations. We also use background fields in axial gauges.

A third option is a truly gauge invariant ERG flow. The theory has to be formulated in gauge invariant variables [5–8,125,127,128,173]. In the present work, we will not further detail such attempts, for a discussion of the related problems see [111]. Let us just briefly mention the possible benefits and short-comings. The obvious benefit is that gauge invariance is encoded trivially. In perturbation theory, one short-coming is the necessary non-locality originating in the non-linearity of the gauge symmetry. Consequently, (IR) singularities are much harder to control. The explicit problem disappears in the ERG equation, precisely due to its infra-red finiteness. Still, there seems to be a feedback. So far, one pays the price in a qualitatively more difficult ERG equation.

Here, we review some work on ERG flows in gauge theories with in gauge-fixed formulations [69,70,107,109–111,119,137,138] together with some new results [141], in particular in Section II B and Section IV.

In Chapter II we present a derivation of the flow equation in the form used throughout this review. The properties of the regulator and of the flow are discussed in detail. Particular emphasis is devoted to the subtleties of the renormalisation of the effective action Γ_k. We discuss the connection between the anomalous dimensions of the full theory to those derived from the k-scaling. It is also shown how standard perturbation theory is recovered in a iterative expansion in loops with the example of two loop diagrams. We close with a discussion of the results and a comparison of the present approach to other non-perturbative methods.

Chapter III deals with the details of the flow equation approach to gauge theories for general linear gauges. The modified Ward-Takahashi identities are derived and their implications are outlined. This is done for the background field gauge as well as for general axial gauges. The results of this chapter provide all necessary tools for embarking on applications of the flow equation approach to non-

Abelian gauge theories, both analytically and numerically. Again the results are briefly summarised and their implications are discussed.

In Chapter IV the background field formulation of ERGs is applied to the calculation of the universal one loop β-function of pure Yang-Mills theory. We also sketch the calculation of the two loop coefficient [140] within the ERG approach. The calculations are important consistency checks of the whole approach. It is well-known that the massless limit of massive theories has to be taken with some care. It is not guaranteed a priori that the theory tends to the massless one in the limit of vanishing mass. For the calculations, we develop analytic methods for background field flows. The one loop nature of the flow enables us to resort to heat-kernel techniques which simplify loop calculations tremendously. Then, these tools are applied to the computation of the β-function. Its present calculation is not only technically very simple way. It also provides informations about approximation schemes within the background gauge.

The analytic means evaluated in this section are also used in Chapter V. Here, we discuss the computation of the one loop effective action in general axial gauges. This provides an explicit check of the absence of spurious singularities in the present approach. Additionally, it offers all necessary ingredients to embark on non-perturbative approximations. We also discuss differences to the background field gauge in view of the inherent approximations within practical calculations in the presence of a background field.

In Chapter VI we review the calculation of instanton-induced fermionic terms in the limit of large cut-off scales. We discuss the scale dependence in leading order of these terms. The impact of the regulator terms for the topologically non-trivial configurations is studied in detail.

Finally, in Chapter VII we propose a thermal flow based on ERGs. Such a flow is introduced as the difference between ERG-flows at zero and finite temperature. We also discuss the possibility of a gauge invariant thermal flow. This flow is defined as a particular sub-case of this proposal, which use axial gauges as well as a mass-like regulator. Possible applications are outlined.

The review closes with a few final remarks and a short outlook. Several Appendices contain some technical details.

II. FLOW EQUATION

In this chapter we derive the Exact Flow Equation (ERG) and discuss its properties. For the issues discussed here we restrict ourselves to a theory with one real scalar field in d Euclidean dimensions also allowing for a background field. The rôle of such a field is twofold. The expansion of the full theory about this configuration simplifies, if this field is chosen appropriately, e.g. related to the physical vacuum. It also can serve as a purely auxiliary field used to simplify symmetry considerations. The results presented here straightforwardly generalise to arbitrary field theories.

In Section II A we define the scale-dependent effective action based upon the path integral representation of the theory. Then, we derive the flow equation and discuss the properties of the regulator, as well as the additional background field dependence introduced by the cut-off term. In case the regulator diverges at a finite UV scale Λ, we also consider the dependence of Γ_k on this UV scale.

Section II B is devoted to a detailed discussion of renormalisation subtleties in this approach. The full theory satisfies a renormalisation group equation which encodes the physical invariance under a variation of the renormalisation conditions. We derive the deviation from the original RG-scaling in the presence of a cut-off term. The impact of particular choices for regulators is discussed.

In Section II C we show how standard perturbation theory is contained in the formalism within the example of two loop diagrams.

We close the chapter with a summary of the results and a structural comparison of the ERG equation to other non-perturbative techniques like Dyson-Schwinger equations and various one-loop improved RG equations.

A. Flow equation

Throughout this review we work in Euclidean space-time. The classical action of a scalar theory in d dimensions is given by

$$S[\phi] = \int d^d x \, \left(\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2} m^2 \phi^2 + V[\phi]\right), \tag{2.1}$$

where we allow for arbitrary potentials $V[\phi]$ that lead to renormalisable theories. Then, the starting point of our considerations is the renormalised Schwinger functional of the theory. For later purposes we also allow for background fields, that is, we split the full field into $\bar\phi + \phi$. Here $\bar\phi$ comprises the background field and ϕ is the fluctuation about the background. The renormalised Schwinger functional reads

$$\exp W[J, \bar\phi] = \int [\mathcal{D}\phi]_{\rm ren} \, \exp\left\{ -S[\bar\phi + \phi] + \int d^d x \, J(x)\, \phi(x) \right\}, \tag{2.2}$$

where the subscript $_{\rm ren}$ indicates that some renormalisation procedure has been employed, rendering $W[J,\bar\phi]$ finite. $W[J,\bar\phi]$ is the generating functional of connected Green functions. A simpler object to deal with is the effective action $\Gamma[\phi,\bar\phi]$, the Legendre transform of $W[J,\bar\phi]$:

$$\Gamma[\phi,\bar\phi] = \sup_J \left\{ \int d^dx\, J(x)\,\phi(x) - W[J,\bar\phi] \right\}, \tag{2.3}$$

being the generating functional of one particle irreducible Green functions. In case $\Gamma[\phi,\bar\phi]$ is differentiable, the field ϕ is given by

$$\phi(x) = \frac{\delta W[J,\bar\phi]}{\delta J(x)}. \tag{2.4}$$

Here, we will not discuss the subtleties which one faces if $\Gamma[\phi,\bar\phi]$ is not differentiable, or if $\frac{\delta^2}{(\delta J)^2}W$ is not positive definite. Interested readers are referred to [2] and the review [150].

As already mentioned, in most cases it is impossible to perform the integral in (2.2), or, alternatively, to explicitly calculate Γ. Either, one has to resort to some expansion scheme like perturbation theory, or to effective models derived from the fundamental theory, or one has to use numerical methods. The flow equation is a method for solving the path integral successively in terms of momentum modes. First, the propagation of small momentum modes is effectively suppressed in the Schwinger functional. We modify the action $S \to S + \Delta S_k$ in the exponent of (2.2). The cut-off term ΔS_k introduces a momentum dependent mass

$$\Delta S_k[\phi,\bar\phi] = \frac{1}{2}\int d^dx\,\phi(x)\,R_k[\bar\phi]\,\phi(x). \tag{2.5}$$

The regulator R_k depends on an infra-red (momentum) scale k which will interpolate from some ultra-violet (UV) scale Λ to the IR limit $k = 0$. The cut-off term (2.5) is quadratic in the fields. Thus, if we add the cut-off term (2.5) to the action in the path integral (2.2), it leads to a modification of the kinetic term. In the effective action, R_k adds to the full field dependent propagator (see (1.1)). As already argued there, R_k should vanish in the UV regime and should behave like a mass (or even diverge) in the IR regime; thus leaving the UV regime unchanged but suppressing the propagation of IR modes. We demand that R_k has the following properties:

(i) It has a non-vanishing limit for $p^2 \to 0$, typically $R_k \to k^2$. This precisely ensures the IR finiteness of the propagator at non-vanishing k even for vanishing momentum p.

(ii) It vanishes in the limit $k \to 0$. In this limit, any dependence on R_k drops out and $\Gamma_{k\to 0}$ reduces to the full quantum effective action Γ.

(iii) For $k \to \infty$ (or $k \to \Lambda$ with Λ being some UV scale much larger than the relevant physical scales), R_k diverges. Thus, the saddle point approximation to the path integral becomes exact and $\Gamma_{k\to\Lambda}$ reduces to the classical action $S[\bar\phi + \phi]$.

Let us now specify regulators R_k with the properties (i)-(iii). Regulators solely dependent on plain momentum squared, are conveniently parametrised as $R_k = p^2\, r(p^2)$, where $r(y)$ a dimensionless function of the dimensionless variable p^2/k^2. We also consider a more general class of regulators with the parametrisation $R_k = z\, r(y)$, where y and z are operators at to our disposal. For the simple choice $z = y = p^2$ we have $R_k = p^2\, r(p^2)$. Particular choices for z, y are discussed in Section II B 2. For now we just specify the limits of the regulator $r(y)$:

$$\lim_{\frac{y}{k^2}\to\infty} (y/k^2)^{d/2} r(y) = 0, \tag{2.6a}$$

$$\lim_{\frac{y}{k^2}\to 0} r(y) \propto \left(\frac{k^2}{y}\right)^n, \quad n \geq 1. \tag{2.6b}$$

Regulators $R_k = z\, r(y)$ with limits (2.6) vanish sufficiently fast in the UV regime and suppress the propagation in the IR regime. A regulator R_k with the limits (2.6) has the properties (i)-(iii). Moreover IR and UV finiteness of the flow equation (see (1.1)) are guaranteed. The k-dependent Schwinger functional W_k is defined as

$$\exp W_k[J, \bar\phi] = \frac{1}{\mathcal{N}_k} \exp\left(-\frac{1}{2}\int d^d x\, \frac{\delta}{\delta J} R_k[\bar\phi] \frac{\delta}{\delta J}\right) \exp W[J, \bar\phi]. \tag{2.7}$$

The normalisation $\mathcal{N}_k$ is a possibly $\bar\phi$-dependent constant. It is worth emphasising that (2.7) is not simply (2.2) with the cut-off term (2.5) added to the action $S[\bar\phi + \phi]$ in the exponential. The integral on the right hand side of (2.2) stands for the *renormalised* Schwinger functional and adding the cut-off term to the exponential within a common renormalisation procedure leads to a k-dependent renormalisation. Then, the t-derivative not only hits the cut-off term but also the implicit scale dependence introduced by the renormalisation. Simply put, (2.7) stands for a renormalised quantity, where the operator R_k is left bare, whereas adding the cut-off term to $S[\phi, \bar\phi]$ in (2.2) also leads to a renormalisation of R_k. Dealing with a bare operator R_k is no problem for the finiteness of W_k, as R_k vanishes sufficiently fast for large momenta. An understanding of these differences is important when it comes to a comparison of the flow equation with other renormalisation group equations. We will further elaborate on this point below.

The flow equation for $W_k[J, \bar\phi]$ follows from the k-independence of $W[J, \bar\phi]$. With $t = \ln k$ we arrive at

$$\left(\partial_t - \int d^d x (\partial_t J) \frac{\delta}{\delta J}\right) W_k = -\frac{1}{2}\int d^d x \left(\frac{\delta}{\delta J} + \frac{\delta W_k}{\delta J}\right) (\partial_t\, R_k)\, \frac{\delta}{\delta J}\, W_k - \partial_t \ln \mathcal{N}_k. \tag{2.8}$$

Here ∂_t stands for the total derivative w.r.t. t. We are more interested in the flow of the effective action, the Legendre transform of W_k. As the cut-off term diverges for $k \to \Lambda$, the Legendre transform of $W_k[J, \bar\phi]$ tends to $\Delta S_k +$ finite terms in this limit. A finite effective action Γ_k is defined as

$$\Gamma_k[\phi,\bar\phi] = \int d^dx\, J\phi - W_k[J,\bar\phi] - \Delta S_k[\phi,\bar\phi], \qquad\qquad \phi(x) = \frac{\delta W_k[J,\bar\phi]}{\delta J(x)}. \tag{2.9}$$

In (2.9), either ϕ or J has to be seen as the independent variable. As a function of ϕ, the current is given by $J = \frac{\delta}{\delta\phi}(\Gamma_k + \Delta S_k)$. The flow of Γ_k is given by

$$\partial_t\Gamma_k[\phi] - \int d^dx\,(\partial_t\phi)J = \int d^dx\,\phi\,\partial_t J - \partial_t W_k - \partial_t\Delta S_k[\phi]. \tag{2.10}$$

The first two terms on the right hand side of (2.10) give minus the right hand side of (2.8). This follows from the definition of ϕ in (2.9). The term of (2.8) proportional to $(\frac{\delta}{\delta J}W_k)^2$ is just $\Delta S_k[\phi]$. Thus, only the term proportional to the second derivative of W_k w.r.t. the current is left. With the definition of Γ_k as a Legendre transformation of W_k we deduce

$$\frac{\delta^2 W_k}{\delta J(x)\delta J(y)} = \left(\frac{\delta^2\Gamma_k}{\delta\phi(x)\delta\phi(y)} + R_k(x,y)\right)^{-1}. \tag{2.11}$$

Then, from (2.8), (2.9) and (2.11) we finally arrive at the flow equation for $\Gamma_k[\phi,\bar\phi]$:

$$\left(\partial_t - (\partial_t\phi)\frac{\delta}{\delta\phi}\right)\Gamma_k[\phi,\bar\phi] = \frac{1}{2}\mathrm{Tr}\, G_k[\phi,\bar\phi]\,\partial_t R_k[\bar\phi] + \partial_t\,\ln\mathcal{N}_k, \tag{2.12}$$

where

$$G_k[\phi,\bar\phi] = \left(\Gamma_k^{(2)}[\phi,\bar\phi] + R[\bar\phi]\right)^{-1} \qquad \text{with} \qquad \Gamma_k^{(n)}[\phi,\bar\phi](x_1,...,x_n) = \frac{\delta^n\Gamma_k[\phi,\bar\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)}. \tag{2.13}$$

The trace Tr denotes a sum over momenta, that is, $\mathrm{Tr}\,A(p,p') = \int \frac{d^dp}{(2\pi)^d}A(p,p)$. Let us briefly discuss the properties of the flow equation (2.12). As the field ϕ is a free variable, usually one chooses $\partial_t\phi = 0$. The flow (2.12) encodes the total t-dependence of the full vertices $\Gamma_k^{(n)}[\phi=\phi_0]$ for any field ϕ_0 with $\partial_t\phi_0 = 0$. In turn, for t-dependent ϕ the right hand side only constitutes a partial derivative at fixed ϕ. For the integration of the flow on needs a total derivative, thus a t-dependent ϕ is not of help here. We conclude that the effective action $\Gamma_k[\phi,\bar\phi]$ is a function of t-independent fields and couplings with only explicit t-dependence. In this sense, we deal with an unrenormalised effective action with respect to the scale t.

Still, given a finite effective action Γ_k the flow equation itself is well-defined, both as a function in the fluctuation field ϕ and the background field $\bar\phi$. This follows straightforwardly from the properties of the regulator function R_k. By construction $\Gamma_{k=0} = \Gamma$, the renormalised quantum effective action. Moreover, it also follows that the integrated flow (integrated over finite intervals) is finite as a consequence of the finiteness of (2.12).

Coming back to the question of renormalisation we realise that the propagator G_k (2.13) is a finite quantity as it originates in the renormalised effective action Γ_k. Loops with G_k, however, are only finite upon an appropriate insertion as R_k. It is instructive to compare (2.12) with a renormalised Callan-Symanzik (CS) equation [39,170], for textbooks see [40,86,175]. Basically, for a renormalised CS-flow we choose a regulator $R_k = k^2(1+\gamma_{\phi^2})$, where γ_{ϕ^2} is the anomalous dimension of the renormalised operator $[\phi^2]$. Instead of G_k the *renormalised* propagator $G_{\rm ren}$ enters. In contradistinction to G_k, loops with $G_{\rm ren}$, i.e. ${\rm Tr}\, G_{\rm ren}$ are finite. In turn, the CS-equation prior to renormalisation is given for the choice $R_k = k^2$ and the bare propagator $G_{\rm unren}$. Given this picture, the flow equation (2.12) is in between these two cases. The regulator is not subject to renormalisation following by the way it is introduced. The propagator G_k is defined in terms of a renormalised quantity but does not lead to renormalised loops. This subtlety has to be kept in mind if one wants to study the limit $R_k \to k^2$.

In turn this leaves us with an option to sneak in an implicit scale dependence. This is done in a similar way to the procedure which turns the unrenormalised to the renormalised CS equation. Here, however, we are more flexible. In the CS equation we are constrained to transformations that render the CS equation finite. The flow equation is finite from the onset and we can use general transformations. t-independent fields and couplings (e.g. $\phi, \bar\phi, g$) can be expressed with t-dependent quantities $\phi[\varphi_t, \bar\varphi_t, g_t; k], \bar\phi[\varphi_t, \bar\varphi_t, g_t; k], g[\varphi_t, \bar\varphi_t, g_t; k]$. Note that the new fields φ_t may be composite fields in terms of $\phi, \bar\phi$. The only constraint is $\partial_t(\phi, \bar\phi, g) = 0$.[1] Such transformations can be used to improve a truncation scheme. Chosen appropriately, they naturally include the transition from fundamental degrees of freedom to effectives ones. An application of this idea to the Nambu-Jona-Lasino model has been attempted in [75,77].

The introduction of a background field dependent regulator $R_k[\bar\phi]$ breaks the symmetry of ϕ and $\bar\phi$ present in the classical action $S[\bar\phi+\phi]$. The scale dependent effective action $\Gamma_k[\phi,\bar\phi]$ in general is not a function of the sum $\bar\phi+\phi$. Instead one has

$$\left(\frac{\delta}{\delta\bar\phi} - \frac{\delta}{\delta\phi}\right)\Gamma_k[\phi,\bar\phi] = \tfrac{1}{2}{\rm Tr}\, G_k[\phi,\bar\phi]\,\frac{\delta}{\delta\bar\phi}R_k[\bar\phi] + \frac{\delta}{\delta\bar\phi}\ln\mathcal{N}_k. \tag{2.14}$$

In the limit where the infrared regulator R is removed, the right hand side of (2.14) tends to zero. Hence, $\Gamma_0[\phi,\bar\phi] = \Gamma_0[\bar\phi+\phi, 0]$. Due to (2.14) it makes sense to distinguish between derivatives w.r.t. ϕ and $\bar\phi$.

Let us close this section with some considerations about the total scale dependence introduced. It is convenient to start an integration of the flow at a trivial initial condition $\Gamma_\Lambda[\phi,\bar\phi] \simeq S[\bar\phi+\phi]$.

---

[1] For general one loop exact flows as defined in [118] one only needs that $\partial_t \ln\big([\mathcal{D}\phi]_{\rm ren}\, e^{-(S+\Delta S_k)+\int d^dx J\, O_k}\big) \simeq \int d^dx\, \big(L_k[J,\bar\phi]\, O_k + O_k\, A_k[J,\bar\phi]\, O_k\big)$. Here the current is coupled to a general operator $O_k[\phi]$ and A_k should lead to a finite flow.

Then, strictly speaking it is required that all propagation is suppressed at this finite UV scale Λ. Necessarily the regulator has to diverge at this scale:

$$\lim_{k\to\Lambda} r(y) \to \infty. \tag{2.15}$$

Consequently the regulator R_k also depends on the scale Λ. Thus, the total scale dependence introduced by R_k is encoded in the flow of Γ_k w.r.t. a variation of both scales k and Λ:

$$\partial_\lambda \Gamma_k[\phi, \bar\phi] = -\tfrac{1}{2}\mathrm{Tr}\, G_k[\phi, \bar\phi]\, \partial_\lambda R_k[\bar\phi] + \partial_\lambda \ln \mathcal{N}_k, \tag{2.16}$$

where $\partial_\lambda = k\frac{\partial}{\partial k} + \Lambda\frac{\partial}{\partial \Lambda}$ is the total derivative w.r.t. t and $\ln\Lambda$. One is tempted to identify the UV scale Λ with the UV cut-off scale of the full theory. Then, however, (2.16) does not represent the total derivative w.r.t. Λ. It would only display the additional explicit Λ-dependence introduced via the regulator R_k. Thus at the present stage we avoid such a premature identification.

B. Flow equation versus RG equation

The last comment brings us to an interesting topic of its own. Ultimately we are interested in the full quantum theory at $k = 0$. Here, anomalous dimensions, critical exponents, or, more generally, the running of Green functions with the renormalisation scale encode non-trivial information about physics. In an ERG-approach, the counterparts of these objects, the running of the Green functions with the infrared scale k, are most easily accessible. A priori this scaling need not agree beyond one loop with the μ-scaling as the regulator introduces an explicit additional mass scale to the theory. It is well-known from standard renormalisation theory, that the relation between these quantities is non-trivial beyond one loop, see e.g. [40,86,175]. Thus, if one is interested in the anomalous dimensions of the underlying theory some work has to be done in order to match the k, Λ-scaling encoded in (2.12,2.16) to the μ-scaling of the full theory at $k = 0$. Still, we would like to know how the RG-scaling of the full theory changes in the presence of the cut-off term. In particular in massless theories it is an important issue to show that the regularised theory tends to the massless theory for $R_k \to 0$. In gauge theories this is intimately related to the question of gauge invariance of observables. An analysis of these questions is closely related to the discussion of the properties of the CS equation. Related questions are also discussed in the recent review on functional RG techniques [150]. The relation between the t-scaling and the RG scaling of the underlying theory has also been studied in [31,55,145]. The approach here is close to the one in [145], where mass-independent RG schemes in the context of flow equations are investigated.

1. RG equation for Γ_k

The ERG equation (2.12) and the flow equation (2.16) describe part of the total scale dependence of Γ_k. For comparing the scalings we also need to know the action of the total μ-derivative on Γ_k. We shall see that the cut-off term in general introduces an anomalous scaling w.r.t. μ. Still, we will call this equation an RG equation, even though it is not homogenous. Our starting point for the derivation of the flow equations (2.12,2.16) was (2.7) for W_k. (2.7) is a simple relation between the *renormalised* Schwinger functional of the full theory and the cut-off dependent Schwinger functional W_k. This can be used to derive the RG equation (w.r.t. μ) for W_k and consequently for Γ_k along the same lines as the flow equation was derived in section II A.

The RG equation for the Schwinger functional $W[J,\bar\phi]$ is given by

$$\mu\frac{d}{d\mu}W[J,\bar\phi]\equiv\left(\mu\frac{\partial}{\partial\mu}+\gamma_g g\partial_g-\gamma_\phi\int d^dx\,J\frac{\delta}{\delta J}+\gamma_{\bar\phi}\int d^dx\,\bar\phi\frac{\delta}{\delta\bar\phi}\right)W[J,\bar\phi]=0, \tag{2.17}$$

where we have introduced the anomalous dimensions

$$\mu\frac{d}{d\mu}g=\gamma_g,\qquad \mu\frac{d}{d\mu}\bar\phi=\gamma_{\bar\phi}\bar\phi,\qquad \mu\frac{d}{d\mu}J=-\gamma_\phi J, \tag{2.18}$$

restricting ourselves to the case of only one coupling g in the potential $V[\phi]$ in the classical action (3.5). Eq. (2.17) displays the fact that the renormalised Schwinger functional has to be independent of the unphysical RG scale μ, that is, its total derivative w.r.t. μ vanishes, e.g. [40,175]. From (2.17) we can deduce $\mu\frac{d}{d\mu}W_k[J,\bar\phi]$. We recall (2.7) and conclude that

$$\mu\frac{d}{d\mu}W_k[J,\bar\phi]=-\tfrac12\left[\mu\frac{d}{d\mu}\,,\int d^dx\left(\frac{\delta}{\delta J}+\frac{\delta W_k}{\delta J}\right)R_k\frac{\delta}{\delta J}\right]W_k[J,\bar\phi]-\mu\frac{d}{d\mu}\ln\mathcal{N}_k. \tag{2.19}$$

In (2.19) we have used that the commutator $[\mu\frac{d}{d\mu}\ ,\ \delta_J\,R_k\delta_J]$ commutes with $\delta_J\,R_k\delta_J$. The two terms on the right hand side of (2.19) stand for the deviation of the μ-scaling of infrared regularised Schwinger functional $W_k[J,\bar\phi]$ from the RG scaling of the full Schwinger functional $W[J,\bar\phi]$. It is related to the non-invariance of a general ΔS_k under an RG-scaling. It is non-zero for all cut-off terms with $\mu\frac{d}{d\mu}\Delta S_k[\phi,\bar\phi]\neq 0$ and $\mu\frac{d}{d\mu}\phi=\gamma_\phi\phi$. In turn, for $\mu\frac{d}{d\mu}\Delta S_k[\phi,\bar\phi]=0$ the right hand side of (2.19) vanishes. To see this more clearly, we calculate the commutator in (2.19) with help of the representation of $\mu\frac{d}{d\mu}$ in parenthesis in (2.17). Alternatively one can directly use the anomalous dimensions (2.18). This leads to

$$-\tfrac12\left[\mu\frac{d}{d\mu}\,,\int d^dx\,\frac{\delta}{\delta J}R_k\frac{\delta}{\delta J}\right]=-\tfrac12\int d^dx\,\frac{\delta}{\delta J}\left[(\mu\frac{d}{d\mu}+2\gamma_\phi)R_k\right]\frac{\delta}{\delta J}. \tag{2.20}$$

In (2.20) we allowed for a μ-dependent R_k (but R_k does not depend on J or ϕ). (2.20) just displays $-\mu\frac{d}{d\mu}\Delta S_k[\phi]$, if we identify $\phi=\frac{\delta}{\delta J}$. With (2.20) we get for (2.19)

$$\mu\frac{d}{d\mu}W_k[J,\bar\phi] = -\tfrac{1}{2}\mathrm{Tr}\,\left[(\mu\frac{d}{d\mu}+2\gamma_\phi)R_k\right]\left(\frac{\delta^2 W_k}{\delta J^2}+\left(\frac{\delta W_k}{\delta J}\right)^2\right)-\mu\frac{d}{d\mu}\ln\mathcal{N}_k. \tag{2.21}$$

Now we proceed to the effective action Γ_k. To that end we recall that $W_k[J,\bar\phi]=\int d^dx\,J\phi-(\Gamma_k[\phi,\bar\phi]+\Delta S_k[\phi,\bar\phi])$. Thus, the total μ-derivative of $W_k[J,\bar\phi]$ is just

$$\mu\frac{d}{d\mu}W_k[J,\bar\phi] = -\left(\mu\partial_\mu|_\phi+\gamma_\phi\int d^dx\phi\frac{\delta}{\delta\phi}\right)(\Gamma_k[\phi,\bar\phi]+\Delta S_k[\phi,\bar\phi]), \tag{2.22}$$

where $\mu\partial_\mu|_\phi$ is the partial μ-derivative at fixed ϕ. Eq. (2.22) follows from the implicit identification $J=\frac{\delta}{\delta\phi}(\Gamma_k+\Delta S_k)$. Then, using (2.19),(2.20) and making the Legendre transformation to the effective action (2.9) we arrive at

$$\left(\mu\frac{\partial}{\partial\mu}+D^\phi\right)\Gamma_k[\phi,\bar\phi]=\frac{1}{2}\mathrm{Tr}\,G_k[\phi,\bar\phi]\left(\mu\frac{\partial}{\partial\mu}+D^\phi+2\gamma_\phi\right)R_k[\bar\phi]+\left(\mu\frac{\partial}{\partial\mu}+D^\phi\right)\ln\mathcal{N}_k, \tag{2.23}$$

where

$$D^\phi=\gamma_g g\partial_g+\gamma_\phi\int d^dx\,\phi\frac{\delta}{\delta\phi}+\gamma_{\bar\phi}\int d^dx\,\bar\phi\frac{\delta}{\delta\bar\phi} \tag{2.24}$$

In a slight abuse of notation we will refer to (2.23) as the RG equation of Γ_k. It displays the μ-dependence of Γ_k, lifted from that of $\Gamma=\Gamma_{k=0}$. Similarly to (2.19), the right hand side of (2.23) is the deviation of the full μ-scaling of Γ_k from that of Γ. Eq. (2.23) and the flow equations (2.12),(2.16) provide the information about the non-trivial scale dependence of Γ_k. An important combination of these equations is the sum of (2.23) and (2.16):

$$\left(\partial_s+D^\phi\right)\Gamma_k[\phi,\bar\phi]=\frac{1}{2}\mathrm{Tr}\,G_k[\phi,\bar\phi]\left(\partial_s+D^\phi+2\gamma_\phi\right)R[\bar\phi]+\left(\partial_s+D^\phi\right)\ln\mathcal{N}_k, \tag{2.25}$$

where $\partial_s=\mu\frac{\partial}{\partial_\mu}+\Lambda\frac{d}{d\Lambda}+k\frac{d}{dk}=\mu\frac{\partial}{\partial_\mu}+\partial_\lambda$. A few comments are in order. Eq. (2.25) displays the full dependence of Γ_k on all scales, implicit and explicit. It is obvious from the derivation that the anomalous dimensions γ_g in (2.23),(2.25) are those of the full theory. The inhomogeneity displayed on the right hand side of (2.25) originates in the fact that we refrained from renormalising the operator R_k. Eq. (2.25) reduces to a variant of the *unrenormalised* Callan-Symanzik equation for $R_k=k^2$. It is only a variant, however, as still the renormalisation of the full theory is present. In this limit the right hand sides of (2.12), (2.16),(2.25) are not well-defined and an additional renormalisation is mandatory. However, as opposed to the usual CS equation the renormalisation of the theory is not changed with a change of k; in other words, the k^2-flow is not directly a flow in the space of theories with mass k^2. Note that this subtle difference plays a rôle when it comes to the identification of anomalous dimensions. For $k\to 0$ the right hand side of (2.23) vanish and we are left with the usual RG-equation for the full effective action Γ.

The results of the previous sections reveal some information about the implications of choosing particular regulators R_k. For dimensional reasons it follows easily for a regulator $z\, r(y)$ that

$$\partial_s R_k = z(\partial_s y - 2y)\, \partial_y r(y) + (\partial_s z)\, r(y) \tag{2.26}$$

as long as $[(\partial_s y)\, ,\, y] = 0$. We already emphasised that in general the RG equation for Γ_k is not homogeneous due to $(\mu\partial_\mu + D^\phi + 2\gamma_\phi)R_k \neq 0$. Eq. (2.23) shows that the original RG equation collects a t-dependent scale anomaly in the presence of a general cut-off term. This was to be expected. The regulator was introduced to differentiate between scales. Of course we would like to decouple μ-scaling and t-scaling. In other words, we would like to keep the homogeneous form of the RG-equation (2.23) even for the effective action Γ_k: $(\mu\partial_\mu + D^\phi)\Gamma_k = 0$. This is achieved for regulators R_k satisfying the following constraint:

$$(\mu\partial_\mu + D^\phi + 2\gamma_\phi)R_k = 0. \tag{2.27}$$

Eq. (2.27) should be interpreted as the anomalous dimension for the regulator R_k which is mandatory for the homogeneous form of the RG equation rendering (2.23) a true RG equation displaying reparametrisation invariance, see e.g. [131]. It introduces a renormalised regulator R_k via the back door. A quite general class of regulators R_k satisfying (2.27) is just proportional to the second derivative of the effective action w.r.t. to the fluctuation fields: $R_k = \Big(\Gamma_k^{(2)}[\phi = 0, \bar\phi] - \Gamma_k^{(2)}[0,0](p=0)\Big)\, r(y)$, where the dimensionless function $r(y)$ depends on an appropriately chosen operator y with $\mu\frac{d}{d\mu}r(y) = 0$. We write

$$R_k[\bar\phi] = \hat\Gamma_k^{(2)}[\bar\phi]\, r(y) \qquad \text{where} \qquad \hat\Gamma_k^{(2)}[\bar\phi] = \Big(\Gamma_k^{(2)}[0,\bar\phi] - \Gamma_k^{(2)}[0,0](p=0)\Big)\,. \tag{2.28}$$

The subtraction at zero momentum $p = 0$ guarantees that the prefactor is vanishing for zero momentum (and zero fields). Note that this subtraction is not necessary but sensible. This choice is not only convenient for explicit calculations but is also in accordance with physical intuition. One *uniformly* suppresses the propagating degrees of freedom in the IR regime. Such a choice should stabilise the flow. Moreover the identification of anomalous coefficients is more straightforward. The choice (2.28) implies, even within simple approximations to Γ_k, the introduction of prefactors Z_k to the regulators R_k. These Z_k essentially mimic wave function renormalisation factors. Such a choice is often used in order to improve results within the flow equation approach. Here, it was shown to bring back RG-invariance w.r.t. the RG-scaling of the full theory.

C. Perturbation theory

Finally we want to show how to recover standard perturbation theory from the flow equation. This should further illuminate the structure of the ERG flow. To that end we start at a high cut-off scale Λ, where the effective action tends to the classical one $\Gamma_{k\to\Lambda} = S_{\text{classical}}$. The latter serves as an initial condition. From there we iteratively compute the one and two loop diagrams for the scalar theory discussed in the previous sections. This analysis extends straightforwardly to theories with arbitrary field content. In order to simplify the subsequent expressions, we introduce a short-hand notation by writing $A_{pqrs\cdots} \equiv A(p,q,r,s\cdots)$ for momentum variables $p,q,r,s,\cdots$, and repeated indices correspond to a momentum integration

$$A_{qp}B_{pq'} \equiv (AB)_{qq'} = \int \frac{d^dp}{(2\pi)^d} A(q,p)\, B(p,q')\,. \tag{2.29}$$

As an example we rewrite the ERG equation (2.12) in this notation,

$$\partial_t \Gamma_k = \tfrac{1}{2} \left(\frac{1}{\Gamma_k^{(2)} + R} \right)_{pq} \partial_t R_{qp}\,, \tag{2.30}$$

where we also dropped the subscript $_k$ of R_k. For the sake of simplicity we assume R to only depend on k and not on other the scales. For the same reason we demand that R is a plain function of momentum: it neither has an implicit scale dependence nor does it depend on the couplings of the theory, in other words, it is tree level. We also have put the background field to zero and dropped the flow of the normalisation $\mathcal{N}_k$ as it is constant in this case. However, even for non-zero background field the contribution of $\partial_t \mathcal{N}_k$ decouples from the flow, as is does not depend on the field. Hence it cannot enter $\Gamma_k^{(2)}$. We stress again that all these simplifications are not necessary but are introduced for the sake of simplicity.

A simple graphical representation for (2.30) is given by Fig. 3.

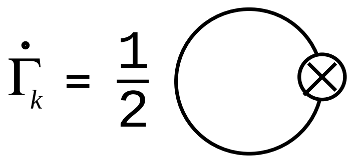

Figure 3: Graphical representation of the ERG equation (2.30).

The closed line in Fig. 3 represents the full field-dependent propagator $(\Gamma^{(2)}[\phi] + R)^{-1}$ and the crossed circle stands for the insertion $\partial_t R$. According to Fig 1, or (2.30), the ERG equation has a simple one loop structure, which should not be confused with a standard perturbative loop as it contains the full propagator. The explicit calculations are most easily carried out within the graphical representation. We introduce the graphical notation as given in Fig. 4.

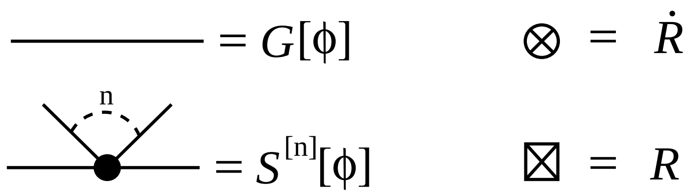

Figure 4: Graphical representation of the propagator $G[\phi]$, the (classical) n-point vertices $S^{(n)}[\phi]$, and the insertions $\partial_t R \equiv \dot{R}$ and R.

The precise expression for the propagator $G[\phi]$ in Fig. 4 depends on the flow studied. The line in Fig. 4 stands for the field dependent perturbative propagator $(S^{(2)}[\phi] + R)^{-1}$, in contrast to Fig. 3. The vertices are the classical ones, but also with full field dependence.

Now let us write the effective action within a loop expansion

$$\Gamma = S + \sum_{n=1}^{\infty} \Delta\Gamma_n, \tag{2.31}$$

where S is the classical action and $\Delta\Gamma_n$ comprises the nth loop order. At one loop, the integrated flow is

$$\Delta\Gamma_1 = \Delta\Gamma_{1,\Lambda} + \int_\Lambda^k \tfrac{dk'}{k'}\, \partial_{t'} \Gamma_{k'}|_{1-\text{loop}} = \Delta\Gamma_{1,\Lambda} + \tfrac{1}{2} \left[\ln\left(S^{(2)} + R\right)\right]_{qq} \Big|_\Lambda^k. \tag{2.32}$$

The expression on the right-hand side of (2.32) approaches the full one loop effective action for $k \to 0$. The subtraction at Λ provides the necessary UV renormalisation, together with $\Delta\Gamma_{1,\Lambda}$. The latter only encodes renormalisation effects. For the two loop calculation we also need $\Delta\Gamma_1^{(2)}$, which follows from (2.32) as

$$\Delta\Gamma^{(2)}_{1,qq'} = \tfrac{1}{2} \left(G_{pp'}\; S^{(4)}_{p'pqq'} - G_{pp'}\; S^{(3)}_{p'rq}\; G_{rr'}\; S^{(3)}_{r'pq'} \right)_\Lambda^k, \tag{2.33}$$

where

$$G_{qp} = \left(\frac{1}{S^{(2)} + R}\right)_{pq} \tag{2.34}$$

as introduced in Fig. 4. The indices q and q' in (2.33) stand for the external momenta. Thus, $\Delta\Gamma_1^{(2)}$ consists of two (subtracted) graphs. Its graphical representation is given in Fig. 5. The double lines stand for subtracted (finite) diagrams. They are introduced in Fig. 6.

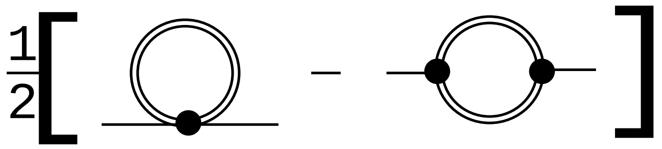

Figure 5: Graphical representation of (2.33). The subtracted diagrams (double lines) are defined in Fig. 6.

Clearly the subtraction at Λ leads to a renormalisation of the diagrams. For our purpose these terms are not interesting since they only provide the details of the renormalisation procedure. Here, however, we are only interested in the graphical structure of the perturbation series, including the combinatorial factors. For this purpose the structure of the subtractions is irrelevant. In other words, we want to focus on diagrams, which are evaluated at k even for sub-diagrams. In most results, both graphical and equations, we will only mention them implicitly.

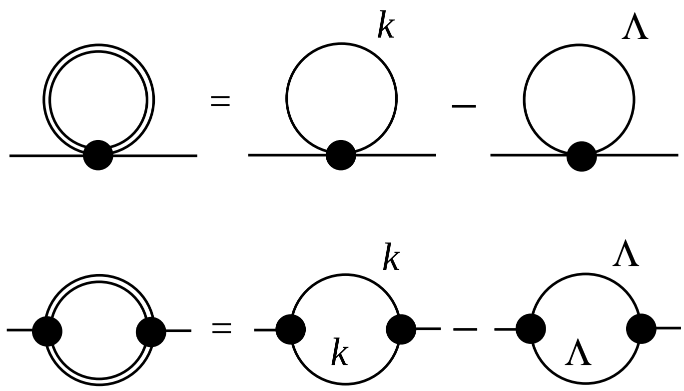

Figure 6: Graphical representation of subtracted diagrams. The scale dependence of the perturbative propagator (full line) is due to the regulator term R_k; hence the index k or Λ.

Now we extract the two loop contribution of (2.30). We notice that the only origin for loop contribution beyond one loop is $\Gamma^{(2)}$ in the denominator as R was demanded to be tree level. As the flow equation itself is one loop, two loop effects only can arise from the one loop correction to the full propagator. It is given by $(\Gamma^{(2)} + R)^{-1}|_{1-\text{loop}} = -G_k\,\Delta\Gamma_1^{(2)}\,G_k$ with G_k defined in (2.34). With these considerations the two loop contribution to the effective action follows as

$$\Delta\Gamma_2 = \tfrac{1}{2} \int_\Lambda^k \tfrac{dk'}{k'}\; \Delta\Gamma^{(2)}_{1,\; pq}\; \partial_{t'} G_{qp}\,, \tag{2.35}$$

where we also used $-G_k\,(\partial_t R)\,G_k = \partial_t G_k$. Now one uses that the only k-dependence of $\Delta\Gamma_1$ or its derivatives with respect to the fields comes from the propagators G within the loops. Graphically, $\partial_t G$ is given in Fig. 7.

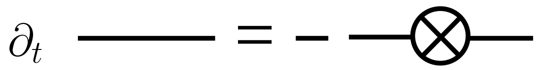

Figure 7: Graphical representation of $\partial_t G = -G\,(\partial_t R)\,G$. The k-dependence of G is only due to the explicit k-dependence of R_k.

This enables us to write (2.35) as a total t-derivative. As in the one loop case, for $k = 0$ we approach usual perturbation theory with the correct combinatorial factors. We get

$$\begin{aligned}
\Delta\Gamma_2 &= \int_\Lambda^k \tfrac{dk'}{k'} \left\{ \tfrac{1}{4} \left(G_{pp'}\; S^{(4)}_{p'pqq'} - G_{pp'}\; S^{(3)}_{p'rq}\; G_{rr'}\; S^{(3)}_{r'pq'} \right)_\Lambda^{k'} \partial_{t'} G_{q'q} \right\} \\
&= \int_\Lambda^k \tfrac{dk'}{k'}\; \tfrac{1}{4}\partial_{t'} \left\{ \tfrac{1}{2}\, G_{pp'}\; S^{(4)}_{p'pqq'}\; G_{q'q} - \tfrac{1}{3}\, G_{pp'}\; S^{(3)}_{p'rq}\; G_{rr'}\; S^{(3)}_{r'pq'}\; G_{q'q} - \text{subtractions} \right\} \\
&= \left[\tfrac{1}{8}\, G_{pp'}\; S^{(4)}_{pp'qq'}\; G_{q'q} - \tfrac{1}{12}\, G_{pp'}\; S^{(3)}_{p'qq'}\; G_{qr}\; S^{(3)}_{prr'}\; G_{r'q'} \right]_{\text{ren.}}\,,
\end{aligned} \tag{2.36}$$

where the subscript $_{\text{ren.}}$ indicates that these are renormalised diagrams due to the subtractions at Λ. Note that the sun-set diagram in (2.36) is completely symmetric under permutations of the propagators, which has led to the factor $\frac{1}{3}$; schematically written as: $(G)^2\partial_t G = \frac{1}{3}\partial_t (G)^3$. For illustration we present in Fig. 8 the diagrams for the term in curly brackets in the first line in (2.36). Employing the identity displayed in Fig. 7 the expression in Fig. 8 is easily rewritten as a total t-derivative. The calculation presented in (2.36) is most easily carried out this way.

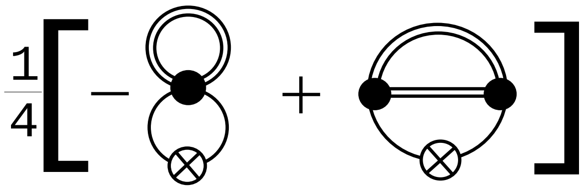

Figure 8: The integrand in curly brackets of (2.36), first line.

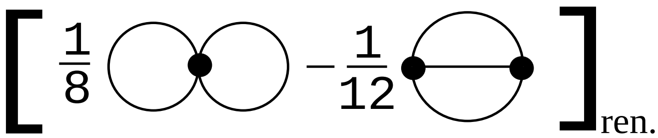

Figure 9: two loop contribution to the effective action as given by (2.36), last line.

From its systematics, this analysis can be straightforwardly extended to any loop order. The integrands can always be rewritten as total t-derivatives. Thus, the result is independent of the regulator R.

The analysis above elucidates that the flow equation offers a way for a handy book-keeping of diagrams and their combinatorial factors (for similar methods, see e.g. [90]).

D. Summary

In section II A we have derived the flow equation. We introduced an inrfa-red cut-off term into the renormalised finite Schwinger functional of the full theory, whose scale dependence was studied. Finally, the ERG flow of the infra-red regularised effective action Γ_k was given by a simple one loop equation. In contradistinction to perturbation theory, one loop means a loop in the full field dependent regulator. The flexibility of an ERG-flow, when it comes to truncations of the full problem, is related to this simple form.

In section II B the relation between the RG-scaling w.r.t the RG scale μ of the full theory and the theory in presence of the cut-off term was detailed. The cut-off term inflicts an anomaly with respect to the RG scaling of the full theory, as a general regulator R_k does not necessarily has the correct μ-scaling for rendering the cut-off term invariant. We have derived the constraint on R_k leading to invariance of Γ_k under a μ-scaling, see (2.27).

We close this chapter with a discussion of the relation between the present flow equation and other non-perturbative methods. We focus on approaches within continuum field theory.

1. Dyson-Schwinger equations

The ERG equation has a close connection to Dyson-Schwinger (DS) equations, see reviews [169,3,161]. DS equations are derived from translation invariance of the path integral measure in (2.2), that is: $[\mathcal{D}\phi]_{\rm ren} = [\mathcal{D}\phi + \delta\phi]_{\rm ren}$. This invariance can be written as

$$\int [\mathcal{D}\phi]_{\rm ren} \frac{\delta}{\delta\phi(x)} \left(\exp\left\{ -S[\phi] + \int d^d x\, J(x)\, (\phi(x) - \bar\phi(x)) \right\} \right) = \left\langle J - \frac{\delta S[\phi]}{\delta\phi(x)} \right\rangle = 0. \qquad (2.37)$$

In an interacting theory (2.37) represents a relation between the expectation values $\langle\phi^n\rangle$ and the vertices $S^{(n)}$, where schematically $S[\phi] = \sum_n 1/n! S^{(n)}\phi^n$. If $S[\phi]$ contains interaction terms with a product of at least four fields, (2.37) is at least a two loop equation in the full field-dependent propagators and vertices. Moreover, as it contains an expectation value of $\frac{\delta S[\phi]}{\delta\phi(x)}$ it also contains bare (unrenormalised) quantities. Still, apart from these differences, the form of the two equations is very similar. In [54] it was shown that ERG equations can be read as differential DS-equations. Upon integration a (quasi-fixed point) solution of (2.12) turns into a solution of (2.37). Consequently this offers a control of results obtained within either DS-equations or ERG equations. Even though the output of both agree for full solutions of the corresponding equations, this is non-trivial for approximate solutions. As a nice side-effect it is possible to transfer technical results like general vertex structures and similar properties with minor modifications from one formalism to the other. Given the vast DS-literature, in particular for non-Abelian gauge theories, this simplifies the application of the flow equation as one has not to start from scratch.

Let us also discuss the differences. To begin with, DS-equations involve loop integrations over the full momentum regime which pose considerable technical problems for both, large momenta and low momenta (for vanishing mass). The momentum integration within the flow (2.12) is peaked about $p^2 = k^2$, see also Fig. 1 and Fig. 2. The full momentum regime is included by integrating the flow. Hence, the problem of the momentum integration is turned into the problem of computing the flow. The flow has no problem with large momenta, the infrared limit is approached in a controlled way. In a recent publication DS equations on the torus have been studied. The torus furnishes an IR cut-off, thus even furthering the similarity of DS equations to ERG equations [64,65].

Furthermore, DS equations depend on both, bare quantities and dressed quantities. This necessitates a detailed discussion of the renormalisation procedure in order to maintain consistency. In flow equations, both sides depend solely on Γ_k and derivatives thereof, where Γ_k approaches the full renormalised effective action in the limit, where R_k tends to zero.

On the side of the ERG-equation, the implementation of gauge invariance is technically more involved as in DS-equations. It is here, where the above mentioned advantages exact their price. To conclude, despite the above mentioned differences, the two approaches share many features and advantages. Moreover the prospect of non-trivial consistency checks offered by results in similar approximations to the full problem at hand is intriguing.

2. One loop improved RG equations

Finally we would like to comment on equations, derived within a one loop improvement. This section comprises a brief summary of the extensive analysis of these equations done recently in [115,117,118]. We begin by reviewing the philosophy of a one loop improved renormalisation group. The starting point is the formal equation for the one loop effective action:

$$\Gamma^{1-\mathrm{loop}} = S_{\mathrm{cl}} + \tfrac{1}{2}\mathrm{Tr}\ln S^{(2)}\,. \tag{2.38}$$

The trace in (2.38) is ill-defined and requires -at least- an UV regularisation. A one loop improved RG is derived from (2.38) by first employing an explicit regularisation, taking the derivative w.r.t. the cut-off scale k and then substituting $S^{(2)}$ by $\Gamma^{(2)}$. There are various methods to achieve a regularisation of the operator trace in (2.38), for non-Abelian gauge theories [104,105]. Hence the resulting one loop improved RG equations differ qualitatively in form. All of them have a one loop form in the full propagator, as is clear from the structure of the derivation. For a summary on the popular choices and their properties we refer to [118].

One can easily show that the flow equation (2.12) can be derived within this philosophy: Adding the infrared regulator R as introduced in (2.5) to $S^{(2)}$ in (2.38) and proceeding according to the one loop improvement philosophy, we arrive at

$$\partial_t\Gamma_k = \tfrac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2)} + R\right)^{-1}\partial_t R\,. \tag{2.39}$$

As we know by now, the flow (2.39) is exact even though here we derived it from a one loop improvement. This fact has fueled hopes that other RG improved equations are also exact. A benchmark test for this hope is a perturbative analysis of these equations as done in Section II C for the flow equation. A necessary requirement for a one loop improved RG equation being exact is that standard perturbation theory is recovered in a loop expansion. In [115,117,118] it has been shown that only the ERG flow meets this constraint. The other choices popular in the literature fail to reproduce perturbation theory. Moreover, one can show that subject to some weak technical assumptions an exact RG equation has to be linear in the full propagator [118]. We conclude, that (2.12) is the most general form of a one loop exact flow.

III. GAUGE THEORIES

In this chapter we evaluate the ERG approach to non-Abelian gauge theories. We focus on general linear gauges, including algebraic ones. The gauge fixing results in non-trivial identities between Green functions, encoded in Ward-Takahashi or BRST identities. In the presence of the cut-off terms these identities get modified. We elaborate on these modifications and on the properties of various linear gauges, in particular the background field gauge and general axial gauges.

In Section III A we define the gauge fixed classical action, which enters the path integral and define some convenient notation. Cut-off terms for gauge fields and ghosts are introduced. Then, along the lines of the derivation in Chapter II the flow equation is derived.

In Section III B we discuss the mWI for the background field gauge. Physical gauge transformations and the auxiliary background gauge transformations are introduced and the related identities are derived. Then, the interpretation of these identities is evaluated. Here, the renormalisation subtleties already discussed in Chapter II become important. In general the mWI are subject to additional renormalisation. We show, how to define a gauge invariant effective action, the object of interest in the later chapters. The consequences of gauge invariance for the non-renormalisation of particular quantities is discussed. We close with the discussion of BRST invariance. As for the mWI we derive modified BRST identities and discuss their relation.

This programme is repeated in Section III C for general axial gauges. We discuss in detail, why spurious singularities are absent in ERG equations. Then, mWI and gauge invariant effective action are derived along the same line as in Section III B. The differences between the two gauges are evaluated.

In Section III D we elaborate on the implications of our findings as well as discussing alternatives.

A. Flow equation for gauge theories

The starting point is the classical action of a non-Abelian gauge theory

$$S_A[A] = \frac{1}{4}\int d^dx\, F^a_{\mu\nu}(A)F^a_{\mu\nu}(A) \quad \text{with} \quad F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu. \tag{3.1}$$

Here, $F_{\mu\nu}$ is the field strength tensor, its components are the color-electric fields $\mathbf{E}_i = F_{0i} = -F_{i0}$ and color-magnetic fields $\mathbf{B}_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. As opposed to electrodynamics already the pure gauge theory is an interacting theory due to the commutator term in F: $g f^a{}_{bc} A^b_\mu A^c_\nu$. The action (3.1) is invariant under the gauge transformation $A \to A + [D, \omega]$ with parameter ω. We resort to a general linear gauge which is given by the following gauge fixing term and the ghost action

$$S_{\rm gf}[\mathcal{Q}, \bar A] = \frac{1}{2\xi}\int d^dx\,(L^{ab}_\mu(\bar A)A^b_\mu)^2 \qquad S_{\rm gh}[\mathcal{Q}, C, \bar C, \bar A] = -\int d^dx\, \bar C^a L^{ab}_\mu(\bar A) D^{bc}_\mu(A) C^c, \tag{3.2}$$

where the linear operator L_μ possibly depends on $\bar{A}$. The gauge field fluctuation is defined as $\mathcal{Q} = A - \bar{A}$. There is no need for a ghost background field, as the ghosts are only auxiliary fields. The covariant derivative $D(A)$ is given by

$$D_\mu^{ab}(A) = \delta^{ab}\partial_\mu + g f^{acb} A_\mu^c \qquad \text{and} \qquad [t^b, t^c] = f_a{}^{bc} t^a, \quad \mathrm{tr_f}\, t^a t^b = -\frac{1}{2}\delta^{ab}, \tag{3.3}$$

where $\mathrm{tr_R}$ denotes the trace in the representation R, R = f stands for the fundamental representation, R = ad for the adjoint representation.

For the sake of brevity we introduce a more condensed notation. This notation will allow us to immediately take over the results obtained for the scalar theory in section III, in particular the discussion concerning the renormalisation in the presence of a cut-off term. We define super fields ϕ, ϕ^* and super currents J, J^* as follows:

$$\begin{aligned} \phi &= (\mathcal{Q}, C, \bar{C}), & J &= (J_\mathcal{Q}, \eta, \bar{\eta}), \\ \phi^* &= (\mathcal{Q}, \bar{C}, -C), & J^* &= (J_\mathcal{Q}, \bar{\eta}, -\eta). \end{aligned} \tag{3.4}$$

Hence, the fields ϕ consist of three component fields: $\phi_1 = \mathcal{Q}$, $\phi_2 = C$, $\phi_3 = \bar{C}$. As this notation allows us to handle all fields in a uniform manner, it makes the expressions remarkably shorter. However, it should be emphasised that this uniformity also bears some danger, as gauge fields and ghosts are of a different nature. This is already indicated by the minus sign in the definition of ϕ^*, J^* which takes care of the fermionic nature of the ghosts. Within this notation, the full gauge fixed action is given by

$$S[\phi, \bar{A}] = S_A[\bar{A} + \mathcal{Q}] + S_{\mathrm{gf}}[\mathcal{Q}, \bar{A}] + S_{\mathrm{gh}}[\phi, \bar{A}], \tag{3.5}$$

We proceed along the lines discussed in section II A for a scalar field. The renormalised Schwinger functional $W[J, \bar{A}]$ of the full theory is given by

$$\exp W[J, \bar{A}] = \int [\mathcal{D}\phi]_{\mathrm{ren}} \exp\left\{ -S[\phi, \bar{A}] + \int d^d x\, J^* \phi \right\}. \tag{3.6}$$

The source term $J\phi$ follows from the definition (3.4) as $J^*\phi = J_\mathcal{Q}\mathcal{Q} + \bar{\eta} C + \bar{C}\eta$. It follows that the expectation value of ϕ, ϕ^* is given by

$$\langle \phi_i(x) \rangle = \frac{\delta W[J, \bar{A}]}{\delta J_i^*(x)}, \qquad \langle \phi_i^*(x) \rangle = \frac{\delta W[J, \bar{A}]}{\delta J_j(x)} \left(2\delta_{j1}\delta_{1i} - \delta_{ji}\right). \tag{3.7}$$

The second equation in (3.7) entails that for the expectation values of the fermionic field in ϕ^* one needs a relative minus sign. To achieve an IR cut-off in the path integral (3.6) we need cut-off terms not only for the gauge field fluctuations $\mathcal{Q}$ but also for the auxiliary ghost fields $C, \bar{C}$.

$$\Delta S_{\mathcal{Q}}[Q,\bar{A}] = \frac{1}{2}\int d^dx\, \mathcal{Q}^b_\mu (R^{\mathcal{Q}}[\bar{A}])^{bc}_{\mu\nu} \mathcal{Q}^c_\nu, \tag{3.8a}$$

$$\Delta S_C[C,\bar{C},\bar{A}] = \int d^dx\, \bar{C}^a (R^C[\bar{A}])^{ab} C^b. \tag{3.8b}$$

As already discussed in the context of one scalar field in Section II A, the cut-off terms (3.8) lead to a modification of the propagators of the fields ϕ. Moreover, they are diagonal in field space: they do not mix different species of fields. For the sake of brevity we would like to write the cut-off terms in a uniform fashion. It is convenient to define the total regulator R_k, R_k^* in line with the vector notation of the super fields introduced in (3.4) as

$$R_k = (R^{\mathcal{Q}}, R^C, R^C) \otimes 1\!\!1_\phi \quad \text{and} \quad R_k^* = (R^{\mathcal{Q}}, -R^C, -R^C) \otimes 1\!\!1_\phi. \tag{3.9}$$

In (3.9) we have implicitly put $R^{\bar{C}} = R^C$ for writing the sum of the cut-off terms as

$$\Delta S_k[\phi,\bar{A}] = \Delta S_{\mathcal{Q}}[Q,\bar{A}] + \Delta S_C[C,\bar{C},\bar{A}] = \int d^dx\, \phi^* R_k \phi, \tag{3.10}$$

which is part of the promised uniformisation. The ghost terms on the right hand side of (3.10) combine to $\frac{1}{2}\int d^dx\, \left(\bar{C}\, R^C\, C - C\, R^C \bar{C}\right) = \int d^dx\, \bar{C}\, R^C\, C$ as the fields anti-commute and R^C should be self-adjoint. In the spirit of the parameterisation of $R_k[\bar{\phi}]$ introduced in Section II A we write

$$R^{\phi_i}[\bar{A}] = z_{\phi_i}\, r^{\phi_i}(y_{\phi_i}), \tag{3.11}$$

where the functions r obey the limits (2.6). Here, the operators z, y possibly depend on the background field $\bar{A}$. For example, the choice (2.28) in SectionII B 2, translated to the present theory, implies $z = \hat{\Gamma}^{(2)}_k[\bar{A}]$. For now, we leave the regulators unspecified. They are specified in the applications, see e.g. Section IV B 2. We continue by transferring the derivation of the flow equation for one scalar field as discussed in Section II A to the present gauge theory. Within the notation introduced above we just can read off the results obtained there. Applying the cut-off terms (3.5) to (3.6) yields $W_k[J,\bar{A}]$:

$$\exp W_k[J,\bar{A}] = \frac{1}{\mathcal{N}_k} \exp\left(-\frac{1}{2}\int d^dx\, \frac{\delta}{\delta J} R_k^*[\bar{A}] \frac{\delta}{\delta J^*}\right) \exp W[J,\bar{A}]. \tag{3.12}$$

The minus sign in the fermionic components of R^* cancels the one in the fermionic components of $\delta_J(J^*\phi)$ leading to $\delta_J(J^*\phi)\, R_k^* = \phi^*\, R_k$. The effective action Γ_k follows by a Legendre transform. As in the case of a scalar theory, we subtract the cut-off terms in order to ensure a finite UV limit. We have

$$\Gamma_k[\phi,\bar{A}] = \int d^dx\, J^*\phi - W_k[J,\bar{A}] - \Delta S_k[\phi,\bar{A}]. \tag{3.13}$$

The ERG flow of $\Gamma_k[\phi,\bar{A}]$ can be read off from (2.12) as

$$\partial_t \Gamma_k[\phi,\bar{A}] = \frac{1}{2}\mathrm{Tr}\, G_k[\phi,\bar{A}]\,\partial_t R_k[\bar{A}] + \partial_t \ln \mathcal{N}_k[\bar{A}], \tag{3.14}$$

where we assumed that the fields $\phi, \bar{A}$ are t-independent. The full field dependent propagator G_k is given by its components

$$G_{k,ij}[\phi,\bar{A}] = \left[\frac{1}{\Gamma_k^{(2)}[\phi,\bar{A}] + R_k[\bar{A}]}\right]_{ij} \quad \text{with} \quad \Gamma^{(2)}_{k,ij}[\phi,\bar{A}](x,y) = \frac{\delta^2 \Gamma_k[\phi,\bar{A}]}{\delta\phi_j^*(y)\delta\phi_i(x)}, \tag{3.15}$$

where we emphasise the order of the derivatives w.r.t. ϕ_i and ϕ_j. This order encodes the relative minus sign of fermionic loops. The trace in (3.14) denotes a sum over momenta indices and species of fields. The trace of a general operator $A^{a_i b_j}_{\mu_i \nu_j}(p,p')$ is given by

$$\mathrm{Tr}\, A = \int \frac{d^d p}{(2\pi)^d} \sum_{i,a_i,\mu_i} A^{a_i a_i}_{\mu_i \mu_i}(p,p). \tag{3.16}$$

Here, i denotes species of fields. The index a_i labels the gauge group representation of the field ϕ_i and μ_i its Lorenz group representation. We emphasise that the field derivatives in (3.15) involve anti-commuting fermionic fields. This results in a relative minus sign for fermionic loops in (3.14). Moreover, as the total regulator R_k is diagonal in field space, only the diagonal parts of $G_k[\phi,\bar{A}]$ contribute to the flow (3.14). These diagonal parts are given by

$$G_k^{\mathcal{Q}} = G_{k,11}, \qquad G_k^{C} = -G_{k,22} \qquad G_k^{\bar{C}} = -G_{k,33} = G_k^{C}, \tag{3.17}$$

where it can be easily checked that G_k^C is nothing but the full field-dependent ghost propagator. It is worth presenting a more explicit form of the right hand side of the flow equation (3.14) by using (3.17). We get $\frac{1}{2}\mathrm{Tr}\, G_k\,\partial_t R_k = \frac{1}{2}\mathrm{Tr}_{\mathcal{Q}}\, G_k^{\mathcal{Q}}\,\partial_t R_k^{\mathcal{Q}} - \mathrm{Tr}_C\, G_k^C\,\partial_t R_k^C$ in the flow equation with the correct relative minus sign and factor two for fermionic loops. Tr_{ϕ_i} stands for the corresponding sub-traces.

B. Background gauge

In the presence of a background gauge field one can employ a gauge fixing which allows the definition of a gauge invariant effective action at vanishing fluctuation fields. This is achieved by the use of a gauge fixing condition which depends on this field in such a way that the condition is invariant under a simultaneous gauge transformation of $\bar{A}$ and of the dynamical fields $\mathcal{Q}, C$ and $\bar{C}$. As the auxiliary field $\bar{A}$ is involved in this transformation it is clear that the invariance of the effective action is, a priori, an auxiliary symmetry. The essential point is that this symmetry for the special choice $\bar{A} = A$ becomes the inherent gauge symmetry of the theory.

The background field gauge is given by the choice $L_\mu(\bar{A}) = \bar{D}_\mu$ with $\bar{D} \equiv D(\bar{A})$. Then the gauge fixing term reads

$$S_{\rm gf}[\mathcal{Q}, \bar{A}] = -\frac{1}{2\xi} \int d^4x \, \mathcal{Q}^a_\mu \, \bar{D}^{ab}_\mu \bar{D}^{bc}_\nu \, \mathcal{Q}^c_\nu \tag{3.18}$$

and the corresponding ghost action is given by

$$S_{\rm gh}[\phi, \bar{A}] = - \int_x \bar{C}_a \, \bar{D}^{ac}_\mu D^{cd}_\mu \, C_d \ . \tag{3.19}$$

We now turn to the symmetries of the action in (3.5) and introduce two different gauge transformations. The first one, given by the generator $\mathfrak{g}$, gauge transforms the fields A, $C\,,\bar{C}$. It generates gauge transformations representing the underlying *physical* gauge symmetry of the theory. Thus, the transformation $\mathfrak{g}_\omega$ with parameter ω is given by

$$\mathfrak{g}_\omega(\phi, \bar{A}) = \big([D(A), \omega] \ , \ [C, \omega] \ , \ [\bar{C}, \omega] \ , \ 0\big) \, . \tag{3.20}$$

In particular, (3.20) implies that $\mathfrak{g}_\omega A = \mathfrak{g}_\omega \mathcal{Q} = [D(A), \omega]$. Hence, the gauge field A is transformed inhomogeneously as a connection, the ghosts transform as tensors according to their representation and the background field is invariant. The covariant derivative $D(A)$ transforms as a tensor

$$\mathfrak{g}_\omega D(A) = [D, \omega]. \tag{3.21}$$

The second gauge transformation, given by the generator $\bar{\mathfrak{g}}$ gauge transforms the background field $\bar{A}$ at fixed fields A, $C\,\bar{C}$:

$$\bar{\mathfrak{g}}_\omega(\phi, \bar{A}) = \big(-[D(\bar{A}), \omega] \ , \ 0 \ , \ 0 \ , \ [D(\bar{A}), \omega]\big) \, , \tag{3.22}$$

which entails $\bar{\mathfrak{g}}_\omega A = \bar{\mathfrak{g}}_\omega \mathcal{Q} + \bar{\mathfrak{g}}_\omega \bar{A} = 0$. Since $\bar{\mathfrak{g}}$ acts on $\bar{A}$ as $\mathfrak{g}$ on A it follows that the covariant derivative $\bar{D}$ transforms as a tensor as displayed in (3.21) replacing A with $\bar{A}$. $\bar{\mathfrak{g}}$ transforms the background field inhomogeneously while leaving the dynamical fields unchanged. The background gauge transformation $\bar{\mathfrak{g}}$ is introduced as an auxiliary transformation which, as it stands, does not carry any physical information.

Let us now study the action of $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ on the action S. The classical action is trivially invariant under both the gauge symmetry (3.20) and under the background gauge symmetry (3.22). In turn, neither the gauge fixing term (3.18) nor the ghost field action (3.19) is invariant under (3.20) or (3.22). Their variation under (3.20) yields

$$\mathfrak{g}_\omega S_{\rm gf} = \frac{1}{2\xi}\int d^4x\, \mathcal{Q}^a_\mu\, \left([\bar D_\mu \bar D_\nu,\, \omega]\right)^{ab}\, \mathcal{Q}^b_\nu. \tag{3.23}$$

$$\mathfrak{g}_\omega S_{\rm gh} = \int_x \bar C_a\, \left([\bar D_\mu,\, \omega]\right)^{cd} D^{cd}_\mu C_d\ . \tag{3.24}$$

Since $\bar{\mathfrak{g}}_\omega \bar D = [\bar D, \omega]$, it follows that (3.23) and (3.24) are just $-\bar{\mathfrak{g}}_\omega S_{\rm gf}$ and $-\bar{\mathfrak{g}}_\omega S_{\rm gh}$ respectively. Thus, each term in the action $S[A, C, C^*; \bar A]$ is *separately* invariant under the combined transformation $\mathfrak{g}+\bar{\mathfrak{g}}$. This brings us to a key point of the background field formalism. The invariance of $S[\phi, \bar A]$ under $\mathfrak{g}+\bar{\mathfrak{g}}$ implies that the action $\hat S[A, C, \bar C] \equiv S[\phi, \bar A = A]$ is invariant under the *physical* symmetry (3.20), $\mathfrak{g}\hat S[\phi, \bar A] = 0$, with $S[\phi, \bar A]$ satisfying the 'classical Ward-Takahashi identity' $\mathfrak{g}S = \mathfrak{g}(S_{\rm gf} + S_{\rm gh})$.

At quantum level these statements turn into gauge invariance of the effective action $\Gamma[\phi, \bar A = A]$ with $\Gamma[\phi, \bar A]$ satisfying the Ward-Takahashi identity of a non-Abelian gauge theory. Note that only the combination of both statements gives a physical meaning to gauge invariance of $\Gamma[\phi, \bar A = A]$.

In order to maintain the invariance of $S + \Delta S_k$ under the combined transformation $\mathfrak{g}+\bar{\mathfrak{g}}$ we have to ensure that both (3.8a) and (3.8b) are invariant: $(\mathfrak{g}+\bar{\mathfrak{g}})(\Delta S_Q + \Delta S_C) = 0$. We find for the action of $\mathfrak{g}$ on the regulator terms

$$\mathfrak{g}_\omega \Delta S_Q[Q, \bar A] = -\frac{1}{2}\int d^dx\, \mathcal{Q}^b_\mu\, \left([R^{\mathcal{Q}}[\bar A],\, \omega]\right)^{bc}_{\mu\nu} \mathcal{Q}^c_\nu, \tag{3.25}$$

$$\mathfrak{g}_\omega \Delta S_C[C, \bar C, \bar A] = -\int d^dx\, \bar C^a\, \left([R^C[\bar A],\, \omega]\right)^{ab} C^b. \tag{3.26}$$

Hence, for $(\mathfrak{g}+\bar{\mathfrak{g}})(\Delta S_Q + \Delta S_C) = 0$ to hold one has to require that R_k transforms as a tensor under $\bar{\mathfrak{g}}_\omega$.

$$\bar{\mathfrak{g}}_\omega R_k[\bar A] = \left[R_k[\bar A], \omega\right]. \tag{3.27}$$

With the parametrisation $R_k = z\, r(y)$ is follows that both variables z, y have to transform as tensors under $\bar{\mathfrak{g}}$:

$$\bar{\mathfrak{g}}_\omega z = [z,\, \omega], \qquad \bar{\mathfrak{g}}_\omega y = [y,\, \omega] \qquad \to \qquad (\mathfrak{g}+\bar{\mathfrak{g}})(\Delta S_Q + \Delta S_C) = 0. \tag{3.28}$$

(3.28) states that y and z should be covariant Laplaceans, for example $y_{\phi_i} = z_{\phi_i} = D^2$ in the representation of the corresponding field ϕ_i.

2. Modified and background Ward-Takahashi identities

We now turn to a detailed discussion of the Ward-Takahashi identities related to the transformations (3.20) and (3.22). Within a Wilsonian approach, the physical Green function are approached in the limit $k \to 0$, where Γ_k approaches the full quantum effective action. We have already pointed out that the statement of *physical* gauge invariance corresponds to background field gauge invariance only if $\Gamma_{k=0}$ satisfies the *usual* Ward-Takahashi identity connected to $\mathfrak{g}$. Therefore it is necessary to keep track of the action of the transformations $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ on Γ_k separately.

Ward-Takahashi identities follow from the invariance of the action S_A in the Schwinger functional under gauge transformations. The non-trivial terms in these identities stem from the non-invariance of gauge fixing term and ghost action. In the Wilsonian formalism, these identities are modified due to the presence of the regulator terms. The identity which follows from considering $\mathfrak{g}\Gamma_k$ is denoted as the *modified* Ward-Takahashi identities (mWI) [24,44,52,69,107,151,158]. A second identity is derived from the background gauge transformations $\bar{\mathfrak{g}}\Gamma_k$, leading to the *background field* Ward-Takahashi identities (bWI).

Here, we first state the mWI and bWI with help of Ward operators $\mathcal{W}_k$ and $\bar{\mathcal{W}}_k$ respectively and outline the proof below. The identities are given by

$$\mathcal{W}_k[A, \bar{A}; \omega] = 0 \tag{3.29a}$$

$$\bar{\mathcal{W}}_k[A, \bar{A}; \omega] = 0 \tag{3.29b}$$

Note, that the Ward operator $\bar{\mathcal{W}}_k$ follows from $\mathcal{W}_k$ with the observation that $\mathfrak{g}_\omega \Gamma_k[\phi, \bar{A}] = -\bar{\mathfrak{g}}_\omega \Gamma_k[\phi, \bar{A}]$. The Ward operator $\mathcal{W}_k$ is defined as

$$\begin{aligned}\mathcal{W}[\phi, \bar{A}; \omega] := \mathfrak{g}_\omega \left(\Gamma_k[\phi, \bar{A}] - S_{\rm gf}[\bar{a}, \bar{A}] - S_{\rm gh}[\phi, \bar{A}]\right) + \frac{1}{2}\mathrm{Tr}\, \omega \left[G_k[\phi, \bar{A}],\, R_k[\bar{A}]\right] \\ -\frac{1}{2}\mathrm{Tr}_{\mathcal{Q}}\, \omega \left[G_k^{\mathcal{Q}}[\phi, \bar{A}],\, \bar{D} \otimes \bar{D}\right] + \mathrm{Tr}_C \left[\bar{D}_\mu,\, \omega\right] D_\mu(A_{\rm op}) \left(G_k^C[\phi, \bar{A}] + C \otimes \bar{C}\right),\end{aligned} \tag{3.30}$$

where we have used the notation (3.17) for terms in the second line of (3.30). We also introduced the abbreviation $A_{\rm op} = A + G_{1i}\frac{\delta}{\delta \phi_i^*}$. We write more explicitly $(D \otimes D)^{ab}_{\mu\nu}(x, y) = D^{ac}_{\mu,x} D^{cb}_{\nu,x} \delta(x - y)$ and $(C \otimes \bar{C})_{ab}(x, y) = C_a(x)\bar{C}_b(y)$. The terms in the second line in (3.30) boil down to sums over momenta and indices in the representation of the corresponding fields $\mathcal{Q}$ and C respectively (see (3.17)). For example the kernel in last term in (3.30), a ghost term, carries the trivial representation of the Lorentz group and the adjoint representation of the gauge group.

For proving the Ward-Takahashi identities (3.29) we apply $\phi \to \phi + \mathfrak{g}_\omega \phi$ to the integration fields variables ϕ: This leaves W_k invariant since the path integral measure is invariant under the action of $\mathfrak{g}$ and hence $\mathfrak{g} W_k = 0$. Collecting all terms and making the Legendre transformation to Γ_k yields

$$\mathfrak{g}_\omega \Gamma_k = \mathfrak{g}_\omega \left(S_{\rm gf} + S_{\rm gh}\right) + \langle \mathfrak{g}_\omega \left(S_{\rm gf} + S_{\rm gh} \Delta S_k\right)\rangle_{\rm ren}, \tag{3.31}$$

where the expectation value $\langle \cdots \rangle_{\rm ren}$ stands for connected Green functions in the external source $J = (\delta_{\mathcal{Q}}\Gamma_k, \delta_C \Gamma_k, \delta_{\bar{C}}\Gamma_k)$. The subscript $_{\rm ren}$ refers to the renormalisation subtleties discussed before. We emphasise that in general multiplicative renormalisation might not be applicable or convenient.

In these cases additional care in the interpretation of (3.31) is required. For the time being we ignore these subtleties and proceed with the proof. Evaluating the expectation values in (3.31) by using (3.23), (3.24) and (3.25) leads to the terms in (3.30).

Now we come back to the renormalisation subtleties and their consequences for (3.30). The term proportional to R_k in the first line in (3.30) is similar to the flow itself and is derived along the same lines. It stems from commuting the generator of gauge transformations $\mathfrak{g}$ with the cut-off term as done in Section II B. The propagator G_k is defined by (3.15). The other terms stem from lifting the loop terms in the usual WT-identity from $k = 0$ to k. In general, this might inflict an additional renormalisation which cannot be absorbed in a multiplicative normalisation. Then, the terms in the second line in (3.30) require an additional renormalisation. Thus, the validity of (3.29a) with (3.30) without further renormalisation is tightly related to the renormalisation procedure applied to the full quantum effective action. In turn, demanding that (3.29a) with (3.30) is satisfied implies a particular renormalisation procedure for Γ. Concerning the terms in the second line, the mWI,bWI in (3.29) with the Ward-operator (3.30) are very similar in spirit to renormalised DS-equations. Indeed, the derivation was done for a particular translation of the fields, see Section II D 1 and [3,169].

For the rest of the discussion of (3.29a) we assume such a renormalisation of Γ. An immediate consequence of (3.29a) is that $\Gamma_k[\phi, \bar{A}]$ is invariant under the action of $\mathfrak{g} + \bar{\mathfrak{g}}$:

$$(\mathfrak{g}_\omega + \bar{\mathfrak{g}}_\omega)\, \Gamma_k[\phi, \bar{A}] = 0, \tag{3.32}$$

which originates in the invariance $(\mathfrak{g}_\omega + \bar{\mathfrak{g}}_\omega)\,(S + \Delta S_k) = 0$. The above considerations allow us to define a gauge invariant effective action in the spirit of the usual background field approach. We have

$$\Gamma_k[A] := \Gamma_k[\phi = 0, A], \qquad \mathfrak{g}_\omega \Gamma_k[A] = 0, \tag{3.33}$$

giving $\bar{A}$ the interpretation of the *physical* mean field $A = \bar{A} + \mathcal{Q}$ (thus taking $\mathcal{Q} = 0$) and setting the unphysical ghost fields to zero. Gauge invariance as displayed in (3.33) follows directly from (3.32). The flow of (3.33) is described by (3.14) for $\phi = 0$.

$$\partial_t \Gamma_k[A] = \frac{1}{2} \mathrm{Tr}\, G_k[0, A]\, \partial_t R_k[A] + \partial_t \ln \mathcal{N}_k[A] \tag{3.34}$$

Since (3.14) depends on the propagators of the fields ϕ, the ERG flow of $\Gamma_k[A]$ requires some knowledge of $\Gamma_k[\phi, A]$. One needs to know the flow (of a subset) of vertices of $\delta^2\Gamma_k[\phi, \bar{A}]/(\delta\phi)^2$ at $\phi = 0$. Still, approximations, where this difference is neglected are of some interest.

Gauge invariance of $\Gamma_{k=0}[A]$ expresses the desired physical gauge invariance. In turn, for $k \neq 0$, physical gauge invariance is encoded in the behaviour of $\Gamma_k[\phi, \bar{A}]$ under the transformation $\mathfrak{g}$. This is also evident from the fact that the flow of $\Gamma_k[\bar{A}]$ is a functional of Γ_k.

An important consequence of gauge invariance of $\Gamma_k[A]$ is the invariance of $g\bar{A}$ for a general flow:

$$\partial_s\,(g\bar{A}) = \partial_t\,(g\bar{A}) = D^\phi(g\bar{A}) = 0. \tag{3.35}$$

This is a key property to be exploited later in the applications since with its help the wave function renormalisation of the background field and the β-function are directly related. It is not surprising that this property will turn out to be helpful. Already in the usual perturbative approach with background fields it is precisely (3.35) which simplifies particular calculations tremendously.

We close this section with a comment on the finiteness of (3.30). The loop corrections in (3.30) that are proportional to R_k are finite. The loop corrections in (3.30) that are not proportional to R_k are the standard terms in the usual WI. Subject to the renormalisation procedure employed, the propagators satisfy some RG conditions and do not completely agree with G_k.

3. Symmetries of the flow and physical gauge invariance

We have seen that in the presence of the cut-off term gauge invariance gets modified. At the formal level it is clear that the original symmetry is restored when the infra-red cut-off scale is removed (see also (3.30)). A more delicate problem is to guarantee that this also happens at the level of an approximate solution to the flow equation.

To understand how gauge invariance is encoded throughout the flow, it is pivotal to also study the action of the symmetry transformations on $\partial_t\Gamma_k$ (see (3.14)). Firstly we determine how the combined transformation $\mathfrak{g}+\bar{\mathfrak{g}}$ acts on $\partial_t\Gamma$ where we only want to argue at the level of the flow equation. The flow equation (3.14) functionally depends on second derivatives of $\Gamma_k[\phi,\bar{A}]$ w.r.t. fields ϕ and on R, $\partial_t R$. Hence, we are interested on the action of $\mathfrak{g}+\bar{\mathfrak{g}}$ on these quantities. We note that

$$(\mathfrak{g}_\omega+\bar{\mathfrak{g}}_\omega)\Gamma_k^{(2)} = [\Gamma_k^{(2)},\,\omega]. \tag{3.36}$$

Here we have used (3.32) and the commutator of $\mathfrak{g}$ and two derivatives w.r.t. the fields ϕ. Eq. (3.36) states that second derivatives of Γ_k w.r.t. the fields ϕ transform as tensors under $\mathfrak{g}+\bar{\mathfrak{g}}$. Together with (3.27) this implies that the propagator G_k transform as a tensor:

$$(\mathfrak{g}_\omega+\bar{\mathfrak{g}}_\omega)G_k = [G_k,\,\omega]. \tag{3.37}$$

With (3.27) and (3.37) we conclude

$$(\mathfrak{g}+\bar{\mathfrak{g}})\partial_t\Gamma_k = 0. \tag{3.38}$$

This implies that $\mathfrak{g}\,\partial_t\Gamma_k[A] = 0$. The only input for (3.38) was the invariance of $\Gamma_k[\phi,\bar{A}]$ and the ERG equation (3.14). Thus, if the initial effective action Γ_Λ is invariant under $\mathfrak{g}+\bar{\mathfrak{g}}$ it follows that the full effective action Γ_0 satisfies $(\mathfrak{g}+\bar{\mathfrak{g}})\Gamma_0 = 0$. In other words, (3.32) and (3.38) prove that the ERG flow commutes with $(\mathfrak{g}+\bar{\mathfrak{g}})$. We conclude that $\mathfrak{g}\Gamma_{k=0}[A] = 0$ displays physical gauge invariance.

The identities (3.29) are consistent with the flow equation (3.14). With consistency, we mean the following. Assume, that a functional Γ_k at some scale k is a solution to both the mWI and the bWI. We perform an infinitesimal integration step from k to $k' = k - \Delta k$ with the flow equation. The question is, whether the functional $\Gamma_{k'}$ again satisfies the required Ward identities (3.29). As for other formulations of Wilsonian flows in gauge theories [44,52,107,111], the flow of the modified Ward-Takahashi identity is proportional to the mWI itself. This identity has the form

$$\left[\partial_t + \frac{1}{2}\mathrm{Tr}\left(G_k\,(\partial_t R_k)\,G_k\,\frac{\delta}{\delta\phi^*}\otimes\frac{\delta}{\delta\phi}\right)\right]\mathcal{W}_k[\phi,\bar{A};\omega] = 0 \tag{3.39}$$

Eq. (3.39) can be derived after some lengthy algebra by using the definition of $\mathcal{W}_k$ in (3.30) and the flow equation (3.14). Here, let us give just present the property, which leads to (3.39). As we know from the derivation of the flow equation, we are dealing with a renormalised theory, where the regulator R_k is left bare. Then, taking the derivative of loop terms w.r.t. k only hits the explicit R_k-dependence as there is no implicit one. However, this is just the same as the insertion of $-G_k(\partial_t R_k)G_k$ in all loops, which is precisely what the operator in (3.39) is doing. The whole situation is reminiscent to the derivation of the unrenormalised CS equation, e.g. [86].

4. Regulator and background field dependence

The flow (3.14) depends on the choice of the regulators R_k. This freedom can be used to optimise the flow within given approximations [112–114,116,120]. Moreover it would be clearly helpful to have an equation which controls the variation of Γ_k if one varies the regulators. Note that the flows themselves could be interpreted as a particular kind of infinitesimal variation. For a general variation δ_{R_k} one proceeds along the same lines and arrives at

$$\delta_{R_k}\Gamma_k[\phi,\bar{A}] = \frac{1}{2}\mathrm{Tr}\,\left(G_k[\phi,\bar{A}]\delta_{R_k}R_k[\bar{A}]\right). \tag{3.40}$$

Note that the variation $\delta_{R_k}R_k$ should also satisfy the conditions (2.6) for the regulator R_k. Moreover a variation should not have a more divergent IR limit than R_k itself, that is $n^\phi[\delta_{R_k}R_k] \le n^\phi[R_k]$. It is obvious that regulator terms (3.8a) with $\bar{A}$-dependent regulators introduce an additional $\bar{A}$-dependence to the effective action. Thus a particularly interesting choice for the variation is $\delta_{R_k} = \mathrm{Tr}\Big(\frac{\delta R_k}{\delta\bar{A}}\frac{\delta}{\delta R_k}\Big)$. For this choice $\delta_{R_k}\Gamma_k$ represents the derivative w.r.t. the $\bar{A}$-dependence of the cut-off term. Using (3.40) we arrive at

$$\mathrm{Tr}\Big(\frac{\delta R_k}{\delta\bar{A}}\frac{\delta}{\delta R_k}\Big)\,\Gamma_k[\phi,\bar{A}] = \frac{1}{2}\mathrm{Tr}\,\left(G_k[\phi,\bar{A}]\frac{\delta R_k[\bar{A}]}{\delta\bar{A}}\right) + \mathrm{Tr}\Big(\frac{\delta R_k}{\delta\bar{A}}\frac{\delta}{\delta R_k}\Big)\,\mathcal{N}_k[\bar{A}]. \tag{3.41}$$

The second term on the right hand side of (3.41) is at our disposal, as we are free to choose $\mathcal{N}_k[\bar{A}]$, even independently from R_k. However, we might use an R_k-dependent normalisation $\mathcal{N}_k$ of Γ_k, in which case the variation w.r.t. R_k does not vanish. Eq. (3.41) has precisely the same form as (2.14) in Section II A. There, the *only* dependence on $\bar{\phi}$ (at fixed $\phi+\bar{\phi}$) came from the cut-off term. Here, already the gauge fixing leads to a differentiation between $\mathcal{Q}$ and $\bar{A}$.

As an analogue of the mWI displayed in (3.29a) we derive an equation for the gauge transformation of the cut-off term only. We choose $\delta_{R_k} = \mathrm{Tr}\Big((\bar{\mathfrak{g}}_\omega R_k)\,\frac{\delta}{\delta R_k}\Big)$. Inserting this definition in (3.40) we get

$$\mathrm{Tr}\Big((\bar{\mathfrak{g}}_\omega R_k)\,\frac{\delta}{\delta R_k}\Big)\Gamma_k[\phi,\bar{A}] = \frac{1}{2}\mathrm{Tr}\,\omega\,\big[G_k[\phi,\bar{A}],R_k[\bar{A}]\big] + \mathrm{Tr}\Big((\bar{\mathfrak{g}}_\omega R_k)\,\frac{\delta}{\delta R_k}\Big)\mathcal{N}_k[\bar{A}], \tag{3.42}$$

where we have used (3.27). The equations (3.40),(3.41) and (3.42) can be used for consistency checks of approximations. The flow of Γ_k is usually calculated with a given Ansatz for Γ_k -apart from possible other approximations. Thus an important issue is whether such an Ansatz is consistent up to the order we are interested. Important consistency checks can be derived from evaluating the commutators below

$$\Big[(\bar{\mathfrak{g}}_\omega R_k)\,\frac{\delta}{\delta R_k},\,\partial_t\Big]\,\Gamma_k, \qquad\qquad \Big[\frac{\delta R_k}{\delta(g\bar{A})}\frac{\delta}{\delta R_k},\,\partial_t\Big]\,\Gamma_k \tag{3.43}$$

in some given approximation. In particular, (3.43) provides consistency checks on the assumed $\bar{A}$-dependence of Γ_k stemming from the regulator. Applying (3.43) to Γ_k and using (3.14),(3.41), and (3.42) lead to a set of consistency equations. Note that these consistency conditions are *not* additional flows one has to calculate but only consistency checks of the flow derived in a given approximation to (3.14).

5. BRST Symmetry

The mWI and bWI with the Ward operator (3.30) are very complicated identities as they involve loop terms. An approach, where non-local symmetry constraints are avoided, would be more advantageous. In perturbation theory, the BRST (Becchi-Rouet-Stora-Tyutin) approach to non-Abelian gauge theories provides such an option. As we shall see, also in the flow equation approach it offers some simplifications. Still, the BRST identities contain loop terms originating in the cut-off term. Here, we discuss BRST symmetries in the background field gauge. However, as it is reviewed here, it straightforwardly applies to general linear gauges. For the standard background field method in perturbative quantum field theory with some interest for the present investigation we refer to the selection [1,14,78–80,96,165] and references therein, for reviews see [12,83]. Modified BRST identities for non-Abelian gauge theories for covariant gauges have been discussed at length in the literature e.g. [24,26,27,44,52–54], for the background field, see Appendix A of [158].

We start the discussion with a brief reminder of the standard BRST formulation. It can be easily shown that the action (3.5) S is invariant under the following symmetry with generator $\mathfrak{s}$:

$$\mathfrak{s}(\phi, \bar{A}) = \Big([D(A), C]\ ,\ -\tfrac{g}{2}\{C\,, C\}\ ,\ \tfrac{1}{\xi}[\bar{D},\ \mathcal{Q}]\ ,\ 0\Big)\,. \tag{3.44}$$

In (3.44), the anti-commutator is given by $\{C\,, C\} = C^a C^b [t^a, t^b] = C^a\, C^b\, f^{abc} t^c$. The Yang-Mills action S_A is invariant as $\mathfrak{s}A$ is a gauge transformation on A. The transformation of the gauge fixing term just cancels the anti-ghost transformation of the ghost term, see e.g. [86]. The operator $\mathfrak{s}$ is nilpotent on $\mathcal{Q}, C$ leading to $\mathfrak{s}^2(\mathcal{Q}, C, \bar{C}) = \Big(0, 0, \tfrac{1}{\xi}\left[\bar{D},\ [D, \mathcal{Q}]\right]\Big)$. This is an important property of the BRST formalism, as it encodes that $\mathfrak{s}$ is a derivative on the field space and allows the use of powerful cohomological methods.[2] The nilpotency of $\mathfrak{s}$ pays off when it comes to the transformation properties of the Schwinger functional. We define

$$\exp W[J, Q, \bar{A}] = \int [\mathcal{D}\phi]_{\rm ren}\, \exp\left\{-S[\phi, \bar{A}] + \int d^d x\, (J^* \phi + Q\mathfrak{s}\phi)\right\}, \tag{3.45}$$

where we have added a source term for the BRST variations of ϕ to the action.

$$\int d^d x\, (Q\mathfrak{s}\phi) \qquad \text{with} \qquad Q = (K, L, \bar{L}), \qquad Q^* = (K, -L, -\bar{L}). \tag{3.46}$$

Thanks to the nilpotency of $\mathfrak{s}$ on $\mathcal{Q}, C$ this additional term is BRST invariant up to the term proportional to $\mathfrak{s}^2\bar{C}$. The BRST identities can be derived by applying the following shift of variables to the Schwinger functional: $\phi \to \phi + \mathfrak{s}\phi\zeta$ where $\mathfrak{s}\phi\zeta = (\mathfrak{s}\mathcal{Q}\zeta\,, \mathfrak{s}C\zeta\,, \mathfrak{s}\bar{C}\zeta)$. The variable ζ is Grassmannian ($\mathfrak{s}\phi\zeta$ has the same ghost number as ϕ). The measure $[\mathcal{D}\phi]_{\rm ren}$ is invariant under this shift[3], as is the action and we conclude that

$$\mathfrak{s}W[J, Q, \bar{A}] = \int d^d x \left(J^* \frac{\delta}{\delta Q} - \frac{1}{\xi}(\bar{D}\bar{L})\frac{\delta}{\delta K}\right) W[J, Q, \bar{A}] \equiv 0, \tag{3.47}$$

As indicated, (3.47) defines the action of $\mathfrak{s}$ on the space of currents J as a linear operation: $\mathfrak{s}f[J, Q] = \int_x (J^* \frac{\delta}{\delta Q} - \frac{1}{\xi}(\bar{D}\bar{L})\frac{\delta}{\delta K})\, f[J, Q]$. The identity (3.47) is supplemented by an equation that encodes the invariance of $W[J, Q, \bar{A}]$ under the arbitrary translation of $\bar{C} \to \bar{C} + \delta\bar{C}$:

$$\eta - \bar{D}\frac{\delta}{\delta K} W[J, Q, \bar{A}] = 0. \tag{3.48}$$

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[2]By further extending the field content $\mathfrak{s}$ can be made nilpotent on all fields, see [12].

[3]Here, we refrain from discussing the subtleties of a non-symmetric renormalisation procedure

Eq. (3.47) and (3.48) translate straightforwardly into the corresponding statement for the effective action $\Gamma[\phi, \bar{A}, Q] = \int d^dx J^*\phi - W[J, Q, \bar{A}]$. The BRST sources are spectators in the Legendre transformation which entails that $\frac{\delta W}{\delta Q} = -\frac{\delta\Gamma}{\delta Q}$. We get

$$\int d^dx \left(\frac{\delta\Gamma[\phi, \bar{A}, Q]}{\delta Q^*(x)} + \frac{1}{\xi}\bar{L}\right)\frac{\delta\Gamma[\phi, \bar{A}, Q]}{\delta\phi(x)} = 0, \tag{3.49a}$$

$$\left(\bar{D}\frac{\delta}{\delta K} + \frac{\delta}{\delta\bar{C}}\right)\Gamma[\phi, \bar{A}, Q] = 0. \tag{3.49b}$$

With the definition (3.12) we can derive a modified BRST identity (mBRST) along the same line we have derived the renormalisation group equation in Section II B. The Schwinger functional with cut-off terms was presented in (3.12). The same definition also applies in the presence of BRST sources

$$\exp W_k[J, Q, \bar{A}] = \frac{1}{\mathcal{N}_k}\exp\left(-\frac{1}{2}\int d^dx \frac{\delta}{\delta J}R_k^*[\phi]\frac{\delta}{\delta J^*}\right)\exp W[J, \bar{A}, Q] \tag{3.50}$$

with $W[J, \bar{\phi}, Q]$ defined in (3.45) and $R_k^* = (R_Q, -R_C, -R_C)$, see (3.9). As mentioned above, the BRST symmetry acts linearly on the space of currents and vanishes on $W[J, \bar{A}, Q]$, see (3.47). Thus, the anomalous term in the mBRST comes from in the commutator

$$\frac{1}{2}\left[\mathfrak{s}\ ,\ \int d^dx \frac{\delta}{\delta J}R_k^*\frac{\delta}{\delta J^*}\right] = -\frac{1}{2}\int d^dx \frac{\delta}{\delta J}R_k\frac{\delta}{\delta Q^*}, \tag{3.51}$$

By using this commutator we arrive at

$$\left(\mathfrak{s} + \frac{1}{2}\int d^dx \frac{\delta}{\delta J}R_k\frac{\delta}{\delta Q^*}\right)W_k[J, Q, \bar{A}] = 0 \tag{3.52}$$

From here we easily proceed to the effective action. The second derivative of W_k w.r.t. both, sources and fields, is given by

$$\frac{\delta^2 W_k[J, Q, \bar{A}]}{\delta J\,\delta Q^*} = \frac{\delta^2\Gamma_k[J, Q, \bar{A}]}{\delta Q\,\delta\phi}\,G_k. \tag{3.53}$$

The leads us finally to the modified BRST (mBRST) identities which we formulate similarly to the mWI (3.29a). They are given by

$$\Delta_{\Gamma_k}[\phi, \bar{A}, Q] = 0, \tag{3.54a}$$

$$\left(\bar{D}\frac{\delta}{\delta K} + \frac{\delta}{\delta\bar{C}}\right)\Gamma_k[\phi, \bar{A}, Q] = 0. \tag{3.54b}$$

where the operator Δ_{Γ_k} is defined as

$$\Delta_{\Gamma_k}[\phi,\bar A,Q] := \int d^dx\,\left(\frac{\delta\Gamma_k[\phi,\bar A,Q]}{\delta Q^*(x)}+\frac{1}{\xi}\bar L\right)\frac{\delta\Gamma_k[\phi,\bar A,Q]}{\delta\phi(x)} - \mathrm{Tr}\,R_k^*\,\frac{\delta^2\Gamma_k[J,Q,\bar A]}{\delta Q\,\delta\phi}\,G_k. \tag{3.55}$$

Again we can derive an equation for the consistency of flow and mBRST [44,52]. Note however, that consistency follows plainly by inspecting the Schwinger functional. Both, flow and $\mathfrak{s}$ act as linear operator which commute. This proves consistency. Indeed, the first proof on an algebraic level was done for the Schwinger functional [52].

Nevertheless, on the level of the effective action, consistency is a non-trivial check, in particular, when it comes to approximations of the flow. The consistency equation has the same form as those for the mWI and bWI. After some lengthy algebra we get

$$\left[\partial_t+\frac{1}{2}\mathrm{Tr}\left(G_k\,(\partial_t R_k)\,G_k\,\frac{\delta}{\delta\phi^*}\otimes\frac{\delta}{\delta\phi}\right)\right]\Delta_{\Gamma_k}[\phi,\bar A]=0. \tag{3.56}$$

We emphasise, that the equations (3.54a) and (3.56) hold true for general linear gauges. It should be also noted that in the background field approach a fully fledged analysis of renormalisation is best done within an extended BRST approach. Then, an additional symmetry of the classical action S_A is taken into account, namely that of shifting $\bar A\to\bar A+\delta A$, $\mathcal{Q}\to\mathcal{Q}-\delta A$. As this takes us too far from the main line of reasoning here, the reader is referred to the BRST-literature mentioned above.

Finally, let us compare the mBRST (3.54a) with the mWI (3.29a) based on the Ward operator (3.30). Both contain loop terms which makes devising consistent approximations a non-trivial task. Clearly the advantage of powerful cohomological methods present in the perturbative BRST analysis [12] is gone in the presence of the cut-off term. Still, the loop term proportional to the cut-off term in (3.54a) is simpler to handle than the loop terms in (3.30). Indeed, a first application of the mBRST in a numerical study was done in [53,54]. The cut-off dependent mass term was calculated with the mBRST, hence in a consistent way. As a result the flow stabilised over a larger scale regime.

C. General axial gauge

General axial gauges have enjoyed considerable attention amongst the non-covariant gauges, especially for computations in QCD at vanishing, or non-vanishing temperature, see [13,101]. The main reason for their popularity stems from the fact that the ghost sector decouples. The number of Feynman diagrams in a perturbative loop expansion is reduced, leading to an important simplification from a technical point of view. Furthermore the problem of possible Gribov copies [81], generically present in covariant gauges, is absent. The price to pay is that the (perturbative) propagator receives spurious poles, which have to be dealt with separately. The question of how to regularise the propagator as to allow for a consistent loop expansion stimulated extensive investigations. The intricacies concerning these regularisations partly spoil the advantage of having fewer diagrams to calculate. Nevertheless it has been an appealing gauge to e.g. calculate expectation values of Wilson loops which serve as order parameters for confinement. In the strong coupling limit they are expected to fulfil Wilson's area law which is correlated to a linear quark potential. The proper calculation of these expectation values may also necessitate the inclusion of topologically non-trivial configurations like instantons. Wilson loop calculations have also been used as a testing ground for the consistency of calculations in general axial gauges.

As for the background gauge, axial gauges allow for the definition of a gauge invariant effective action. The crucial differences to the background gauge formulation are the following. Firstly the ghosts decouple in the axial gauge. Secondly, for the theory without cut-off term, the effective action solely depends on the sum of fluctuation and background field: $\Gamma[\mathcal{Q},\bar{A}] = \Gamma[\mathcal{Q}+\bar{A},0]$. Gauge theories in axial gauges share this property with scalar theories, see (2.14). The only deviation from this property in the presence of the cut-off term originates in the background field dependence of the regulator. We expect, that this simplifies the control of the additional background field dependence. Moreover for $k \to 0$ the regulator tends to zero, as does the dependence on the background field. Thus, approximations in this regime based on an identification of the fluctuation $\mathcal{Q}$ and the full field $A = \bar{A} + \mathcal{Q}$ are more likely to work than in the background gauge.

1. Gauge fixing, cut-off terms and symmetries

In the flow equation approach to axial gauge Yang-Mills it is more convenient, to keep the full field $A = \bar{A} + \mathcal{Q}$ and $\bar{A}$ as independent variables. Then, the latter is truly an auxiliary variable. As it only enters the theory via the regulator $R_k[\bar{A}]$ it can be seen as an index labelling different regulators.

A general axial gauge fixing for the (fixed) Lorentz vector n_μ is given by $\mathrm{Ł}^{ab}_\mu = \delta^{ab} n_\mu$ in (3.2). Then, the gauge fixing term reads

$$S_{\rm gf} = \frac{1}{2}\int d^dx\; n_\mu A^a_\mu \,\frac{1}{\xi n^2}\, n_\nu A^a_\nu. \tag{3.57}$$

The gauge fixing parameter ξ has mass dimension -2 and may also be momentum dependent for $k \neq 0$. In particular, the case $\xi = 0$ ($\xi p^2 = -1$) is known as the axial (planar) gauge. The ghost action in axial gauges decouples and the action S (3.5) is given by

$$S[A] = S_A[A] + S_{\rm gf}[A]. \tag{3.58}$$

The price to pay for this property are additional spurious singularities in the propagator of the gauge field. The propagator $P_{\mu\nu}$ related to $S = S_A + S_{\rm gf}$ is

$$P_{\mu\nu} = \frac{\delta_{\mu\nu}}{p^2} + \frac{n^2(1+\xi p^2)}{(np)^2}\frac{p_\mu p_\nu}{p^2} - \frac{1}{p^2}\frac{(n_\mu p_\nu + n_\nu p_\mu)}{np}. \tag{3.59}$$

It displays the usual IR poles proportional to $1/p^2$. In addition, we observe additional divergences for momenta orthogonal to n_μ. These poles appear explicitly up to second order in $1/np$ and can even be of higher order for certain np-dependent choices of ξ. For the planar gauge, the spurious divergences appear only up to first order.

This artifact makes the application of perturbative techniques very cumbersome as an additional regularisation for these spurious singularities has to be introduced.

2. Absence of spurious singularities

We will show that in the presence of a cut-off term the above mentioned difficulties, present in perturbation theory, disappear. The cut-off term was introduced in (3.8a). For now, as we are interested in its impact on the momentum dependence of the propagator, we set the background field to zero $\bar{A} = 0$. For Wilsonian flows in axial gauges we drop the superscript $^{\mathcal{Q}}$ for the regulator of the gauge field as it is the only regulator and specify the regulator as

$$R^{ab}_{k,\mu\nu}(p) = \delta^{ab}\left[r(p^2)\, p^2\delta_{\mu\nu} - \tilde{r}(p^2)\, p_\mu p_\nu\right] \tag{3.60}$$

In (3.60) we did not introduce terms with tensor structure $(n_\mu p_\nu + n_\nu p_\mu)$ and $n_\mu n_\nu$. For example the interesting class of regulator (2.28), designed to keep the original renormalisation group scaling, contains such terms as they are present in $\hat{\Gamma}^{(2)}_k$. For the present purpose, the discussion of spurious singularities, (3.60) suffices. Indeed, even $\tilde{r}$ plays no rôle for the absence of spurious singularities in the flow equation approach. The only important term for the discussion of spurious singularities is that proportional to $p^2\delta_{\mu\nu}$. This term has to be present anyway, as it guarantees the suppression of all momentum modes for large cut-off. The other tensor structures are proportional to projection operators and cannot lead to a suppression of all modes. With a regulator obeying (3.60) the propagator takes the form

$$P_{k,\mu\nu} = a_1\frac{\delta_{\mu\nu}}{p^2} + a_2\frac{p_\mu p_\nu}{p^4} + a_3\frac{n_\mu p_\nu + n_\nu p_\mu}{p^2(np)} + a_4\frac{n_\mu n_\nu}{n^2p^2}, \tag{3.61}$$

with the dimensionless coefficients

$$a_1 = 1/(1+r), \qquad a_2 = (1+\tilde{r})(1+\xi p^2(1+r))/z \tag{3.62}$$

$$a_3 = -(1+\tilde{r})s^2/z, \qquad a_4 = -(r-\tilde{r})/z \tag{3.63}$$

and

$$s^2 = (np)^2/(n^2p^2) \tag{3.64}$$
$$z = (1+r)[(1+\tilde r)s^2 + (r-\tilde r)(1+p^2\xi(1+r))]. \tag{3.65}$$

Let us now evaluate the different limits in p^2 and k important for the approach. To keep things simple we restrict ourselves to the case $\tilde r = 0$. For this choice we deduce from (3.61) that $P_{k,\mu\nu}$ has the limits

$$\lim_{p^2/k^2\to\infty} P_{k,\mu\nu} = P_{\mu\nu}, \qquad \lim_{p^2/k^2\to 0} P_{k,\mu\nu} = \frac{1}{k^2}\left(\delta_{\mu\nu} + \frac{n_\mu n_\nu}{n^2}\frac{1}{1+\xi k^2}\right)\delta_{n^A 1}, \tag{3.66}$$

with $P_{\mu\nu}$ given by (3.59) and $R_k \propto (p^2)^{(1-n^A)}$ for $p^2/k^2 \to 0$ (2.6). The IR limit for $P_{k,\mu\nu}$ is only non-vanishing for regulators r with a mass-like IR behaviour, that is $n^A = 1$: $\lim_{p^2/k^2\to 0} r(p^2) = k^2/p^2$. Moreover, for $k \to \infty$ the propagator vanishes and all propagation is suppressed. By construction, the propagator (3.61) is IR finite for any $k > 0$. Now, the important observation is the following: In contrast to the perturbative propagator $P_{\mu\nu}$, the limit of $P_{k,\mu\nu}$ for $np \to 0$ is finite. This holds true even for an arbitrary choice of $\xi(p,n)$ and leads to

$$P_{k,\mu\nu} = \frac{1}{1+r}\frac{\delta_{\mu\nu}}{p^2} + \frac{1+\tilde r}{(1+r)(r-\tilde r)}\frac{p_\mu p_\nu}{p^4} - \frac{1}{(1+r)(1+p^2\xi(1+r))}\frac{n_\mu n_\nu}{n^2p^2}. \tag{3.67}$$

Thus (3.67) is perfectly well-behaved and finite for all momenta p as long as the regulators r and $\tilde r$ have not been chosen to be identical. However, in the infrared region $\tilde r$ has to be smaller than r in order to have a suppression of longitudinal modes at all. So we discard the option of identical r and $\tilde r$.

It is noteworthy that the spurious divergences are absent as soon as the infra-red behaviour of the propagator is under control. Still, for $np = 0$ and large momenta $y = p^2$ the regulator vanishes and the second term in (3.67) diverges in the limit $y \to \infty$ proportional to $y^{-1}(r-\tilde r)^{-1} > y^{d/2-1}$ following from (2.6a). Hence, even though the term only diverges for $y \to \infty$, a more careful analysis is needed for proving the finiteness of the flow equation. As a detailed proof goes beyond the scope of the present review, we just briefly discuss the necessary ingredients and the main statement. Finiteness of the flow equation is proven by deriving an upper bound for the flow following a bootstrap approach. Our starting point for the derivation of the flow equation was the existence of a renormalised *finite* Schwinger functional $W[J]$. Note that this only implies the *existence* of a renormalisation procedure for axial gauges, an explicit systematic renormalisation procedure is not required. The latter is a problem in perturbative field theory: no renormalisation procedure is know so far, which can be proven to be valid to all orders of perturbation theory. We recall the flow equation (3.14) and write it with variables $A = \mathcal{Q} + \bar A$ and $\bar A$:

$$\partial_t \Gamma_k[A,\bar A] = \frac{1}{2}\mathrm{Tr}\,\frac{1}{\Gamma_k^{(2)}[A,\bar A] + R_k[\bar A]}\,\partial_t R_k[\bar A], \tag{3.68}$$

where we can safely assume, that for large momenta the full propagator $\Gamma_k^{(2)}$ is dominated by its classical part (possibly with some multiplicative renormalisation constants). Hence for large momenta we can estimate $\Gamma_k^{(2)}(S_A^{(2)}+S_{\rm gf}^{(2]})^{-1} < C[A,\bar A]$ with $C[A,\bar A]>0$. Consequently the field independent part of the flow provides a bound on the full flow. The only terms that could produce divergences are related to the terms in (3.61) proportional to a_2 and a_3, the source for divergences being z^{-1}. The coefficient a_4 of last term in (3.61) also contains z^{-1} but also an additional factor r. Hence the limit $np\to 0$ can be safely done in this term.

We do not go into the detail of the computation but quote the result for $\tilde r=0$. Upon integrating the angular s-part of the momentum integration we get an estimate from the part of $\mathrm{Tr}\, P_k\,\partial_t R_k$ with the slowest decay for $y\to\infty$

$$\text{bound} \propto \left| \frac{1}{y_0^{d/2+1}} \int_{y_0}^{\infty} dy\, y^{d/2} \frac{\sqrt{1+y\xi}}{1+y_0\xi}\, \frac{r'(y)}{\sqrt{r(y)}} \right|, \tag{3.69}$$

The square root terms stem from an integration $\int_{-1}^{1} ds/(s^2+(1+\xi y)r(y))$, where we have introduced $y=p^2$. Since the possible problem only occurs from an integration over large momenta squared $y=p^2$, we have restricted the y-integral to $y\geq y_0$ where y_0 is up to our disposal. The bound (3.69) stems from the second term in (3.61) proportional to a_2. Eq. (3.69) is finite for regulators r that decay faster than $y^{-(d+1)}$. Without spurious singularities, r has to decay stronger than $y^{-d/2}$, see (2.6). Hence we have a mild additional constraint.

However, the bound (3.69) clearly marks the use of Callan-Symanzik (CS) type flows ($R_k\propto k^2$) as questionable in axial gauges. As already discussed, such an option requires an additional renormalisation. The presence of contributions from all momenta at every flow step makes the limit $k\to 0$ an extremely subtle one. This limit is very sensitive to the proper fine-tuning. In axial gauges, however, this problem for CS flows gets even worse by the spurious singularities. We know that a consistent renormalisation procedure in the axial gauge is certainly non-trivial. Hence, for CS type flows, one is back to the original problem of spurious singularities in perturbation theory. Moreover, one is in an even worse situation, as masses spoil the original gauge symmetry. Indeed, a recent calculation has shown [134,135], that formulations in axial gauge with a mass term for the gauge field meet problems. In [134,135], perturbative corrections to the Wilson loop have been calculated in the presence of a mass-term. The massless limit of this observable did not coincide with the well-known result. In turn, for regulators decaying faster than $y^{-(d+1)}$, the problem is cured.

3. Modified Ward-Takahashi Identities

The above analysis already hints that the control of gauge invariance in approximations might be even more important in the present approach. Here, we repeat the analysis of modified WT identities done in Section III B 2 for the background gauge. Again, the presence of the background field makes it necessary to deal with two kinds of modified Ward-Takahashi Identities. The first one is related to the requirement of gauge invariance for physical Green functions, the modified Ward Identity (mWI). The second one has to do with the presence of a background field $\bar{A}$ in the regulator term R_k, the background field Ward-Takahashi Identity (bWI).

The generator of the physical gauge transformation $\mathfrak{g}$ has been defined in (3.20), that of the background gauge transformation $\bar{\mathfrak{g}}$ in (3.22). The ghosts are missing for axial gauges and we simply have the transformations of the gauge field $\mathcal{Q}$ and the background field $\bar{A}$. As we have introduced the full field A as independent variable instead of the fluctuation $\mathcal{Q}$, we recall the action of $\mathfrak{g}, \bar{\mathfrak{g}}$ on $A, \bar{A}$:

$$\mathfrak{g}_\omega(A,\ \bar{A}) = ([D(A),\omega]\ ,\ 0) \tag{3.70a}$$

$$\mathfrak{g}_\omega(A,\ \bar{A}) = \big(0\ ,\ [D(\bar{A}),\omega]\big) \tag{3.70b}$$

The action of the gauge transformations $\mathfrak{g}_\omega$ and $\bar{\mathfrak{g}}_\omega$ on the effective action Γ_k can be computed straightforwardly. For stating the WT identities we defined the Ward operators

$$\mathcal{W}_k[A,\bar{A};\omega] \equiv \mathfrak{g}_\omega \Gamma_k[A,\bar{A}] - \mathrm{Tr}\,(n_\mu \partial_\mu \omega)\ \frac{1}{n^2\xi} n_\nu A_\nu + \frac{1}{2}\mathrm{Tr}\,\omega\,\big[G_k[A,\bar{A}], R_k[\bar{A}]\big] \tag{3.71a}$$

$$\bar{\mathcal{W}}_k[A,\bar{A};\omega] \equiv \bar{\mathfrak{g}}_\omega \Gamma_k[A,\bar{A}] - \frac{1}{2}\mathrm{Tr}\,\omega\,\big[G_k[A,\bar{A}], R_k[\bar{A}]\big]\ . \tag{3.71b}$$

In terms of (3.71), the behaviour of $\Gamma_k[A,\bar{A}]$ under the transformations $\mathfrak{g}_\omega$ and $\bar{\mathfrak{g}}_\omega$, respectively, is given by

$$\mathcal{W}_k[A,\bar{A};\omega] = 0 \tag{3.72a}$$

$$\bar{\mathcal{W}}_k[A,\bar{A};\omega] = 0 \tag{3.72b}$$

In (3.72b) we have used that $\mathfrak{g}S[A] = \mathrm{Tr}\,(n_\mu \partial_\mu \omega)\ \frac{1}{n^2\xi} n_\nu A_\nu$. Furthermore, for the validity of (3.72) it is required that the regulator function transforms as a tensor under $\mathfrak{g}_\omega$,

$$\bar{\mathfrak{g}}_\omega R_k[\bar{A}] = \big[R_k[\bar{A}],\omega\big]\ . \tag{3.73}$$

We keep the notation of the background gauge and refer to (3.72a) as the modified Ward-Takahashi identity, and to (3.72b) as the background field Ward-Takahashi identity.

It is worth emphasising an important difference between the mWI (3.72) in axial gauges and general covariant gauges, for example in the background field gauge (3.29a). Comparing the latter with (3.72) one realises that the renormalisation subtleties discussed in the context of (3.29a) are not present here. The only loop terms in (3.72) are proportional to R_k and stem from the commutators of $\mathfrak{g}, \bar{\mathfrak{g}}$ and the cut-off term in the definition of the Schwinger functional W_k. Thus there is no additional renormalisation involved as opposed to the general case in the background gauge as discussed in Section III B 2. Amongst others, this is a big advantage when devising approximations to Γ_k consistent with the mWI and bWI. This property is related to the fact that in axial gauges gauge symmetry is realised linearly on the Schwinger functional $W[J]$. Then the mWI just follows from the commutator of the cut-off terms with the representation of $\mathfrak{g}$ on the space of currents.

The consistency of the identities (3.72) follows along the same lines discussed already in Section III B, particular Section III B 5. The derivation, however, is far simpler. On the level of the Schwinger functional, as for the consistency of the mBRST, it follows from the commutator of ∂_t and $\mathfrak{g}$ on the space of currents.

The result is the same as in (3.39) without the ghosts terms, namely

$$\partial_t \mathcal{W}_k[A,\bar{A};\omega] = -\frac{1}{2}\mathrm{Tr}\left(G_k \frac{\partial R_k}{\partial t} G_k \frac{\delta}{\delta A}\otimes\frac{\delta}{\delta A}\right)\mathcal{W}_k[A,\bar{A};\omega] \tag{3.74a}$$

$$\partial_t \bar{\mathcal{W}}_k[A,\bar{A};\omega] = \frac{1}{2}\mathrm{Tr}\left(G_k \frac{\partial R_k}{\partial t} G_k \frac{\delta}{\delta A}\otimes\frac{\delta}{\delta A}\right)\bar{\mathcal{W}}_k[A,\bar{A};\omega], \tag{3.74b}$$

where $\left(\frac{\delta}{\delta A}\otimes\frac{\delta}{\delta A}\right)^{ab}_{\mu\nu}(x,y) = \frac{\delta}{\delta A^\mu_a(x)}\frac{\delta}{\delta A^\nu_b(y)}$. (3.74) states that the flow of mWI is zero if the mWI is satisfied for the initial scale. The required consistency follows from the fact that the flow is proportional to the mWI itself (3.74a), which guarantees that (3.72a) is a fixed point of (3.74a). The same follows for the bWI by using (3.74b). There is no fine-tuning involved in lifting a solution to (3.72a) to a solution to (3.72b). It also straightforwardly follows from (3.74a) and (3.74b).

Again it is interesting to compare the mWI (3.72a) with the mBRST (3.54a) as done in the background gauge. In axial gauges, both have the same anomalous term proportional to the regulator. This is due to the fact that already gauge symmetry is linearly implemented in axial gauges as already mentioned above.

4. Gauge invariant effective action

The results of the previous section permit the definition of a gauge invariant effective action by identifying $\bar{A} = A$. It is a straightforward consequence of the mWI (3.72a) and the bWI (3.72b) that the effective action $\Gamma_k[A,\bar{A}]$ is gauge invariant – up to the gauge fixing term – under the combined transformation

$$(\mathfrak{g}_\omega + \bar{\mathfrak{g}}_\omega)\Gamma_k[A,\bar{A}] = \mathrm{Tr}\, n_\mu(\partial_\mu\omega)\frac{1}{n^2\xi}n_\nu A_\nu. \tag{3.75}$$

We define the effective action $\Gamma_k[A]$ as

$$\Gamma_k[A] = \Gamma_k[A,\bar{A}=A]. \tag{3.76}$$

The action $\Gamma_k[A]$ is gauge invariant up to the gauge fixing term, to wit

$$\mathfrak{g}_\omega \Gamma_k[A] = \mathrm{Tr}\left\{ n_\mu(\partial_\mu\omega)\frac{1}{n^2\xi} n_\nu A_\nu \right\}. \tag{3.77}$$

This follows from (3.75). Because of (2.6), the effective action $\Gamma_{k=0}[A]$ is the full effective action. The flow equation for $\Gamma_k[A]$ can be read off from the basic flow equation (3.14),

$$\partial_t \Gamma_k[A] = \frac{1}{2}\mathrm{Tr}\left\{ G_k[A,A]\, \partial_t R_k[A] \right\}, \tag{3.78}$$

As for the background gauge, the right-hand side of (3.78) is not a functional of $\Gamma_k[A]$. The flow depends on the full propagator $G_k[A,A]$, which is the propagator of A in the background of $\bar{A}$ taken at $\bar{A} = A$. Approximations where this difference is neglected are of some interest. In axial gauges, this approximation becomes exact in the limit where the infrared scale k tends to zero. Hence, for the calculation of non-perturbative effects in the low energy regime using background field techniques the axial gauge is favoured.

As in the background gauge, the scale independence of gA follows from (3.77) as is well-known for axial gauges. $\Gamma_k[A]$ is gauge invariant up to the plain breaking due to the gauge fixing term. We define its gauge invariant part as

$$\Gamma_{k,\mathrm{inv}}[A] = \Gamma_k[A] - S_{\mathrm{gf}}[A] \tag{3.79a}$$

$$\mathfrak{g}_\omega \Gamma_{k,\mathrm{inv}}[A] = 0\ . \tag{3.79b}$$

(3.79) implies that the combination gA is invariant under arbitrary scalings, e.g. $k\partial_t(gA) = 0$, see also (3.35).

5. Background field dependence

By construction, the effective action $\Gamma_k[A,\bar{A}]$ at some finite scale $k \neq 0$ will depend on the background field $\bar{A}$. This dependence disappears for $k = 0$. The effective action $\Gamma_k[A]$ is the simpler object to deal with as it is gauge invariant and only depends on one field. As we have already mentioned below (3.78), its flow depends on the the propagator $\delta_A^2\Gamma_k[A,\bar{A}]$ at $A = \bar{A}$. Eventually we are interested in approximations where we substitute this propagator by $\delta_A^2\Gamma_k[A]$. The validity of such an approximation has to be controlled by an equation for the background field dependence of $\Gamma_k[A,\bar{A}]$. The flow of the background field dependence of $\Gamma_k[A,\bar{A}]$ can be derived in two ways. $\delta_{\bar{A}}\partial_t\Gamma_k$ can be derived from the flow equation (2.12),

$$\frac{\delta}{\delta\bar{A}}\partial_t\Gamma_k[A,\bar{A}] = \frac{1}{2}\frac{\delta}{\delta\bar{A}}\,\mathrm{Tr}\,\left\{ G_k[A,\bar{A}]\partial_t R_k[\bar{A}] \right\}\ . \tag{3.80}$$

The flow $\partial_t \delta_{\bar{A}} \Gamma_k$ follows the observation that the only background field dependence of Γ_k originates in the regulator. Thus, $\delta_{\bar{A}}\Gamma_k$ is derived along the same lines as the flow itself and we get

$$\partial_t \frac{\delta}{\delta \bar{A}} \Gamma_k[A, \bar{A}] = \frac{1}{2} \mathrm{Tr}\, \partial_t \left\{ G_k[A, \bar{A}] \frac{\delta}{\delta \bar{A}} R_k[\bar{A}] \right\}, \tag{3.81}$$

which turns out to be important also for the derivation of the universal one loop β-function in Sect. V B 2. The difference of (3.80) and (3.81) has to vanish

$$[\frac{\delta}{\delta \bar{A}}\, ,\, \partial_t]\, \Gamma_k[A, \bar{A}] = 0 \tag{3.82}$$

since $\partial_t \bar{A} = 0$. Eq. (3.82) combines the flow of the intrinsic $\bar{A}$-dependence of $\Gamma_k[A, \bar{A}]$ (3.81) with the $\bar{A}$-dependence of the flow equation itself (3.80). It provides a check for the validity of a given approximation. Using the right hand sides of (3.80) and (3.81) the consistency condition (3.82) can be turned into

$$\mathrm{Tr} \left\{ G_k \frac{\delta \Gamma_k^{(2)}}{\delta \bar{A}} G_k\, \partial_t R_k \right\} = \mathrm{Tr} \left\{ G_k \frac{\delta R_k}{\delta \bar{A}}\, G_k \partial_t \Gamma_k^{(2)} \right\}, \tag{3.83}$$

where

$$\Gamma_k^{(2)}[A, \bar{A}]^{\,ab}_{\mu\nu}\,(x, x') = \frac{\delta^2 \Gamma_k[A, \bar{A}]}{\delta A_a^\mu(x) \delta A_b^\nu(x')}\,. \tag{3.84}$$

With (3.83), we control the approximation

$$\left. \frac{\delta^2 \Gamma_k[A, \bar{A}]}{\delta A\, \delta A} \right|_{\bar{A}=A} = \frac{\delta^2 \hat{\Gamma}_k[A]}{\delta A\, \delta A} + \text{sub-leading terms} \tag{3.85}$$

For this approximation the flow (3.78) is closed and can be calculated without the knowledge of $\Gamma_k^{(2)}$, but with $\hat{\Gamma}_k^{(2)}$. Amongst others, the approximation (3.85) is implicitly made within proper-time flows, where the use of heat-kernel methods is even more natural [104]. This is discussed in [115]. Let us finally comment on the domain of validity for the approximation (3.85). In the infrared $k \to 0$, the dependence of the effective action $\Gamma_k[A, \bar{A}]$ on the background field $\bar{A}$ becomes irrelevant, because the regulator $R_k[\bar{A}]$ tends to zero. Therefore we can expect that (3.85) is reliable in the infrared, which is the region of interest.

D. Discussion

In this section we have presented an extensive analysis of structural aspects of the flow equation approach to gauge theories. Before we come to applications of the method in the next chapters, we would like to summarise the important results and their implications.

We have derived so called modified Ward-Takahashi and BRST identities, (3.29a), (3.54a) and (3.72a), which encode the information about gauge invariance of physical Green functions. All these identities have in common that they differ from the identities for $R_k = 0$ by a one loop term proportional to R_k. Here, one loop means a loop in the full field dependent propagator. This imposes a non-algebraic constraint upon the terms in the effective action. One can say that the simplicity of the flow which only consists of one loop terms in the full propagator, exacts its price here. The identities were shown to be maintained along the flow: Given an initial effective action Γ_{k_I} at some scale k_I obeying the modified WT or BRST identities, then Γ_k obeys these identities for all k if it evolves according to the flow equation (3.14). On a very basic level this is nothing else than the statement, that the operator ∂_t generating the flow (on the Schwinger functional) commutes with the operator generating gauge transformations or BRST transformations. For Γ_k given in some approximation to the full effective action, this consistency is a non-trivial check on the validity of the approximation employed.

The introduction of a background field $\bar{A}$ in the cut-off term is a response to this situation, as it establishes an auxiliary gauge invariance for the effective action in both, the background gauge and in axial gauges. It has to be emphasised that this procedure does not remove the problem of the non-trivial mWI and BRST identities. In this respect the situation is identical to the usual perturbative background field formulation. Moreover, as in perturbation theory, the background field simplifies the extraction of physical information from the effective action. This will be exploited later in Chapter IV and Chapter V.

1. Fixed point of gauge fixing parameter and approximate solutions

We continue our analysis of the consequences of the mWI/mBRST as stated in (3.29), (3.54a) and (3.72). We shall argue that the choice $\xi = 0$ corresponds to a fixed point of the flow.

Note, that $\xi = 0$ corresponds to a δ-function $\delta[L_\mu A_\mu]$ in the path-integral. The insertion of the cut-off term with $S \to S + \Delta S_k$ cannot change this δ-function. However, we have argued that the flow equation does not correspond precisely to this substitution but rather to (3.12). Of course it is unlikely, that this subtlety changes the above conclusion. Still ,we want to present an argument which does not rely on the path-integral representation. The argument presented here makes use only of the mWI or, alternatively, of the mBRST.

First note that ξ enters the mWI/mBRST 'implicitly' through derivatives of the action and explicitly, via $\mathfrak{g}_\omega S_{\rm gf}$ and $\mathfrak{s}^2\bar{C}$. The explicit ξ-dependence corresponds to the choice of ξ in the full theory, or, alternatively at some initial scale. The question is, whether the flow, for $\xi = 0$, can soften the singular term $\frac{1}{\xi}\int_x \mathrm{tr}(L_\mu A_\mu)^2$. Now, let us choose $\xi(\Lambda) = 0$ with Γ_Λ solving (3.29a), (3.72a) or

(3.54a) and *assume* that $\xi(k) \neq 0$ for some $k < \Lambda$. This means in particular that Γ_k will no longer contain a singular term $\sim 1/\xi$. Thus the only singular term appearing in the mWI or mBRST is the term explicitly proportional to $1/\xi$. One can always find an A^a such that $\partial\, L_\mu A_\mu$ does not vanish. This corresponds to a non-vanishing $\bar{L}$ in (3.54a). Therefore Γ_k with $\xi(k) \neq 0$ cannot be a solution of (3.72a) for $\xi(\Lambda) = 0$. However, this cannot be true as the compatibility of the flow equation and the mWI (3.74) implies that Γ_k solves the mWI. It follows that $\xi(k) = 0$ for $\xi(\Lambda) = 0$. Hence $\xi = 0$ is indeed a fixed point of the flow equation. Note that this argument does not involve any approximations regarding Γ_k.

We conclude that the $\xi = 0$ gauge is singled out and appears to be a natural choice. Furthermore, $\xi = 0$ is well-suited for actual computations as both the flow equation and the mWI are comparatively simple in that case. It is for this reason that in covariant gauges, the Landau gauge $\xi = 0$ is the preferable one, when it comes to practical computations within flow equations and also Dyson-Schwinger equations.

2. One-loop effective action

The above analysis is intimately linked to the question of the control of gauge invariance for an *approximate* solution to the flow equation. It has been already emphasised that the task of solving the flow equation for gauge theories consistent with gauge invariance is a non-trivial one: One has to find a solution of the mWI or mBRST at some initial scale $k = \Lambda$ in order to ensure gauge invariance of the physical Green function in the limit $k \to 0$. Usually this is done within some systematic expansion. However, as the solution to the flow is an approximate one, it is no longer guaranteed that mWI,mBRST are maintained during the flow. This has to be checked independently.

Here we want to illustrate this procedure within a simple example. The emphasis is on the term proportional to R_k in the mWI or mBRST. In order to avoid the discussion of the usual loop terms present already in perturbation theory we restrict ourselves to axial gauges. Then, we have to deal with the mWI (3.72a). Consider the action $\Gamma_0 = S_A + S_{\rm gf}$, where we restrict ourselves to $\xi = 0$ at the scale $k = 0$. Obviously, Γ_0 is a solution of the mWI (3.72a). The flow equation can be integrated analytically in leading order of perturbation theory, replacing Γ_k through Γ_0 on the r.h.s. of (3.14)

$$\Gamma_k = \Gamma_0 + \frac{1}{2}{\rm Tr}\ln\left(\Gamma_0^{(2)} + R_k\right) - \frac{1}{2}{\rm Tr}\ln\Gamma_0^{(2)}. \tag{3.86}$$

The mWI without the R_k-dependent term is solved by Γ_0. For the remaining terms in (3.72a) we obtain in leading order

$$\frac{1}{2}D_\mu^{ab}(x)\frac{\delta({\rm Tr}\ln(\Gamma_0^{(2)} + R_k) - {\rm Tr}\ln\Gamma_0^{(2)})}{\delta A_\mu^b(x)} - g\int d^dy\; f^{abc}R^{cd}_{k,\mu\nu}(x,y)G^{db}_{k,\nu\mu}(y,x) = 0. \tag{3.87}$$

$\Gamma_0^{(2)}$ is the second derivative of Γ_0 with respect to the gauge field. We use the commutator $[D\frac{\delta}{\delta A}, \frac{\delta^2}{(\delta A)^2}]$ and (3.72a) for Γ_0 to obtain

$$\frac{1}{2} D_\mu^{ab}(x) \frac{\delta \mathrm{Tr} \ln(\Gamma_0^{(2)} + R_k)}{\delta A_\mu^b(x)} = -g \int d^d y \; f^{abc} \Gamma_{0,\mu\nu}^{(2)cd}(x, y) \left(\frac{1}{\Gamma_0^{(2)} + R_k} \right)_{\nu\mu}^{db} (y, x). \tag{3.88}$$

For $k = 0$, (3.88) is simply zero. Using $(\Gamma_0^{(2)} + R_k)^{-1} = G_k + \mathcal{O}(g)$ and inserting (4.27) into (3.87) results in

$$-g \int d^d y \; f^{abc} \left(\Gamma_{0,\mu\nu}^{(2)cd} + R_{k,\mu\nu}^{cd} \right) (x, y) G_{k,\nu\mu}^{db}(y, x) \; \sim -g^2 f^{abc} \delta^{bc} = 0. \tag{3.89}$$

Thus we have shown that the compatibility of the flow and the mWI can be maintained even within an approximate solution. It is straightforward to show that this holds true systematically even for higher orders within a perturbative loop expansion [44].

In [52,53] the one loop compatibility of the scale dependent gluon mass parameter was checked explicitly (for covariant gauges) within the BRST formulation. This is a rather non-trivial task since the flow equation as computed directly from (3.14) or from the corresponding Slavnov-Taylor identity receives contributions from quite different diagrams (involving ghosts and gauge fields). However, in our formulation (without ghosts) the consistency check is rather simple and is done without problems for the entire effective action.

In more general situations, and especially within non-perturbative regions, it is not obvious how the compatibility between flow and mWI of a given truncation can be maintained. However, it is still possible to exploit the compatibility condition and to use it as a control mechanism for the Ansatz itself. A natural implementation would be to use the mWI as a flow equation for some of the relevant couplings (like mass terms) of the theory. Comparing the flow of these operators with the flow as derived directly from (3.14) allows one to control the domain of validity of a given truncation.[4]

For approximations beyond perturbation theory the flow equation (3.14) and the mWI (3.72a) can also be used to control the dependence on n_μ of the effective action and thus generalise the observations of [73] to the case with a cut-off term.

3. Numerical implementation

The above discussion brings us to the question of systematic approximations in numerical applications. The flow equation (2.12), the mWI (3.29a) and a suitably truncated (initial) effective action

[4]See [52,53] where a similar line of reasoning has been applied to QCD in covariant gauges.

are the starting points for numerical applications [15,53,54,76,158]. The first step is to introduce a parametrisation of the effective action in terms of some couplings γ. In an expansion in the powers of the fields, these couplings are just the momentum-dependent vertex functions. The key point is that the mWI introduces relations between the different couplings γ. Thus only a subset of couplings $\{\gamma_{\rm ind}\}$ can be independently fixed, whereas the other couplings $\gamma_{\rm dep}$ can be derived with the mWI and the set $\{\gamma_{\rm ind}\}$. It is worth mentioning that the splitting $\{\gamma\} = \{\gamma_{\rm ind}, \gamma_{\rm dep}\}$ is not unique.

Now we chose a truncation of the flow equation such that only the couplings γ related to operators important for the problem under investigation are included. For QCD, these typically include the gauge coupling, a gluonic mass term, and the 3- and 4- point (and higher) vertices. In general, only a (finite) subset of $\{\gamma_{\rm dep}(k)\}$ is taken into account. The flow equation will then be integrated as follows: After integration of an infinitesimal momentum shell between $k + \Delta k$ and k we obtain couplings $\gamma_{\rm ind}(k)$. With the mWI one derives the (finite) subset of $\{\gamma_{\rm dep}(k)\}$ which together with the $\gamma_{\rm ind}(k)$ serve as the input for the next successive integration step. By employing this procedure for the integration of the (truncated) flow equation from the initial scale k_0 to a scale k one obtains a set of $\{\gamma(k)\} = \{\gamma_{\rm ind}(k), \gamma_{\rm dep}(k)\}$ for any scale k. These $\gamma(k)$ parametrise an effective action Γ_k which by construction does satisfy the mWI at any scale k, in particular for $k = 0$ (see section III D 2). Thus the full effective action $\Gamma_{\rm trunc}$ calculated with the truncated flow equation satisfies the usual Ward identity. The truncation does not imply a breaking of gauge invariance but rather a neglecting of the back-reaction of the truncated couplings on the flow of the system.

For a validity check of the truncation we have to exploit the fact that the system is over-determined. The set $\{\gamma_{\rm dep}\}$ may be also directly calculated with the flow equation (2.12) itself. Only for the full system both equations (flow equation and mWI) are compatible, as shown in section III D 2, (3.74). Thus as long as the results for the $\gamma_{\rm dep}$, which are obtained by either using (2.12) or using (3.29a), do not deviate from each other, the truncation remains valid. The validity bound of a truncation is reached when these independently determined results for $\gamma_{\rm dep}$ no longer match. Typically, this defines a final cut-off scale $k_{\rm fin} \ll k_0$. Such a check has been done with the gluonic mass [53,54]. Even though this was only a partial consistency check –and thus not entirely satisfactory– it essentially gives the flavour of what has to be done in practice: For non-perturbative truncations the mWI is employed both for the consistency check *and* as a tool in order to calculate the value of the $\gamma_{\rm dep}$. For a fully controlled calculation one additionally has to find a suitable expansion parameter which can be employed for general validity checks of the truncation.

This leaves us with the question whether one can improve on the situation described above. Obviously devising a gauge invariant flow would rid us of the problems with non-algebraic identities. Then, the problem of the validity of a truncation boils down to the same problem one already encounters in scalar theories. There, much work has been devoted to the investigation of quality checks and optimisation of a truncation scheme [112–114,116,120].

A gauge invariant or standard gauge fixed formulation for any scale k can only be achieved by relaxing at least one of the key assumptions leading to the flow equation itself. There are essentially two options available. One either relaxes the constraints regarding the regulator function R_k, or starts with a completely different mechanism for introducing the regularisation.

We start with discussion the first option, that is to change the requirements regarding R_k. It is straightforward to observe that a necessary and sufficient condition for a standard gauge fixed formulation even during the flow is just the vanishing of the commutator $[R_k, G_k] = 0$ (see (3.29a,3.72a)). The only solution to this constraint is $R_k =\propto k^{d-2d_\phi}$ where d_ϕ is the mass dimension of the field ϕ. Then, R_k comprises mass terms proportional to the cut-off scale k. Even though this choice satisfies (2.6a) (with $n = 1$), the second condition (2.6b), which guarantees that the ultraviolet (UV) behaviour of the theory is unaltered in the presence of the cut-off term, is no longer satisfied. [166–168]

The kernel of the trace in the flow equation (2.12) is no longer peaked at momenta about k, if (2.6a) is violated. Even more so, (2.12) is not well-defined as it stands and needs some additional UV renormalisation. To be consistent, this has to be done on the level of the effective action rather than on the level of the flow equation. Otherwise the connection between the flow equation and the original –even though only formal– path integral becomes unclear. Furthermore, since this additional UV renormalisation has to be k-dependent, one may ultimately lose the 1-loop structure of the flow equation. This depends on how the actual renormalisation is done.

Moreover the interpretation of the flow equation is now completely different from the original Wilsonian idea. The flow equation no longer describes a successive integrating-out of momentum modes, but rather a flow in the space of massive theories. Even though the suppression of low momentum modes still works at every step of the flow, all parameters of the theories change for *all* momenta larger than k. This can be considered as a loss of locality, in the sense mentioned earlier. It has also to be pointed out that the result is *not* what is usually denoted by a massive gauge theory. The difference stems from the fact that the cut-off term –in the Wilsonian approach– is introduced *after* the gauge fixing has been done. In the case of a massive gauge theory the Fadeev-Popov mechanism is applied to the path integral, where the action already includes the mass term. The difference between these two approaches are those terms stemming from $\int dg \exp -k^2 \mathrm{Tr}(A^g)^2$, where

$g(x)$ is a space-time dependent gauge group element. These are exactly the terms which usually are made responsible for the breakdown of renormalisability in massive gauge theories. A simple way to see this is as follows: Introduce $\chi, g = e^{\chi}$ as a new field and do the usual power counting with respect to the fields (A, χ). Dropping these terms changes the content of the theory, and although it looks superficially like a massive gauge theory, it is *not*.

Apart from these conceptual problems, it appears that a numerical implementation is essentially out of reach. The momentum integrals involved would receive contributions from *all* momenta larger than k instead of only being peaked within a small momentum shell at about $p^2 \sim k^2$. Note that the numerical applicability of the Wilsonian flow equation may be seen as one of its most attractive features, and losing it is a stiff penalty for gaining formal gauge invariance during the flow.

It should be mentioned that a mass-like regulator remains an interesting option for a first approximative computation, consistency checks or conceptual issues, as it typically simplifies analytic calculations tremendously. For more involved and non-trivial truncations in general, one has to use more elaborate regulators, though.

A more attractive possibility is the proposal of changing the starting point of the derivation, but to stick to the 1-loop nature of the resulting flow equation. This can only be done by mapping the degrees of freedom from the original fields to another set in a non-linear way, e.g. to a representation in terms of Wilson loops [5–8,125,127,128,173]. This approach necessitates the extension of the field space from $SU(N)$ to $SU(N|N)$ and the introduction of auxiliary Pauli-Villars fields. It was shown that in the limit where the cut-off is removed, the propagating degrees of freedom are those of the original gauge theory. The corresponding flow equation has not been published yet, so a conclusive judgement is not possible. Still, it seems, that gauge invariance during the flow exacts a high price.

IV. APPLICATIONS IN THE BACKGROUND GAUGE

A. Introduction

In this chapter we discuss analytic computations within the background gauge in four dimensions $d = 4$. The methods proposed here are used for the calculation of the universal one loop. We also sketch the computation of the two loop coefficient [140,141]. These perturbative properties are well-known but provide crucial consistency checks for the method. We have explained in the last chapter that during the flow, gauge invariance is encoded in the mWI or mBRST identity. Even though for $k \to 0$ the original identities are recovered, it has to be shown that this can be achieved in a smooth way. This problem is not so much a problem of the full solution of the flow equation. However, for any practical purpose one has to rely on truncations to the full flow. Then, the discussion of the limit $k \to 0$ poses a non-trivial task. Equally important is the question of the reliability of the approximation at hand. The systematic control of approximations has alway been a hot topic in the community. Only recently some progress has been made in this direction [112–115,117,118,120], see also [106]. As it stands, a safe option for getting access to some information about the reliability is the calculation of universal properties in a given approximation. Such a computation serves as a benchmark test for the approximation. It also reveals structural information about the flow.

In gauge theories there is great need for such information. So far, the one loop β-function has been obtained within several computations, a particularly simple one using the background field approach was given in [151]. It has to be said, however, that such a computation only offers a first rough check. In the background field approach this is most clearly seen and we shall come back to this point later. The correct two loop coefficient result has not been computated yet in the ERG approach. So far, the two loop coefficient to the running coupling w.r.t. t in the ERG equation has been obtained in given approximations [76,53,151,158].[5] None of the results match the correct two loop β-function. This is not too surprising, as the approximations done in the above papers neglect contributions to the flow that are important at two loop.

In scalar theories the two loop β-function has been first achieved with ERG equations in [136]. The computations was redone in a number of subsequent publications [31,126,95,145]), where the two loop β-function served as a check for investigations on different aspects of the flow.

It is worth elaborating on the different methods. In [136,126,95] the result was achieved within a iterative solution of the flow equation. In [31,145] the correct renormalisation of perturbative graphs

[5]The closest result so far is obtained in [76], where, with a *particular choice* of R_k, one gets as close as 99% of the correct coefficient.

in the presence of the cut-off terms has been discussed prior to taking the scale derivative of these objects. The renormalisation was done in the spirit of BPHZ-subtractions. Both methods have their merits as they focus on different aspects of the approach. For practical applications of ERG flows iterative solutions reveals more informations. Within an iterative procedure one can also try to use the flexibility of the flow equation to your favour. In turn, the computation of perturbative graphs in the presence of the regulator terms complicates the usual perturbative calculation.

Below, we proceed in the spirit of an iterative solution of the flow equation. We use the flexibility of the approach in order to simplify the problem as much as possible. The explicit computations turn out to be extremely simple, the non-trivial task is the proper interpretation of the results. As a side effect we get some insight into the structure of inherent approximation inevitable in practical applications of the present background field approach. We hope that this will help future non-perturbative approximations.

We briefly discuss the approximations applied so far in applications of the ERG flow in the background gauge [76,151,158]. Firstly, the identification

$$\frac{\delta^2\Gamma_k}{(\delta\mathcal{Q})^2}[\phi=0,A] \stackrel{!}{=} \frac{\delta^2\Gamma_k[\phi,\bar{A}]}{(\delta A)^2}, \tag{4.1}$$

was made. Eq. (4.1) is almost inevitable for iterative computations. As already discussed in Section III B, the flow equation for $\Gamma_k[A]=\Gamma_k[\phi=0,\bar{A}]$ depends on the propagator of the fluctuation fields at $\phi=0$, $G^{\mathcal{Q}}[0,A]$ and $G^C_k[A]$. Hence it is not closed: Its output $\partial_t\Gamma_k[\bar{A}]$ is not the input of the flow on the right hand side. Without truncation we have to calculate the flow of $G_k[0,\bar{A}]$ for getting the result of $\partial_t\Gamma_k[A]$. As G_k couples to all higher vertices of $\Gamma_k[\phi,\bar{A}]$ at $\phi=0$, the advantage of the background field formalism is partially gone.

The truncation (4.1) simplifies the calculations as we directly determine $G^{\mathcal{Q}}_k[0,A]$ from the flow of $\Gamma_k[A]$. Then, however, it is a crucial question for the reliability of the results, to get some control of the missing terms. These terms are controlled by the mWI, bWI or mBRST derived in the last chapter.

As a second truncation the ghost sector was left trivial:

$$\Gamma_k[\phi,\bar{A}] \stackrel{!}{=} \hat{\Gamma}_k[\mathcal{Q},\bar{A}] + S_{\rm gh}[\phi,\bar{A}] \tag{4.2}$$

rendering its contribution to the flow (3.14) the perturbative one loop one. In total these two truncations (4.1) and (4.2) leave us with the task to solve a tremendously simplified flow equation. This flow equation was solved upon further truncation of the most general form of $\Gamma_k[A]$.

The truncations (4.1) and (4.2) are used for the whole flow trajectory. This implies that the full effective action has the form

$$\Gamma_0[\phi,\bar{A}] = \hat{\Gamma}_0[\mathcal{Q}+\bar{A}] + S[\phi,\bar{A}] + O(\phi^3), \tag{4.3}$$

which fails already at one loop. The two main truncations (4.1),(4.2) rely on an assumed triviality of the gauge fixing sector. This is a highly non-trivial assumption, even more so, as the background gauge, at its heart, is just a covariant gauge. It is well-known, that in the Landau gauge, DS-studies [3] support the Kugo-Ojima confinement scenario, in which the non-perturbative enhancement of the ghost propagator plays a pivotal rôle whereas the gauge field propagation is suppressed. Remarkably enough, the results of [76] of an infrared fixed point match qualitatively those of DS-studies [3], for very recent results, see also [4,103]. This is a non-trivial cross check on the result, as the truncation schemes are completely different. To conclude, these facts underline the need for a more detailed study of the background field flow equation.

The object of interest is the gauge invariant effective action $\Gamma_k[A]$ as defined in (3.33). Due to the one loop nature of the flow equation we can resort to heat kernel methods for solving the flow equation (3.14) within approximations adapted to these techniques. Such an approach mirrors a derivative expansion in scalar theories. Here, impressive results have been obtained in the flow equation approach within this expansion [16]. In gauge theories, the expansion is one in covariant derivatives.

It is our aim to relate the scaling w.r.t. the IR cut-off scale to the renormalisation group scaling the full theory. The information about the full scale dependence of Γ_k is encoded in (2.25) which we detail for gauge theories:

$$\left(\partial_s + D^\phi\right)\Gamma_k[\phi,\bar{A}] = \frac{1}{2}\mathrm{Tr}\, G_k[\phi,\bar{A}]\left(\partial_s + D^\phi + 2\gamma_\phi\right) R[\bar{A}] + \left(\partial_s + D^\phi\right)\ln\mathcal{N}_k[\bar{A}], \tag{4.4}$$

where the operator D^ϕ is given by

$$D^\phi = \gamma_g g\partial_g + \gamma_\xi \xi\partial_\xi + \sum_i \gamma_{\phi_i}\int d^4x\, \phi_i \frac{\delta}{\delta\phi_i} + \gamma_A \int d^4x\, \bar{A}\frac{\delta}{\delta\bar{A}} \tag{4.5}$$

and $\partial_s = \mu\partial_\mu + \partial_\lambda$ with $\partial_\lambda = \partial_t + \Lambda\partial_\Lambda$. The flow $\partial_\lambda\Gamma_k$ follows directly from (3.14) by substituting $k\partial_t$ with ∂_λ.

B. Consistent Approximations

1. Ansatz for $\Gamma_k[\phi,\bar{A}]$

Before going to the calculations we need to discuss the last missing ingredient of the approach, the choice of an Ansatz for the effective action $\Gamma_k[\phi,\bar{A}]$ which includes the full information (in its coefficients) relevant for the issues under investigation. The strategy we pursue is the following: we choose an initial effective action Γ_Λ which facilitates the calculation and identification of the terms on the left hand side of (4.4). At first sight it looks counter intuitive that we are simplifying the left hand side. However, whereas the explicit computation of the operator traces appear to be simple for the present application, the identification of the left hand side is non-trivial beyond one loop.

In order to consistently implement this procedure we have to discuss a subtle difference of the renormalisation programme in the present approach as opposed to the usual background field formalism used in perturbation theory. This subtlety has not been taken into account in previous approaches which is one of the reason why these attempts failed to produce correct two loop properties.

It is well-known that the only renormalisation factors Z needed in the usual background field formalism are the wave function renormalisation of the background field $\bar{A} \to Z_A^{1/2} \bar{A}$, the renormalisation of the coupling $g \to Z_g g$ and the renormalisation of the gauge fixing parameter $\xi \to Z_\xi \xi$. The wave function renormalisations $\phi_i \to Z_{\phi_i}^{1/2} \phi_i$ of the fluctuations, cancel in diagrams with only external $\bar{A}$-legs. Each ϕ protruding from a vertex is coupled to a propagator of ϕ. Thus the total product of Z_{ϕ_i}'s cancels (for a discussion of this property see e.g. [1]).

Hence one usually just drops these constants, that is $Z_{\phi_i} = 1$ even though they are *non-trivial*. However the flow equation (3.14) and the RG equation (4.4) depend on the full mean field dependent propagators of the internal degrees of freedom ϕ. The insertion of some scale derivative of R^{ϕ_i} in the diagrams in general invalidates the perturbative argument of irrelevance of Z_{ϕ_i}. Thus a calculation in this approach additionally necessitates the wave function renormalisations Z_{ϕ_i}. Note that if one wants to rescue as much as possible of the usual perturbative properties these considerations suggest to define R_k as in (4.18) proportional to $\hat{\Gamma}_k^{(2)}[0, \bar{A}]$.

We now continue with our programme by taking the above subtlety into account. Let us split the effective action as follows

$$\Gamma_k[\phi, \mathcal{Q}] = \Gamma_{k,\text{rel}}[\phi, \mathcal{Q}] + \Gamma_{k,\text{irr}}[\phi, \mathcal{Q}], \tag{4.6}$$

where $\Gamma_{k,\text{rel}}$ contains all terms with power counting relevant couplings. $\Gamma_{k,\text{irr}}$ stands for contributions which are higher order in fields and/or momenta. These terms have (power counting) irrelevant couplings. We consider the following parametrisation for $\Gamma_{k,\text{rel}}[\phi, \bar{A}]$:

$$\Gamma_{k,\text{rel}}[\phi, \bar{A}] = S[Z_\phi^{1/2}\phi,\, Z_A^{1/2}\bar{A};\, Z_g g,\, Z_\xi \xi] - \text{Tr}\,\left(a_{1,k}\bar{D}\otimes\bar{D} + a_{2,k}F[g\bar{A}] + m_k^2\right)\mathcal{Q}\otimes\mathcal{Q} + O[\phi^3] \tag{4.7}$$

where $S[\phi, \bar{A}; g, \xi] = S_A[\mathcal{Q} + \bar{A}] + S_{\text{gf}}[\mathcal{Q}, \bar{A}] + S_{\text{gh}}[\phi, \bar{A}]$. The operator trace Tr is in the fundamental representation of the gauge group and the single contributions to the classical action are defined in (3.1),(3.18) and (3.19) respectively. Note that here the factors Z_ϕ, Z_g, Z_ξ are constants evolving with the ERG flow. We introduce the anomalous dimensions w.r.t. λ:

$$\eta_{\phi_i} = \partial_\lambda \ln Z_{\phi_i}, \qquad \eta_A = \partial_\lambda \ln Z_A, \qquad \eta_g = \partial_\lambda \ln Z_g = -\tfrac{1}{2}\eta_A, \qquad \eta_\xi = \partial_\lambda \ln Z_\xi, \tag{4.8}$$

where we have used that $\partial_\lambda(g\bar{A}) = 0$, (3.35). The η are likely to be identified with the anomalous dimensions of the full theory. A priori, neither η, $\mu\partial_\mu \ln Z$, $\partial_t \ln Z$ nor combinations thereof are directly related to the anomalous dimensions γ of the full theory, as defined in (4.5). We would also like to emphasise that (4.7) is general, its structure is constrained by gauge invariance of $\Gamma_k[A]$ for all k. It follows from (3.36) that $\Gamma_{k,ii}[0,\bar{A}]$ must transform as a tensor under gauge transformations.

For a two loop calculation we only need $\Gamma^{(2)}_{\Lambda,ii}[0,\bar{A}]$ at one loop. In order to simplify our calculations and the interpretation of the left hand side of (4.4), we demand a particularly simple form of Γ_Λ. In the following we set $a_{1,\Lambda} = a_{2,\Lambda} = 0$, $m_\Lambda = 0$ and demand a vanishing $\Gamma_{\rm irr}$ at $k = \Lambda$ and at one loop:

$$a_1 = a_2 = 0, \qquad m^2_\Lambda = 0, \qquad \Gamma_{\Lambda,{\rm irr}} = 0. \tag{4.9}$$

The condition (4.9) (together with $Z = 1$) means that all propagation is suppressed at $k = \Lambda$, representing decoupling. It implies restrictions on the class of regulators r. These restrictions and compatibility can be derived from the mWI or mBRST. We defer the derivation to the next section where we specify the form of R_k. Eq. (4.9) simplifies the right hand side of (4.4) at two loop. It implies that all power counting irrelevant contributions to Γ_k vanish at $k = \Lambda$ with powers of $(\Lambda^2 - k^2)$. It follows immediately that s-derivatives of these contributions also vanish at $k = \Lambda$ ($\partial_s = \mu\frac{\partial}{\partial_\mu} + \Lambda\frac{d}{d\Lambda} + k\frac{d}{dk}$).

We conclude that for a fully consistent 2-loop calculation at $k = \Lambda$ we only have to consider (4.7) with (4.4), that is the classical action, where general multiplicative renormalisations Z_i are taken into account. The initial effective action still depends on the yet undetermined constants Z_i introduced in (4.7). Their value at Λ implicitly determines the UV renormalisation of the full theory. A natural choice would be

$$Z_i|_{k=\Lambda} = 1, \qquad m^2_\Lambda = 0, \qquad \Gamma_{\Lambda,{\rm irr}} = 0 \tag{4.10}$$

which precisely mimics the usual RG procedure for the full theory. For the computation of the right hand side of the RG-equation (3.14) we need the full propagator $G_k[0,A]$ (3.15). We first note that $\Gamma^{(2)}_k[0,A]$ as defined in (3.15) follow from (4.6) and (4.7) as

$$\Gamma^{(2)}_{k,11}[0,A] = Z_{\mathcal{Q}}\left(D_T + (1 - \frac{1}{Z_\xi \xi})D\otimes D\right) + m^2_k + O_{\mathcal{Q}}[\Gamma_{k,{\rm irr}}], \tag{4.11a}$$

$$\Gamma^{(2)}_{k,22}[0,A] = -\left(-Z_{\bar{C}}D^2 + O_C[\Gamma_{k,{\rm irr}}]\right), \tag{4.11b}$$

where we have introduced

$$D_{T\,\mu\nu}^{\ ab} = -D^{2\,ab}\delta_{\mu\nu} - 2gf^{cab}F^c_{\mu\nu}, \qquad D^{2\,ab} = D^{ac}_\rho D^{cb}_\rho. \tag{4.12}$$

Both are covariant Laplaceans, D^2 with spin zero, and D_T carries spin one. The minus sign in (4.11) originates in the definition of the second derivative of Γ_k in (3.15). It is responsible for the relative minus signs of fermions loops as mentioned before. The terms $O_\phi[\Gamma_{k,{\rm irr}}]$ only can contribute beyond two loop at $k = \Lambda$.

Let us now discuss different regulators R_k and their properties. Of course we have to satisfy the constraint (3.27) that R_k transforms as a tensor under $\bar{\mathfrak{g}}_\omega$. We also recall the property (2.26) for a regulator $R_k = z\, r(y)$ with $r = (r^{\mathcal{Q}}, r^C, r^C)$:

$$\partial_s R_k = z(\partial_s y - 2y)\, \partial_y r(y) + (\partial_s z)\, r(y). \tag{4.13}$$

for $[\partial_s y\, ,\, y] = 0$. The limits of r are

$$\lim_{\frac{y}{k^2}\to\infty} y^{d/2} r(y) = 0, \qquad\qquad \lim_{\frac{y}{k^2}\to 0} r^{\phi_i}(y_i) \propto \left(\frac{k^2}{y_i}\right)^{n^{\phi_i}}, \quad n \geq 1. \tag{4.14}$$

A common choice for a regulator r^{ϕ_i} with $n^{\phi_i} = 1$ is

$$r^{\phi_i}(y) = \left(\exp\frac{y}{k^2} - \exp\frac{y}{\Lambda^2}\right)^{-1}, \tag{4.15}$$

It is easy to check that the choice (4.15) satisfies the limits (4.14). It also has the property (4.13). Even more important is the tensor structure of the regulators. A straightforward choice for the R_k is $R_{k,0}$ with

$$R_0^{\mathcal{Q}}[\bar{A}] = \left(\bar{D}_T + (1 - \frac{1}{\xi})\bar{D} \otimes \bar{D}\right) r^{\mathcal{Q}}(\bar{D}_T), \qquad R_0^C[\bar{A}] = \bar{D}^2\, r^C(\bar{D}^2). \tag{4.16}$$

For $\xi = 1$ the regulator $R_0^{\mathcal{Q}}$ behaves like a mass for small momenta, for $\xi \neq 1$ it has a non-trivial tensor structure. We have already shown in Section II B 1, that for the choice (4.16), the RG equation for Γ_k is not homogeneous due to $(\mu\partial_\mu + D^\phi + 2\gamma_\phi)R_{k,0} = 2\gamma_\phi R_{k,0} \neq 0$ (see (2.23)). This was to be expected. As we want to calculate anomalous dimensions, we would like to decouple UV-scaling and IR-scaling. As argued in Section II B 2, this is achieved for

$$(\mu\partial_\mu + D^\phi + 2\gamma_\phi)R_k = 0, \tag{4.17}$$

which leads to $(\mu\partial_\mu + D^\phi)\Gamma_k = 0$, see (2.23). Then, the following regulator R_k uniformly suppresses the propagating degrees of freedom (transversal and longitudinal modes) in the IR regime:

$$R^{\mathcal{Q}}[\bar{A}] = \hat{\Gamma}_{k,11}[\bar{A}]\, r^{\mathcal{Q}}(\bar{D}_T), \qquad R^C[\bar{A}] = -\hat{\Gamma}_{k,22}[\bar{A}]\, r^C(-\bar{D}^2), \tag{4.18}$$

where

$$\hat{\Gamma}_k^{(2)}[\bar{A}] = \left(\Gamma_k^{(2)}[0, \bar{A}] - \Gamma_k^{(2)}[0, 0](p = 0)\right). \tag{4.19}$$

The class of regulators in (4.18) leads to an uniform suppression of the propagating modes, as it is proportional to $\Gamma_k^{(2)}$. Moreover, the identification of anomalous coefficients in the present calculation at $k = \Lambda$ is more straightforward.

Now, we come back to the determination of the one loop mass m_k with the mWI/mBRST . This has been considered in [52] for covariant gauges. For the present purpose the background field plays no rôle and the results of [52] can be used directly. We need the regulator R_k at tree level, where the classes (4.16) and (4.18) agree. The mass is given by

$$m_k^2 = g^2 \frac{N}{16\pi^2} \int dy\, y^{d/2-2} \frac{r(y)}{(1+r(y))^2} \left(\frac{11}{2} - d - \frac{5}{d} + \xi(1-\frac{1}{d}) + \left(\frac{7}{2} - \frac{6}{d}\right) \frac{y\partial_y r}{1+r(y)} \right). \qquad (4.20)$$

For a general regulator $r(y)$ and all k this mass is in general non-zero. It has been shown in [52], that for a standard choice of $r = (\exp y/k^2 - 1)^{-1}$ the mass is proportional to $(1-\xi)$ and vanishes for $\xi = 1$. For our purpose we demand that the regulator r has the limit

$$\lim_{k\to\Lambda} r(y) = \frac{\Delta_k}{y} \exp\left\{-\frac{y}{\Delta_k}\right\} + O(\Delta_k^0) \exp\left\{-\frac{y}{\Lambda^2}\right\} \qquad \text{with} \qquad \Delta_k = \frac{k^2\Lambda^2}{\Lambda^2 - k^2}. \qquad (4.21)$$

Eq. (4.21) is the constraint that the cut-off should rapidly approach a diverging mass. The exponentials ensure the UV properties of the regulator. Inserting (4.21) into (4.20) leads to a vanishing mass for $\xi = 1$:

$$m_\Lambda^2(\xi = 1) = 0. \qquad (4.22)$$

This is not surprising as for the choice $\xi = 1$ the plain propagator has a simple tensor structure $p^2\delta_{\mu\nu}$. Note that (4.20) expresses the necessity of matching m_Λ with $m_0 = 0$ for the full theory. This implies that the integrated flow contributing to the mass has to vanish, which in general would require an oscillating R (see e.g. [142]).

3. Covariantly constant fields and heat kernels

For the calculation of the propagators related to (4.11) we restrict ourselves to fields A with covariantly constant field strength similarly to [53,76,151,158]. These configurations have the following properties.

$$[D_\mu, F_{\nu\rho}] = 0, \qquad (4.23a)$$

$$[D^2, D_\mu] = -2gF_{\mu\rho}D_\rho, \qquad (4.23b)$$

$$D_{T,\mu\rho}D_\rho = -D_\mu D^2. \qquad (4.23c)$$

Furthermore we take regulators of the form (4.18). For these regulators we get from (4.11) with (4.23) that

$$G^{\mathcal{Q}}[0,A] = \frac{1}{Z_{\mathcal{Q}}} \frac{1}{D_T(1+r^{\mathcal{Q}}(D_T)) + m_k^2} \left(1\!\!1 + D \otimes \frac{(1-Z_\xi \xi)(1+r^{\mathcal{Q}}(-D^2))}{-D^2(1+r^{\mathcal{Q}}(-D^2) + Z_\xi \xi\, m_k^2} \otimes D \right) + O^{\mathcal{Q}}_{\text{irr}}, \quad (4.24a)$$

$$G^C[0,A] = -\frac{1}{Z_{\bar{C}}} \frac{1}{1+r^C(D^2)} \frac{1}{D^2} + O^C_{\text{irr}}, \quad (4.24b)$$

where $O^{\mathcal{Q}}_{\text{irr}}, O^C_{\text{irr}}$ stem from $\Gamma_{k,\text{irr}}$ and vanish at $k = \lambda$. For the subsequent calculation we use heat kernel techniques. However, it is worth emphasising that the heat kernel is only used as a technical tool and *not* for the regularisation of infinities since the flow equation is finite. We will need the heat-kernel of the closely related operators D^2 and $D_T = -D^2 - 2gF$. Let us remind the reader of the definition of the heat-kernel as $K_{\mathcal{O}}(\tau) = \exp\{\tau\mathcal{O}\}(x,x)$

$$K_{D^2}(\tau) = \int \frac{d^4p}{(2\pi)^4} e^{\tau X_\mu X_\mu}, \qquad K_{-D_T}(\tau) = e^{2\tau g F} K_{D^2}(\tau), \quad (4.25)$$

where $X_\mu = ip_\mu + D_\mu$. Here we have used that $2gF$ commutes with X_μ for covariantly constant fields. Both kernels are tensors in the Lie algebra and K_{-D_T} is a non-trivial Lorentz tensor because of the prefactor. For the calculation of the momentum integral we just refer the reader to the literature (e.g. [121]) and quote the result for covariantly constant field strength

$$K_{D^2}(\tau) = \frac{1}{16\pi^2\tau^2} \det \left[\frac{\tau g F}{\sinh \tau g F} \right]^{1/2}, \quad (4.26)$$

where the determinant is performed only with respect to the Lorentz indices. $K_{-D_T}(\tau)$ follows by multiplication of (4.26) with $\exp 2\tau g F_{\mu\nu}$. For our present purpose we only need $K(\tau)$ up to order F^2 (equivalently order τ^0). Expanding K_{D^2} in $\tau g F$ we get

$$K_{D^2}(\tau) = \frac{1}{16\pi^2} \left(\frac{1}{\tau^2} - \frac{1}{12} g^2 (F^2)_{\rho\rho} \right) + O[\tau, (gF)^3]. \quad (4.27)$$

With (4.27) and the expansion $\exp 2\tau g F_{\mu\nu} = (1 + 2\tau g F + 2\tau^2 g^2 (F^2)_{\mu\nu}) + O[\tau, (gF)^3]$ we read off the coefficient of the $K(\tau)$ proportional to F^2.

$$\text{Tr}_{\mathcal{Q}} K_{D^2} \simeq -\frac{1}{16\pi^2} \frac{4}{3} N g^2 S_A[A], \quad (4.28a)$$

$$\text{Tr}_{\mathcal{Q}} K_{-D_T} \simeq \frac{1}{16\pi^2} \frac{20}{3} N g^2 S_A[A] \quad (4.28b)$$

We have used that $S_A[A] = \frac{1}{2}\int \text{tr}_{\text{f}} F^2$. Since the operators D_T and D^2 carry the adjoint representation the trace Tr includes tr_{ad} with $2N \text{tr}_{\text{f}}\, t^a t^b = \text{tr}_{\text{ad}}\, t^a t^b$. The factor 4 in (4.28a) comes from the summation over Lorentz indices in the trace $\text{Tr}_{\mathcal{Q}}$. For the ghosts, living in the trivial representation of the Lorentz group we have $\text{Tr}_C K_{D^2} = \frac{1}{4} \text{Tr}_{\mathcal{Q}} K_{D^2}$.

C. One loop

1. Generalities

We start with a calculation of the one loop contributions to the RG-equation (4.4). This helps to clarify some of the conceptual points discussed in the previous sections, in particular those that are also important for the two loop calculation. Additionally it is a warm-up for the two loop calculation. At one loop the right hand side of the ERG equation (3.14) (for ∂_t substituted by ∂_λ) and of the scale equation (4.4) are identical (for $\mu\partial_\mu R_k = 0 + O(g)$). Moreover it is not necessary to restrict ourselves to $k = \Lambda$ for a fully consistent calculation. This is so because all contributions related to $Z \neq 1$, $\alpha \neq 0$, $m_k \neq 0$ and $\Gamma_{k,\mathrm{irr}} \neq 0$ (see (4.6) and (4.7)) in (4.24) can only contribute at higher loops. Thus the right hand side of the RG-equation (4.4) consists of traces of operators dependent either solely on D_T or on D^2. Now we insert (4.24) into (4.4) and use that for covariantly constant fields $D_\mu \left(D_T r^{\bar a}(D_T)\right)_{\mu\nu} D_\nu = \left(-D^2 r^{\bar a}(-D^2)\right)(D^2)$. This property follows from the last equation in (4.23). These considerations lead to

$$\left[\left(\partial_s + D^\phi\right)\Gamma_k[A] - \partial_s \ln \mathcal{N}_k\right]_{1-\mathrm{loop}} = \frac{1}{2}\mathrm{Tr}_{\mathcal{Q}}\frac{\partial_s r^{\mathcal{Q}}(D_T)}{1+r^{\mathcal{Q}}(D_T)} - \mathrm{Tr}_C\frac{\partial_s r^C(-D^2)}{1+r^C(-D^2)}. \tag{4.29}$$

The right hand side of (4.29) can be straightforwardly calculated. The left hand side, however, bears some intricacies.

Before we proceed with the calculation in the next section, let us describe what goes wrong with a naive identification of terms in (4.29): Naively one would just take a trivial $\mathcal{N}_k$, as it certainly is an irrelevant term put in by hand. Then, $D^\phi\Gamma_k[A]|_{1-\mathrm{loop}} = 2\gamma_A S_A[A] = -2\gamma_g\, S_A[A]$, which can be easily read off from the parametrisation (4.6),(4.7) restricted to the 1-loop effective action. Furthermore, if one goes to the limit $k \to \Lambda$, one concludes that

$$\partial_s\Gamma_k[A]|_{1-\mathrm{loop}} = 0, \tag{4.30}$$

in particular there is no term proportional to S_A. All terms with positive momentum dimension vanish with powers of $(\Lambda^2 - k^2)$ for $k \to \Lambda$. For the terms with dimensionless coefficients as S_A (4.30) follows if implicit scale dependences (at tree level) are missing. Usually this is the case. Thus, the coefficient of S_A only depends on ratios of the explicit scales μ, Λ, k. This implies that the term proportional to S_A on the right hand side of (4.29) is directly related to the one loop β-function. It turns out, however, that the right hand side depends on n^{ϕ_i} as defined in (4.14). Only for $n^{\phi_i} = 1$ we get the correct expression. We shall see, that the expectation that $\partial_s\Gamma_k|_{1-\mathrm{loop}} = 0$ is wrong in general, as the cut-off term introduces an additional field dependence leading to an implicit scaling for $n^{\phi_i} \neq 1$. The additional dependence on the background field $\bar A$ is controlled by (3.41). The derivative of (3.41) w.r.t. μ vanishes at one loop for general R_k with $\mu\partial_\mu R^\phi = 0+O(g)$. This is already a strong hint, that for regulators leading to non-vanishing (3.81) we have introduced an *implicit* s-dependence even at one loop. Note that it is not a proof yet as also the other field dependent terms could scale with s.

2. One loop β-function

Now we proceed with the calculation. The explicit expression on the right hand side of (4.29) is evaluated with help of the heat kernel expressions (4.28). We notice that the flow equation (4.29) is well-defined in both the IR and the UV region. This allows us to take advantage of the following fact: Given the existence (convergence, no poles) of the Taylor expansion of a function $f(z)$ about $z = 0$ we can use the representation

$$f(-\mathcal{O}) = f(-\partial_\tau)\exp\{\tau\mathcal{O}\}|_{\tau=0} \tag{4.31}$$

Due to the infrared regulator the terms in the flow equation (4.29) have this property, where $\mathcal{O} = D_T, D^2$. Hence we can rewrite the arguments D_T and $-D^2$ in (4.29) as derivatives w.r.t. τ of the corresponding heat kernels $K_{-D_T}(\tau)$ and $K_{D^2}(\tau)$. Applying this to the flow equation (4.29) we arrive at

$$\begin{aligned}\partial_s\left(\Gamma_k[A] - \ln\mathcal{N}_k\right) - 2\gamma_g\, S_A[A] &= \left[\frac{1}{2}\frac{\partial_s r^{\mathcal{Q}}(-\partial_\tau)}{1+r^{\mathcal{Q}}(-\partial_\tau)}\mathrm{Tr}_{\mathcal{Q}}K_{-D_T}(\tau) - \frac{\partial_s r^C[-\partial_\tau]}{1+r^C(-\partial_\tau)}\mathrm{Tr}_C K_{D^2}(\tau)\right]_{\tau=0}\\ &= \left(10n^{\mathcal{Q}} + n^C\right)\frac{N}{24\pi^2}g^2 S_A[A] + O[(gF)^3],\end{aligned} \tag{4.32}$$

where it is understood that (4.32) is only valid at one loop. We also have used that $D^\phi\Gamma_k[A]|_{1-\mathrm{loop}} = 2\gamma_A S_A[A] = -2\gamma_g\, S_A[A]$. As already mentioned in section IV B 3, the heat kernels are only used as a technical tool. The expression (4.29) is finite as is (4.32) without necessity of a renormalisation. Indeed we could have calculated the traces in (4.29) directly with methods similar to those used in [121]. However, it is of course more convenient to use existing results. Note also the right hand side of (4.32) is just the t-flow as

$$d_t\left(\Gamma_k[A] - \mathcal{N}_k\right)_{1-\mathrm{loop}} = \left(10n^{\mathcal{Q}} + n^C\right)\frac{N}{24\pi^2}g^2 S_A[A] + O[(gF)^3] + O[(gF)^3] \tag{4.33}$$

for all k and expanded about vanishing D^2. This statement comes from the fact that (4.32) is valid for general k. Hence, (4.33) leads to $d_t \ln Z_A = \left(10n^{\mathcal{Q}} + n^C\right)\frac{N}{24\pi^2}g^2$. Only for $\left(10n^{\mathcal{Q}} + n^C\right) = 11$ the t-scaling agrees with $\gamma_A = -\gamma_g$. Of course regulators with a mass-like IR-limit ($n^{\phi_i} = 1$) satisfy this condition.

For solving this puzzle we take a closer look at the one loop terms in the effective action. At one loop the flow equation can be easily integrated (see Section II C) and we have

$$\Gamma_k^{1-\text{loop}}[0,A] = \tfrac{1}{2}\text{Tr}\ln\left(S^{(2)}[0,A] + R[A]\right) - \tau^*\Gamma_0^{1-\text{loop}}[0,A], \tag{4.34}$$

where $\tau^*\Gamma_0^{1\text{loop}}$ stands for the subtraction terms in the renormalised effective action at one loop with renormalisation scale μ. Here τ^* refers to the subtraction operator.[6] The renormalised effective action at $k=0$ would be $(1-\tau^*)\Gamma_0$, see e.g. [40,175]. The particulars of the subtraction operator τ^* are not important at one loop. Only beyond one loop they come into play.

The crucial ingredient is that the terms proportional to field derivatives of $S^{(2)}$ in the first term on the right hand side are made finite by the subtraction. At one loop the subtractions are a power series in external momenta. Thus, their contributions proportional to S_A trivially depend on ratios of the scales μ, k, Λ and logarithms thereof. No terms proportional to $\ln p^2/\text{scale}^2$ are present. Thus the s-derivative of these terms vanish. In turn, terms proportional to A-derivatives of $R_k[A]$ have no counterpart in $\tau^*\Gamma_0^{1\text{loop}}[0,A]$. In general, they are responsible for an implicit scale dependence. This scale dependence is controlled by (3.41). Below we shall calculate its contribution.

In case we want to rescue the absence of implicit scaling at $k=\Lambda$ (4.30), the normalisation $\mathcal{N}_k[\bar A]$ has to cancel the non-zero contribution encoded in (3.41). This is achieved by demanding that

$$\partial_s \text{Tr}\Big(\frac{\delta R_k}{\delta A}\frac{\delta}{\delta R_k}\Big)\,\Gamma_k[0,A] = 0 \tag{4.35}$$

for $k\to\Lambda$. Eq. (4.35) can be rewritten as a defining equation for $\mathcal{N}_k$:

$$\partial_s \frac{\delta}{\delta A}\ln\mathcal{N}_k[A] = -\frac{1}{2}\text{Tr}\,\partial_s\left(G_k[0,A]\frac{\delta}{\delta A}R_k[A]\right), \tag{4.36}$$

where it is implied that the only dependence of $\mathcal{N}_k$ on A comes from $R_k[A]$. Note that at higher loop order and $k\neq\Lambda$ (4.36) nearly inevitably contains cross terms that mix the $\bar A$-dependence of R_k and of $\Gamma_k^{(2)}$ in the flow. It is left to calculate $\ln\mathcal{N}_k$ as defined in (4.36). After some algebra it follows that

$$\begin{aligned}-\frac{1}{2}\text{Tr}\,\partial_s\left(G_k[0,A]\,\frac{\delta}{\delta \bar A}R_k[A]\right) = &-\Bigg[\frac{1}{2}\partial_s\left(\frac{[-\partial_\tau r^{\mathcal{Q}}(-\partial_\tau)]'}{1+r^{\mathcal{Q}}(-\partial_\tau)}\frac{1}{\partial_\tau}\right)\frac{1}{\tau}\frac{\delta}{\delta A}\text{Tr}_{\mathcal{Q}}K_{-D_T}(\tau)\\ &-\partial_s\left(\frac{[-\partial_\tau r^{C}(-\partial_\tau)]'}{1+r^{C}(-\partial_\tau)}\frac{1}{\partial_\tau}\right)\frac{1}{\tau}\frac{\delta}{\delta A}\text{Tr}_C K_{D^2}(\tau)\Bigg]_{\tau=0},\end{aligned} \tag{4.37}$$

where $(-\partial_\tau r[-\partial_\tau])'$ denotes the derivative of the expression in the brackets w.r.t. the argument $-\partial_\tau$. The computation of (4.37) is done within a few technical steps. We rely on the representation $\tau^{-1} = \int dz\, \exp -\tau\, z$ which leads to

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[6] Usually, this operation is called $R^* = (1-\tau^*)$, which could be confused with the regulator.

$$f(-\partial_\tau)\frac{1}{\tau} = \int_0^\infty dz\, f(z), \tag{4.38}$$

if $f(z)$ decays fast enough for $z \to \infty$. For our problem (4.37), f decays fast enough. Moreover, for the integration we also can use the properties of r, when hit with an s-derivative (4.13). For general regulators r we have $\partial_s r(x) = -2\partial_x\, r(x) + O(g^2)$. This results in

$$-\left[\partial_s\left(\frac{[-\partial_\tau\, r(-\partial_\tau)]'}{1+r(-\partial_\tau)}\frac{1}{\partial_\tau}\right)\frac{1}{\tau}\right]_{\tau=0} = \left[\frac{\partial_s r(x)}{1+r(x)} + 2\frac{1}{1+r(x)}\right]_0^\infty = 2-2n. \tag{4.39}$$

Eq. (4.39) vanishes for $n^{\phi_i}=1$. In this case $\mathcal{N}_k$ is trivial. Thus for regulators R_k with mass-like IR limit the analysis boils down to the naive one as already mentioned before. Indeed, for $n=1$, (4.32) gives the correct result for $\partial_s\Gamma_k = 0$ and trivial $\mathcal{N}_k$. The general $\mathcal{N}_k$ leading to $\partial_s\Gamma_k|_{1-\rm loop} = 0$ at $k=\Lambda$ follows from (4.37) and (4.39) as

$$(\partial_s \frac{\delta}{\delta\bar A}\ln\mathcal{N}_k)[A] = -\left(10(n^{\mathcal{Q}}-1)+(n^C-1)\right)\frac{N}{24\pi^2}g^2\frac{\delta}{\delta\bar A}S_A[A] + \frac{\delta}{\delta\bar A}O[(gF)^3]. \tag{4.40}$$

Inserting (4.40) into (4.32) we arrive at the correct result for the one loop β-function for general regulator R_k with general n^{ϕ_i}:

$$\gamma_g = -\frac{11N}{48\pi^2}g^2. \tag{4.41}$$

Note that due to the non-trivial $\mathcal{N}_k$ we have $d_t\Gamma_k[A]|_{1-\rm loop} = 2\gamma_A S_A[A] + O[(gF)^3]$ for arbitrary n^ϕ. This is a trivial consequence of (4.30).

This somewhat extensive analysis of the one loop β-function was presented for several reasons. First of all, it gives an impression of how the method evaluated here can be used. Its flexibility allows for a variety of convenient short-cuts. Note, that the above subtleties are not seen for standard choices of cut-offs ($n^{\phi_i}=1$). Nevertheless they are important properties, even more so if one discusses universal properties of the underlying theory. If one just would have evaluated the flow w.r.t. k (or t) without paying attention to the connection between UV and IR flow one would have just ended up with the disquieting result that the anomalous dimension η_A is n^{ϕ_i}-dependent and not universal. With the naive identification of IR and UV flow at one loop one would have stayed without explanation for this result. In turn, only the full analysis presented here enables us to extract universality properties in the full theory at $k=0$ from the t- or λ-scaling at non-zero k. However, it should be also mentioned, that partially these intricacies are a peculiarity of the background field approach. They are, at this level, related to the additional dependence on the background field introduced via the regulator. Still, the occurrence of additional implicit scalings introduced via the regulator is by no means restricted to the background field approach. Indeed, it is related to non-trivial renormalisation factors of terms quadratic in the field. Due to the background field this shows up in a more stringent way. Related problems in scalar theories have been discussed in [9,10]. There, translated into the language here, a field dependend cut-off is used. On the level of a one loop flow, this directly corresponds to a background field.

D. Conclusions

Let us close with a summary and outlook. We have discussed the relation between the RG-equation in the full theory and the flow equation at general k. This enabled us to relate the anomalous dimensions in the full theory to the Wilsonian anomalous dimensions. For arbitrary scales these coefficients can be quite different. There are, however, flow regions were their relation is simple. The first to mention is the scaling region of the theory where the different scalings have to match. Beyond one loop, however, full knowledge of the momentum dependence of the coefficients of the relevant operators is necessary to calculate the (universal) anomalous dimensions. This requires either a quite complicated calculation within the flow equation approach or one has to rely on perturbative loop calculations with regularised propagators. Both programmes require an input which contains more information than necessary for the problem under investigation.

The work done above pays off at two loop [140,141]. The computation itself is simple and is done along the same lines as the one loop calculation. The hard work is the interpretation of the result. The right hand side of (4.4) depends on the chosen regulator. For the relation of t-scaling and μ scaling the operator τ^* becomes relavant. Simply put, the implementation of mass independent RG conditions becomes non-trivial. We hope to report on this matter in near future.

Important for the present purpose is the following conceptional issue: the choice of the appropriate renormalisation related to the quantum fluctuations. This part of the renormalisation plays an important rôle in the flow equation approach as opposed to the usual perturbative background field approach [1]. This is originated in the modification of gauge symmetry during the flow. Note that this does *not* mean that we have changed the theory. Indeed we have shown that only the renormalisation procedure or scheme is changed in comparison to the usual perturbative treatment. Only by properly taking this change into account we are able to compute the two loop coefficient.

A second important issue concerns the background field dependence introduced via the cut-off. This dependence is controlled with equation (3.81). The use of this equation is twofold. It allows consistency checks of the approximations under investigation, thus closing conceptual gaps present in the background field approach to ERG flows as presented in earlier calculations. Moreover its use in the definition of the *physical* part of the flow is pivotal for the calculation. Without (3.81) no statement about universality can be made in the background field approach to ERG flows even at one loop. On the contrary it has been shown that for regulators with $n^\phi \neq 1$ (3.81) is even necessary to reach the correct result. In the light of the present contribution the choice of regulators and the gauge fixing parameter in [151,158,76] –apart from the fact that the necessary renormalisation of the quantum fluctuations is neglected– minimises the unphysical part of the flow equation.

If one aims at more general results (e.g. the full effective action within a non-trivial truncation) (3.81) certainly is non-vanishing. However the form of these consistency equation is similar to the flow equation itself and can be tackled with the same methods. Moreover the *integrated* background field dependence related to the cut-off vanishes at $k = 0$. Thus, if one is only interested in the final result at $k = 0$ one may as well start with $\mathcal{N}_k = 1$. Note that this does not change the result but only changes the flow trajectory.

The results of the next chapter will allow us to extend the current computations to the full effective action. There, we compute the one loop effective action for axial gauges. The analysis can be directly translated into the background gauge.

V. APPLICATIONS IN AXIAL GAUGES

A. Introduction

The computations of the last chapter have revealed some of the intricacies of the flow equation approach in the background gauge. The requirement of further investigations and improved truncation schemes could be bypassed or supplemented by having an independent check of the results obtained there. Flow equations in axial gauges provide such an independent check apart from having some advantages concerning the approximations necessary for practical computations with background fields. We have already discussed in SectionIII C, that in contradistinction to the background gauge, the only obstacle for the approximation (4.1) is the background field dependence of the cut-off term, see Section III C 5. Hence the approximation (4.1) is getting better the deeper in the infrared region we are. This property makes the axial gauge preferable for the use of background field methods as the approximation (4.1) is difficult to circumvent without loosing the advantages of the background field formulation of ERG flows. The second approximation discussed in the last chapter (4.2) is exact for axial gauges, as the ghosts decouple.

In the present chapter we develop analytical methods to study flow equations for gauge theories in general axial gauges based on the heat kernel results of section IV B 3. Even at one loop we shall see interesting differences which support the picture painted above. We compute the scale dependent one loop effective action and discuss the implications of the results, in particular in view of non-perturbative approximations put upon the present result. Furthermore possible expansion schemes are outlined.

1. Propagator for covariantly constant fields

For the computation of the flow (3.14) in a general axial gauge we restrict ourselves to specific field configurations, similar to the procedure in the background field gauge, see section IV B 3. Here, however, we have introduced an additional tensor structure with the gauge vector n_μ. In order to use simple properties for covariant derivatives and the field strength we further restrict our configurations in comparison to section IV B 3. We start with field configurations with covariantly constant field strength, namely $D_\mu F_{\nu\rho} = 0$. Then, we also demand that $n_\mu A^\mu = 0 = n_\mu F_{\mu\nu}$, that is, the gauge field component in direction of the gauge fixing vector vanishes. This means that even for $\xi \neq 0$ we only use gauge fields obeying the gauge. A prominent well-investigated example for such a gauge is the Weyl gauge: $A_0 = 0$. That this further constraint is not too strong is seen by the explicit example of $n_\mu = \delta_{\mu 0}$ and $(A_\mu) = (A_0 = 0, A_i(\vec{x}))$, where A_i is a three-dimensional field with covariantly constant field strength. The constraints are summarised in

$$[D_\mu, F_{\nu\rho}] = 0, \tag{5.1a}$$

$$n_\mu A_\mu = 0, \tag{5.1b}$$

$$n_\mu F_{\mu\nu} = 0. \tag{5.1c}$$

To keep finiteness of the action of such configurations we have to go to a theory on a finite volume. The volume dependence will drop out in the final expressions and we smoothly can take the limit of infinite volume.[7] For the configurations satisfying (5.1) we get

$$[D^2,\, D_\mu] = -2gF_{\mu\rho}D_\rho, \tag{5.2a}$$

$$D_{T,\mu\rho}D_\rho = -D_\mu D^2\ , \tag{5.2b}$$

$$[n_\mu D_\mu,\, D_\nu] = n_\mu F_{\mu\nu} = 0, \tag{5.2c}$$

where we quoted (4.23) and used (5.1c) for (5.2c). Defining the projectors P_n and P_D with

$$P_{n,\mu\nu} = \frac{n_\mu n_\nu}{n^2}, \tag{5.3a}$$

$$P_{D,\mu\nu} = D_\mu \frac{1}{D^2} D_\nu, \tag{5.3b}$$

we establish that

$$P_D D_T = -P_D D^2 P_D, \qquad P_n D_T = -P_n D^2 \tag{5.4}$$

holds true. After these preliminary considerations we consider the gauge-fixed classical action given in (3.1). We need the propagator on tree level to obtain the traces at one loop. The initial action reads

$$\Gamma_\Lambda[A] = S_A + S_{\rm gf}\ . \tag{5.5}$$

As opposed to the discussion in the background field gauge (4.18) we restrict ourselves to the following tensor structure

$$R_k[\bar{A}] = \bar{D}_T\, r(\bar{D}_T) \tag{5.6}$$

with the yet unspecified function r satisfying (4.14). From (5.5) and (5.6) we derive the full inverse propagator as

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[7] it should be mentioned, that the Weyl gauge for finite time direction has its problems. Only variants as the Polyakov gauge, is properly defined if some basic properties are demanded, see e.g. [66,102,123]

$$\Gamma^{(2)ab}_{k,\mu\nu}[A,A] = \left(D^{ab}_{T,\mu\nu} + (D_\mu D_\nu)^{ab} + \frac{1}{\xi n^2} n_\mu n_\nu \delta^{ab} \right) + O(g^2; D_T, D_\mu D_\nu) \ . \tag{5.7}$$

The inverse propagator (5.7) is an operator in the adjoint representation of the gauge group. We now turn to the computation of the propagator (3.15) for covariantly constant fields. Using (5.7), (5.1) and (5.2), we find

$$G_k[A,A]^{ab}_{\mu\nu} = \left(\frac{a_1}{D_T}\right)_{\mu\nu} - D_\mu \frac{a_2}{D^4} D_\nu - n_\mu \frac{a_3}{D^2(nD)} D_\nu - D_\mu \frac{a_3}{D^2(nD)} n_\nu - n_\mu \frac{a_4}{n^2 D^2} n_\nu \ , \tag{5.8}$$

with the dimensionless coefficient functions

$$a_1 = \frac{1}{1+r_T} \ , \tag{5.9a}$$

$$a_2 = \frac{1-\xi D^2(1+r_D)}{(1+r_D)} \left(s^2 + r_D[1 - D^2\xi(1+r_D)]\right)^{-1} \ , \tag{5.9b}$$

$$a_3 = -\frac{s^2}{(1+r_D)} \left(s^2 + r_D[1 - D^2\xi(1+r_D)]\right)^{-1} \ , \tag{5.9c}$$

$$a_4 = -\frac{r_D}{(1+r_D)} \left(s^2 + r_D[1 - D^2\xi(1+r_D)]\right)^{-1} \ . \tag{5.9d}$$

Notice that a_1 is a function of D_T while a_2, a_3 and a_4 are functions of both D^2 and $(nD)^2$. We also introduced the convenient short-hand notation

$$r_T \equiv r(D_T), \qquad r_D \equiv r(-D^2), \qquad s^2 \equiv \frac{(nD)^2}{(n^2 D^2)}. \tag{5.10}$$

The regulator, as introduced in (5.6), depends on D_T. The dependence on D^2, as apparent in the terms a_2, a_3 and a_4, comes into play due to the conditions (5.1) and (5.2). They imply

$$r(D_T) D_\mu = D_\mu r(-D^2), \qquad r(D_T) n_\mu = n_\mu r(-D^2) \ , \tag{5.11}$$

which can be shown term by term for a Taylor expansion of r_k about vanishing argument. For vanishing field $A = 0$ the propagator (5.8) reduces to the one already discussed in [107]. There, it has been shown that the regularised propagator (5.8) (for $r \neq 0$) is not plagued by the spurious propagator singularities as encountered within standard perturbation theory, and in the absence of a regulator term ($r = 0$). For the axial gauge limit $\xi = 0$ the expression (5.8) simplifies considerably. With (5.7) and (5.10) we get

$$\begin{aligned} G_{k,\mu\nu}[A,A]] =& \left(\frac{1}{D_T(1+r_T)}\right)_{\mu\nu} - D_\mu \frac{1}{D^4(1+r_D)(s^2+r_D)} D_\nu + \frac{n_\mu}{n^2} \frac{nD}{D^4(1+r_D)(s^2+r_D)} D_\nu \\ &+ D_\mu \frac{nD}{D^4(1+r_D)(s^2+r_D)} \frac{n_\nu}{n^2} + \frac{r_D}{D^2(1+r_D)(s^2+r_D)} P_{n,\mu\nu} \ . \end{aligned} \tag{5.12}$$

The propagators (5.8) and (5.12) are at the basis for the following computations. Notice that this analysis straightforwardly extends to approximations for $\Gamma_k[A,\bar{A}]$ beyond the one-loop level. Indeed, it works for any $\Gamma_k[A,\bar{A}]$ such that $\Gamma^{(2)}_{k,\mu\nu}[A,A]$ is of the form

$$\Gamma^{(2)}_{k,\mu\nu}[A,A] = f_k^{D_T}\, D_{T\,\mu\nu} + D_\mu\, f_k^{DD}\, D_\nu + n_\mu\, \frac{f_k^{nD}}{nD}\, D_\nu + D_\mu\, \frac{f_k^{nD}}{nD}\, n_\nu + n_\mu\, f_k^{nn}\, n_\nu\ . \tag{5.13}$$

Here, the scale-dependent functions $f_k^{D_T}$ and f_k^{DD} can depend on D_T, D^2 and nD. In turn, the functions f_k^{nD} and f_k^{nn} can depend only on D^2 and nD. An explicit analytical expression for the full propagator, similar to (5.8), follows from (5.13). Such approximations take the full (covariant) momentum dependence of the propagator into account. The inverse propagator (5.7) corresponds to the particular case $f^{D_T} = f^{DD} = 1$, $f^{nD} = 0$, and $f^{nn} = 1/\xi$.

2. Expansion in the fields

Even for analytic calculations one wishes to include more than covariantly constant gauge fields, and to expand in powers of the fields, or to make a derivative expansion. Eventually one has to employ numerical methods where some sort of approximation has to be made. Therefore it is important to have a formulation of the flow equation which allows for simple and systematic expansions.

Here we are arguing in favour of a different splitting of the propagator which makes it simple to employ very general approximations. For the present purpose it is more convenient to use regulators $R_k[D^2(\bar{A})]$. We split the inverse propagator into

$$\Gamma^{(2)ab}_{k,\mu\nu}[A,A] = \Delta^{ab}_{\mu\nu} - \left(2gF^{ab}_{\mu\nu} - (D_\mu D_\nu)^{ab}\right) \tag{5.14}$$

with

$$\Delta^{ab}_{\mu\nu} = \left\{-D^2(1+r_D)\right\}^{ab}\delta_{\mu\nu} + \frac{1}{\xi n^2} n_\mu n_\nu \delta^{ab}. \tag{5.15}$$

The operator Δ can be explicitly inverted for any field configuration (and $A = \bar{A}$). We have

$$\Delta^{-1} = -\frac{1}{D^2(1+r_D)}\mathbb{1} + \frac{1}{D^2(1+r_D)}\frac{1}{1+\xi D^2(1+r_D)} P_n. \tag{5.16}$$

With (5.14) and (5.16) we can expand the propagator as

$$G_k[A,A] = \Delta^{-1}\sum_{n=0}^{\infty}\left[(2gF - D\otimes D)\,\Delta^{-1}\right]^n. \tag{5.17}$$

where $(D\otimes D)^{ab}_{\mu\nu}(x,y) = D^{ac}_\mu D^{cb}_\nu \delta(x-y)$. For $\xi = 0$ (the axial gauge), Δ^{-1} can be neatly written as

$$\Delta^{-1}(\xi=0) = -\frac{1}{D^2(1+r_D)}(\mathbb{1}-P_n), \tag{5.18}$$

which simplifies the expansion (5.17). The most important points in (5.17) concern the fact that it is valid for arbitrary gauge field configurations and each term is convergent for arbitrary gauge fixing parameter ξ. As opposed to similar expansions in covariant gauges this expansion does not involve the projector P_D which is non-local.

B. Applications

In order to put the methods to work we consider in this section the full one-loop effective action for $SU(N)$ Yang-Mills theory which entails the universal one-loop beta function for arbitrary regulator function.

1. Effective action

For the right hand side of the flow we need

$$\Gamma_k[A,\bar{A}] = \frac{1}{2}\int Z_A(t)\,\mathrm{tr}_{\mathrm{f}}\,F^2(A) + S_{gf}[A] + O[(gA)^5, g^2\partial A, (g\bar{A})^5, g^2\partial\bar{A}], \tag{5.19}$$

Only the classical action can contribute to the flow, as n-loop terms in (5.19) lead to $n+1$-loop terms in the flow, when inserted on the right hand side of (3.78). This Ansatz leads to the propagator (5.12) which together with our choice for the regulator (5.6) is the input in the flow equation (3.78). We also use the following identitiy in the evaluation of the different terms in (3.78):

$$\mathrm{tr}\,D^2 = 4\mathrm{tr}\,D\otimes D. \tag{5.20}$$

Finally we choose $\mathcal{N}_k = 1$ and arrive at

$$\partial_\lambda\hat{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left\{\frac{\partial_\lambda r(D_T)}{1+r(D_T)} - \frac{1}{2}\frac{\partial_\lambda r(-D^2)}{1+r(-D^2)} + \frac{1}{4}\frac{\partial_\lambda r(-D^2)}{s^2+r(-D^2)}\right\}, \tag{5.21}$$

where the trace Tr contains also the Lorentz trace and the adjoint trace tr_{ad} in the Lie algebra. The first term on the right-hand side in (5.21) has a non-trivial Lorentz structure, while the two last terms are proportional to $\delta_{\mu\nu}$. We notice that the flow equation (5.21) is well-defined in both the IR and the UV region. We apply the heat-kernel results of section IV B 3 to the calculation of (5.21). To that end we take advantage of the following fact: given the existence (convergence, no poles) of the Taylor expansion of a function $f(x)$ about $x=0$ we can use the representation (4.31). Due to the infrared regulator the terms in the flow equation (5.21) have this property, where $\mathcal{O} = D_T, D^2$. Hence we can rewrite the arguments D_T and $-D^2$ in (5.21) as derivatives w.r.t. λ of the corresponding heat kernels $K_{-D_T}(\tau)$ and $K_{D^2}(\tau)$. Applying this to the flow equation (5.21) we arrive at

$$\begin{aligned}\partial_t\Gamma_k[A] = \frac{1}{2}\bigg[&\frac{\partial_t r(-\partial_\tau)}{1+r(-\partial_\tau)}\,\mathrm{Tr}K_{-D_T}(\tau) - \frac{1}{2}\frac{\partial_t r(-\partial_\tau)}{1+r(-\partial_\tau)}\,\mathrm{Tr}K_{D^2}(\tau)\\ &+\frac{1}{4}\int dp_n \frac{(p_n^2-\partial_\tau)\partial_t r(p_n^2-\partial_\tau)}{p_n^2+(p_n^2-\partial_\tau)r(p_n^2-\partial_\tau)}\frac{\tau^{1/2}}{\sqrt{\pi}}\mathrm{Tr}K_{D^2}(\tau)\bigg]_{\tau=0}\end{aligned} \tag{5.22}$$

The two terms in the first line follow from (5.21). The last term is more involved because it depends on both D^2 and nD due to $s^2 \equiv (nD)^2/n^2D^2$. We note that $nD = (n\partial)$ holds for configurations satisfying (5.1a) and only depends on the momentum parallel to n_μ. Furthermore it is independent of the gauge field. Now we use the splitting of $(p_\mu) = (p_n, \vec{p})$ where $p_n = P_n p$ and $\vec{p} = (1-P_n)p$. The heat kernel related to $\vec{D}^2$ follows from the one for D^2 via the relation $K_{\vec{D}^2}(\tau) = \frac{\tau^{1/2}}{\sqrt{\pi}}K_{D^2}(\tau)$ as can be verified by a simple Gaußian integral in the p_n-direction.

With these prerequisites at hand, we turn to the full effective action at the scale k, which is given by

$$\Gamma_k[A] = \Gamma_\Lambda[A] + \int_\Lambda^k dk'\,\frac{\partial\Gamma_{k'}[A]}{\partial k'}\,, \tag{5.23}$$

where Λ is some large initial UV scale. We start with the classical action $\Gamma_\Lambda = S_A + S_{\mathrm{gf}}$. Performing the k-integral in (5.23) we finally arrive at

$$\begin{aligned}\Gamma_k[A] = &\Big(1+\tfrac{Ng^2}{16\pi^2}\big(\tfrac{22}{3}-7(1-n^A)\big)\ln k/\Lambda\Big)\,S_A[A]\\ &+S_{\mathrm{gf}}[A] + \sum_{m=1}^{\infty} C_m(k^2/\Lambda^2)\;\Delta\Gamma^{(m)}[gF/k^2] + \mathrm{const.}\end{aligned} \tag{5.24}$$

The combination $S_A + S_{\mathrm{gf}}$ on the right-hand side of (5.24) is the initial effective action. All further terms stem from the expansion of the heat kernels (4.26) in powers of τ. The terms $\sim \tau^{-2}$ give field-independent contributions, while those $\sim \tau^{-1}$ are proportional to tr$F$ and vanish. The third term on the right-hand side of (5.24) stems from the τ^0 coefficient of the heat kernel. This term also depends on the regulator function through the coefficient n^A (4.14). All higher order terms $\sim \tau^m, m > 0$ are proportional to the terms $C_m(k^2/\Lambda^2)\,\Delta\Gamma^{(m)}[gF/k^2]$. These terms have the following structure: they consists of a prefactor

$$C_m(x) = -\frac{1}{4m}\frac{(-)^m}{m!}\,(1-x^m) \tag{5.25a}$$

and scheme-dependent functions of the field strength, $\Delta\,\Gamma^{(m)}[gF]$, each of which is of the order $2+m$ in the field strength gF. They are given explicitly as

$$\Delta\Gamma^{(m)}[gF] = B_m^{D_T}\ \mathrm{Tr}\, K^{(m)}_{-D_T}(0) + \left(B_m^{D^2} + B_m^{nD}\right) \mathrm{Tr}\, K^{(m)}_{D^2}(0)\ . \tag{5.25b}$$

Here, $K^{(m)}_{D^2}(0)$ and $K^{(m)}_{-D_T}(0)$ denote the expansion coefficients of the heat kernels. We use the following identity

$$f^{(m)}(0) = f(\partial_\tau)\tau^m|_{\tau=0}\,, \tag{5.26}$$

and $f^{(m)}(x) = (\partial_x)^m f(x)$. In addition, the terms in (5.25b) contain the scheme-dependent coefficients

$$B_m^{D_T} = \left(\frac{\dot r_1}{1+r_1}\right)^{(m)}(0)\ , \tag{5.27a}$$

$$B_m^{D^2} = -\frac{1}{2}B_m^{D_T}\ , \tag{5.27b}$$

$$B_m^{nD} = \frac{(-1)^{m+1}}{4}\int_0^\infty dx \left(\partial_x - \frac{1}{x}\alpha\partial_\alpha\right)^{m+1} \left.\frac{\dot r_1(x)}{\sqrt{r_1(x)}\sqrt{r_1(x)+\alpha}}\right|_{\alpha=1}\ . \tag{5.27c}$$

The coefficients B^{D_T}, B^{D^2} and B^{nD} follow from the first, second and third term in (5.21). We introduced dimensionless variables by defining $r_1(x) = r(xk^2)$ and $\dot r_1(x) \equiv \partial_t r_1(x) = -2xk^2 r'(xk^2) = -2xr_1'(x)$, in order to simplify the expressions and to explicitly extract the k-dependence into (5.25a). The explicit derivation of B^{nD} is tedious but straightforward and given – together with some identities useful for the evaluation of the integral and the derivatives – in appendix A A.1. All coefficients B^{D_T}, B^{D^2} and B^{nD} are finite. In particular, we can read off the coefficients for $m = 0$ which add up to the prefactor of the classical action in (5.24):

$$B_0^{D_T} = 2n, \qquad B_0^{D^2} = -n, \qquad B_0^{nD} = -\frac{1}{2}(1-n), \tag{5.28}$$

where we have used (A.5) in the appendix. Together with the heat kernel terms proportional to τ^0 given in (4.28) this leads to (5.24).

This application can be extended to include non-perturbative truncations. The flow of the coefficients (5.25b) becomes non-trivial, and regulator-dependent due to the regulator-dependence of the coefficients (5.27). Then, optimisation conditions for the flow can be employed to improve the truncation at hand [112].

Finally, we discuss the result (5.24) in the light of the derivative expansion. Typically, the operators generated along the flow have the structure $F\, f_k[(D^2+k^2)/\Lambda^2]\, F$, and similar to higher order in the field strength. For dimensional reasons, the coefficient function $f_k(x)$ of the operator quadratic in F develops a logarithm $\sim \ln x$ in the infrared region. An additional expansion of this term in powers of momenta leads to the spurious logarithmic infrared singularity as seen in (5.24). To higher order in the field strength, the coefficient function behave as powers of $1/(D^2+k^2)$, which also, at vanishing momenta, develop a spurious singularity in the IR, and for the very same reason. All these problems are absent for any finite external gluon momenta, and are an artifact of the derivative expansion. A second comment concerns the close similarity of (5.24) with one-loop expressions found within the heat-kernel regularisation. In the latter cases, results are given as functions of the proper-time parameter τ and a remaining integration over $d\ln\tau$. Expanding the integrand in powers of the field strength and performing the final integration leads to a structure as in (5.24), after identifying $\tau \sim k^{-2}$. In particular, these results have the same IR structure as found in the present analysis.

2. Running coupling

We now turn to the computation of the beta function at one loop. We prove that the result is independent of the choice of the regulator and agrees with the standard one. However, it turns out that the actual computation depends strongly on the precise small-momentum behaviour of the regulator, which makes a detailed discussion necessary.

Naively we would read off the β-function from the t-running of the term proportional to the classical action S_A in (5.24). Using $\partial_t(gA)=0$ leads to $\partial_t \ln Z_g = -\frac{1}{2}\partial_t \ln Z_A$. We get from (5.24)

$$Z_A = 1+\left(\tfrac{22}{3}-7(1-n^A)\right)\frac{Ng^2}{16\pi^2}t \quad \to \quad \partial_t \ln Z_g = -\left(\tfrac{11}{3}-\tfrac{7}{2}(1-n^A)\right)\frac{Ng^2}{16\pi^2}+O(g^4). \qquad (5.29)$$

We would like to identify $\beta = \partial_t \ln Z_g$. This relation, however, is based on the assumption that at one loop one can trade the IR scaling encoded in the t-dependence of this term directly for a renormalisation group scaling. This assumption is based on the observation that the coefficient of $S_A[A]$ is dimensionless and at one loop there is no implicit scale dependence. It is the latter assumption which in general is not valid. A more detailed analysis of this fact is given in [140]. Here, we observe that the background field dependence of the cut-off term inflicts contributions to $\partial_t Z_A S_{\rm cl}$. These terms would be regulator-dependent constants for a standard regulator without $\bar A$. As mentioned below (2.5), one should see the background field as an index for a family of different regulators. We write the effective action as

$$\Gamma_k[A,\bar A] = \Gamma_{k,1}[A]+\Gamma_{k,2}[\bar A]+\Gamma_{k,3}[A,\bar A]\ . \qquad (5.30)$$

The second term only depends on $\bar A$ and is solely related to the $\bar A$-dependence of the regulator. The last term accounts for gauge invariance of Γ_k under the combined transformation $\mathfrak{g}_\omega+\bar{\mathfrak{g}}_\omega$. This term vanishes in the present approximation, because of the observation that our Ansatz is invariant – up to the gauge fixing term – under both $\mathfrak{g}_\omega$ and $\bar{\mathfrak{g}}_\omega$ separately. The physical running of the coupling is contained in the flow of $\Gamma_{k,1}[A]$. This leads to

$$\beta = -\tfrac{1}{2}\partial_t Z_A + \tfrac{1}{2}\partial_t Z_{\bar{A}}, \tag{5.31}$$

where $Z_{\bar{A}}$ is the scale dependence of $\Gamma_{k,2}[A] \propto Z_{\bar{A}} S_A[A]$. We rush to add that this procedure is only necessary because we are interested in extracting the universal one-loop β-function from the flow equation. For integrating the flow itself this is not necessary since for $k = 0$ the background field dependence disappears anyway. For calculating $\partial_t \ln Z_{\bar{A}}$ we use (5.12) and (5.20) and get

$$\begin{aligned}\partial_t \frac{\delta}{\delta \bar{A}^a_\mu}\Gamma_k[A, \bar{A} = A] =& \frac{1}{2}\mathrm{Tr}\partial_t\Big\{ \frac{R'_k[D_T]}{D_T + R_k[D_T]}\frac{\delta D_T}{\delta \bar{A}^a_\mu} + \frac{1}{2}\frac{R'_k(-D^2)}{-D^2 + R_k[-D^2]}\frac{\delta D^2}{\delta \bar{A}^a_\mu} \\ & -\frac{1}{4}\frac{R'_k[-D^2]}{(-nD)^2 + R_k[-D^2]}\frac{\delta D^2}{\delta \bar{A}^a_\mu}\Big\},\end{aligned} \tag{5.32}$$

where we have introduced the abbreviation

$$R'_k(x) = \partial_x R_k(x). \tag{5.33}$$

For the derivation of (5.32) one uses the cyclic property of the trace and the relations (4.23). We notice that (5.32) is well-defined in both the IR and the UV region. The explicit calculation is done in appendix A A.2. Collecting the results (A.2),(A.3),(A.4) we get

$$\partial_t \delta_{\bar{A}} \Gamma_k[A, \bar{A} = A]|_{F^2} = -\frac{Ng^2}{16\pi^2}\, 7(1 - n^A)\, \delta_A S_A[A] \to \partial_t Z_{\bar{A}} = -\frac{Ng^2}{16\pi^2}\, 7(1 - n^A) \tag{5.34}$$

We insert the results (5.29) for $\partial_t Z_A$ and (5.34) for $\partial_t Z_{\bar{A}}$ in (5.31) and conclude

$$\beta = -\frac{11}{3}\frac{Ng^2}{16\pi^2} + O(g^4). \tag{5.35}$$

which is the well-known one-loop result. For regulators with a mass-like infrared limit, $n^A = 1$, there is no implicit scale dependence at one loop. It is also worth emphasising an important difference to Lorentz-type gauges within the background field approach. In the present case only the transversal *physical* degrees of freedom scale implicitly with $t = \ln k$ for $n^A \neq 1$. This can be deduced from the prefactor $7(1 - n^A)$ in (5.34). Within the background gauge, this coefficient is $\frac{22}{3}(1 - n^A)$ [140], see section IV C 2. The difference has to do with the fact that in the axial gauge one has no auxiliary fields but only the physical degrees of freedom. In a general gauge, this picture only holds true after integrating-out the ghosts. This integration leads to non-local terms. They are mirrored here in the non-local third term on the right hand side of the flow (5.22) and in the third term on the right hand side of (5.32) [see also (A.4)].

C. Conclusions

The results of this chapter show that axial gauges are well-suited for background field methods. As a by-product, the finiteness of the ERG flow in axial gauges was verifyied within an explicit example.

It is obvious that no singuarity will reappear in a more elaborated approximation. Moreover, the approximation (4.1) is not problematic here. As (4.1) is at the heart of a practical use of background field methods in ERG flows, this is good news. Finally we would like to emphasise, that the present results are the starting point for a non-perturbative analysis simply by taking the back-reaction of the t-dependent coefficients in (5.23).

VI. INSTANTON-INDUCED TERMS AND THE $U(1)$-PROBLEM

A. Introduction

In this chapter we discuss the use of ERG equations in the presence of topological configurations [137,138], see also [122]. The instanton liquid model has been successfully used to get an even quantitative insight into chiral symmetry breaking [162]. We want to use ERG flows to further our information about how the evolution of this situation starting in the perturbative sector. Equally interesting is the flow of the θ-parameter. As a starting point this necessitates the computation of the leading order to instanton-induced terms in the effective action. These terms trigger the non-trivial running of the related fermionic couplings responsible for the anomalous high η' mass. In the following we study the modification of the standard analysis in the presence of infra-red cut-off terms.[8] We show that fermionic zero modes are still present in an instanton background if the fermionic cut-off term has global chiral symmetry. In ERG flows, chiral symmetry [33,85] is treated very similarly to Ginsparg-Wilson fermions on the lattice, see e.g. [87]. Then we derive the T'Hooft determinant and comment on first numerical studies.

B. Effective action for Yang-Mills theory with fermions

So far, we have only dealt with pure Yang-Mills theory. Now we add the topological term S_θ and the fermionic classical action. The full gauge fixed action of an $SU(N)$-gauge theory coupled to fermions is given by

$$S[\phi, \bar{A}] = S_A[A] + S_\theta[A] + S_{\rm gf}[\mathcal{Q}, \bar{A}] + S_{\rm gh}[\mathcal{Q}, C, \bar{C}, \bar{A}] + S_\psi[A, \psi, \bar{\psi}] \tag{6.1}$$

Here ϕ is the super field as in (3.4), but including the fermions $\psi, \bar{\psi}$. We have

$$\begin{aligned} \phi &= (\mathcal{Q}, C, \bar{C}, \psi, \bar{\psi}), \qquad & J &= (J_\mathcal{Q}, J_C, J_{\bar{C}}, \eta, \bar{\eta}), \\ \phi^* &= (\mathcal{Q}, -\bar{C}, C, \bar{\psi}, -\psi), \qquad & J^* &= (J_\mathcal{Q}, -J_{\bar{C}}, J_C, \bar{\eta}, \eta), \end{aligned} \tag{6.2}$$

where the currents $\eta, \bar{\eta}$ are those of the physical fermions and not those of the ghosts as in the previous sections. The ghosts are solely spectators. The gauge fixed action of pure Yang-Mills in the background field gauge is defined in (3.5), (3.18) and (3.19). The fermionic action S_ψ is

$$S_\psi[\psi, \bar{\psi}, A] = \int d^4x \; \bar{\psi} \not{D}(A) \psi. \tag{6.3}$$

[8]For instantons in the presence of an infra-red cut-off, see [67,68]

The fermions $\psi = (\psi^A_{s,\xi})$ live in the fundamental representation, where A denotes the gauge group indices, $s = 1, ..., N_f$ the flavors and ξ the spinor indices. The Dirac operator is given by

$$\not{D}^{AB}_{\xi\xi'}(A) = (\gamma_\mu)_{\xi\xi'} D^{AB}_\mu, \qquad D^{AB}_\mu(A) = \partial_\mu \delta^{AB} + A^c_\mu (t^c)^{AB}, \qquad \{\gamma_\mu, \gamma_\nu\} = -2\,\delta_{\mu\nu}. \tag{6.4}$$

We have also introduced the topological θ-term in (6.1)

$$S_\theta[A] = \frac{i}{16\pi^2} \int d^4x\ \theta(x) \mathrm{tr}_{\mathrm{f}}\, F\tilde{F}(A), \tag{6.5}$$

where $\tilde{F}$ denotes the dual field strength: $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. For constant θ only the global properties of the gauge field enter into S_θ. It only can take values $S_\theta = i\theta n,\ n \in Z$ as $S_{\theta=i}[A]$ is the topological charge of A, e.g. [132]. In this case we may lose differentiability of S_θ with respect to the gauge field. The effective action Γ_k is defined by the Legendre transformation of the infrared regularised Schwinger functional W_k. For non-differentiable W_k one has to use the general definition of the Legendre transformation

$$\Gamma_k[\phi, \bar{A}] = \sup_J \left\{ \int d^4x\ J^*\phi - W_k[J, \bar{A}] \right\}. \tag{6.6}$$

However, it is difficult to use this definition for practical purposes. We circumvent this problem by allowing for an x-dependent θ. θ can be seen as a source of the index [154]. In this context let us mention a little subtlety: In [154] it was noted that the flow for θ did not commute with the limit where θ becomes space-time independent. We believe, that in order to resolve this problem such a study would require the use of (6.6), for related issues see [150].

The total cut-off term is defined by

$$\Delta S_k[\phi, \bar{A}] = \int d^4x\ \phi^* R_k \phi \qquad \text{where} \qquad R_k = (R^Q,\ R^C\ ,\ R^C\ ,\ R^\psi\ ,\ R^\psi) \otimes 1\!\!1_\phi. \tag{6.7}$$

The cut-off terms for gauge fields and ghosts have been discussed in great detail in the previous sections. As mentioned before, we want to use a fermionic cut-off term having the same Dirac structure as the Dirac operator. However, we would like to emphasise that the results obtained here extend to more general cut-off terms. A convenient choice is

$$\Delta S_\psi[\psi, \bar{\psi}, \bar{A}] = \int d^4x \bar{\psi} R^\psi[\bar{A}]\psi, \qquad R^\psi[\bar{A}] = \bar{\not{D}}\, r^\psi(\bar{\not{D}}^2), \tag{6.8}$$

where $\bar{\not{D}} = \not{D}(\bar{A})$. The Λ-dependence of the effective action is not specified, because we are working in the limit $\Lambda \to \infty$. In Appendix B B.1 we discuss the properties of the ultraviolet cut-off in more detail.

In the following we rely on the path integral representation of W_k and Γ_k. We write

$$\exp W_k[J,\bar{A}] = \int [\mathcal{D}\varphi]_k \, \exp\left\{ -(S[\varphi,\bar{A}] + \Delta S_k[\varphi,\bar{A}]) + \int d^4x \, J^* \varphi \right\}, \tag{6.9}$$

where the super field $\varphi = (a, c, \bar{c}, \chi, \bar{\chi})$ is the integration variable with mean value $\phi = \langle\varphi\rangle$. The subscript $_k$ refers to the sublteties of the renormalisation when adding the cut-off term to the classical action as discussed in Section II A. For the present investigation this plays no rôle. Note that (6.9) has to reproduce the flow equation (3.14) upon a *total* scale derivative w.r.t. k. In the following, we do not persue this point any further, but keep the subsript as a reminder. Since we are interested in the contribution of fermionic zero modes to the effective action, we rely on a path integral representation of Γ_k. It follows from its definition $\Gamma_k[\phi, \bar{A}] = \int d^dx \, J^*\phi - W[J, \phi] - \Delta S_k[\phi, \bar{A}]$ as a Legendre transformation of the W_k, that it can be written as

$$\exp\{-\Gamma_k[\phi,\bar{A}]\} = \frac{1}{\mathcal{N}_k} \int [d\varphi]_k \exp\{-S[\varphi+\phi,\bar{A}] + \Delta S_k[\varphi+\phi,\bar{A}] + \int d^4x \, J^*\varphi\}, \tag{6.10}$$

where we have shifted the integration variable $\varphi \to \varphi + \phi$. Later, we need the fermionic sources as functions of the fields. They follow from the definition of Γ_k as

$$\eta = \frac{\delta(\Gamma_k + \Delta S_k)}{\delta\bar{\psi}}, \qquad \bar{\eta} = -\frac{\delta(\Gamma_k + \Delta S_k)}{\delta\psi}. \tag{6.11}$$

Eq. (6.10) is the definition of Γ_k in terms of a functional integro-differential equation. The limits of Γ_k have been discussed in section II A. Here we are interested in the ultraviolet limit $k \to \infty$, where Γ_k tends to the classical action, possibly with Z-factors and non-vanishing marginal terms, see section IV.

C. Instanton-induced terms

However, so far we have not discussed the impact of topologically non-trivial configurations. Now we take into account $U_A(1)$-violating terms in the ultraviolet region. Even though these terms are suppressed by powers of $1/k$ they cannot be neglected as they introduce terms with chiral symmetry breaking to the effective action. For the choice of the fermionic cut-off term (6.8) it can be shown that the infrared regularised Dirac operator still has a non-trivial zero mode in the one instanton sector (see Appendix B B.1). This serves as the source for $U_A(1)$-violating terms.

For the calculation of these terms we rely on the dilute gas approximation: we consider gauge field configurations $a + \mathcal{Q} + \bar{A}$ with topological charge ± 1, where $a + \mathcal{Q} + \bar{A}$ is the argument of $S_A[a + \mathcal{Q} + \bar{A} + \delta a]$ in (6.10) with fluctuation δa. We achieve this by choosing a as a configuration with instanton number 1, but not necessarily an instanton. With topologically trivial $\mathcal{Q}, \bar{A}$ this leads to $a + \mathcal{Q} + \bar{A}$ with topological charge ± 1. Then, the full mean field $A = \bar{A} + \mathcal{Q}$ is in the topologically trivial sector.

When performing the path integral in the limit $k \to \infty$ we can rely on a saddle point approximation, as in the perturbative sector. Because of the infrared cut-off ΔS_A, there are no gauge field zero modes. The scale invariance of the action is broken and the minimum of the action is at vanishing instanton width:

$$S_A[a + \mathcal{Q} + \bar{A}] \geq \frac{8\pi^2}{g^2}, \qquad \Delta S_A[a, \bar{A}] \geq 0. \tag{6.12}$$

The first equation follows from $S_A[A] \geq 8\pi^2 |S_{\theta=i}[A]|$, following from $\mathrm{tr}(F \pm \tilde{F})^2 \geq 0$. This bound is saturated for (anti-) self-dual configurations a_I: $F(a_I) = \pm \tilde{F}(a_I)$. In addition, ΔS_A is vanishing only for gauge fields with vanishing norm (see Appendix B B.3). Here, a subtlety comes into play. For $a = a_I + \delta a$ with non-vanishing topological charge we have to decide whether we insert the full field a into the cut-off term or whether we only consider δa: $\Delta S_A = \Delta S_A[\delta a, \bar{A}]$. However, the latter is natural, as the reason for introducing an infrared cut-off term is to regularise momentum fluctuations. We also couple only δa to the current: $\varphi = (\delta a, c, \bar{c}, \chi, \bar{\chi})$.

Thus the gauge field sector has no infrared problems even if topologically non-trivial gauge field configurations are considered. In the limit $k \to \infty$ the gauge field integration becomes trivial. This remains valid for $\mathcal{Q}$, $\bar{A}$ with arbitrary instanton number. First we have a closer look at the fermionic part of the action. We shall argue by using the limit $\Gamma_k \stackrel{k\to\infty}{\longrightarrow} S$ that only the source terms couple to the fermionic zero mode. Therefore the zero mode integration can be done explicitly. The fermionic part of the exponent (6.10)) reads for vanishing fluctuation δa:

$$-S_\psi[\chi' + \psi', \bar{\chi}' + \bar{\psi}', a_I + \mathcal{Q} + \bar{A}] - \Delta S_\psi[\chi' + \psi', \bar{\chi}' + \bar{\psi}', \bar{A}] + \Delta S_\psi[\psi, \bar{\psi}, \bar{A}] + \int_x (\bar{\eta}\,\chi + \bar{\chi}\,\eta)\,, \tag{6.13}$$

where the primed fermionic fields are the non-zero modes of the infrared regularised Dirac operator $\not{D}(a_I + A + \bar{A}) + R^\psi[\bar{A}]$ and the zero modes are denoted by χ_0, ψ_0. In the limit $k \to \infty$ we get for the fermionic sources

$$\eta \to \frac{\delta}{\delta \bar{\psi}}(S_\psi + \Delta S_\psi), \qquad \text{and} \qquad \bar{\eta} \to -\frac{\delta}{\delta \psi}(S_\psi + \Delta S_\psi). \tag{6.14}$$

Using (6.14) we deduce from (6.13)

$$\begin{aligned} -S_\psi[\psi', \bar{\psi}', a_I + \mathcal{Q} + \bar{A}] - S_\psi[\chi', \bar{\chi}', a_I + \mathcal{Q} + \bar{A}] - \Delta S_\psi[\chi', \bar{\chi}', \bar{A}] \\ + \int_x (\bar{\eta}\,\chi_0 + \bar{\chi}_0\,\eta) + \Delta S_\psi[\psi, \bar{\psi}, \bar{A}] - \Delta S_\psi[\psi', \bar{\psi}', \bar{A}]. \end{aligned} \tag{6.15}$$

The terms linear in $\bar{\chi}_0$, χ_0 remain. There is no counter-term in the action, since $S_\psi + \Delta S_\psi$ does not depend on the zero mode. The cross terms $\Delta S_\psi[\bar{\chi}', \psi', \bar{A}]$, $\Delta S_\psi[\bar{\psi}', \chi', \bar{A}]$ have been cancelled by the source terms. Moreover, we have dropped the cross terms $S_\psi[\psi', \bar{\chi}', a_I + \mathcal{Q} + \bar{A}]$, $S_\psi[\chi', \bar{\psi}', a_I + \mathcal{Q} + \bar{A}]$, since they are suppressed with $1/k$ in the limit $k \to \infty$. We have, with $\not{D}\psi_0 = -R^\psi \psi_0$,

$$S_\psi[\psi',\bar\psi',a_I+\mathcal{Q}+\bar A]-\Delta S_\psi[\psi,\bar\psi,\bar A]+\Delta S_\psi[\psi',\bar\psi',\bar A]=S_\psi[\psi,\bar\psi,a_I+\mathcal{Q}+\bar A]. \tag{6.16}$$

The final result for the fermionic part of the exponent in (6.10) is

$$-S_\psi[\psi,\bar\psi,a_I+\mathcal{Q}+\bar A]-S_\psi[\chi',\bar\chi',a_I+\mathcal{Q}+\bar A]-\Delta S_\psi[\chi',\bar\chi',\bar A]+\int_x(\bar\eta\,\chi_0+\bar\chi_0\,\eta) \tag{6.17}$$

The last term in (6.17) depends on the instanton a_I via the zero mode. However only instantons a_I with width $\rho\sim 1/k$ contribute. The infrared regularisation of the gauge field suppresses instantons with width $\rho\gg 1/k$ (see Appendix B B.3). We split the gauge field measure into a measure of collective coordinates of the instanton and a measure of fluctuations about the instanton. Let $da_{I,k}$ be the (k-dependent) measure of the collective coordinates of the $SU(N)$-instanton [171,17]. The zero mode contribution factorises in the limit $k\to\infty$. Hence taking into account the trivial sector and the ±1 instanton sectors the effective action is given by

$$\exp\left\{-\Gamma_k[\phi,\bar A]\right\}=\exp\left\{-S[\phi,\bar A]\right\}\left(1+\left[\int d\mu_1(\theta)\,d\bar\chi_0\,d\chi_0\,e^{\int(\bar\eta\,\chi_0+\bar\chi_0,\eta)}+\text{h.c.}\right]\right)+O(1/k) \tag{6.18}$$

with

$$d\mu_1(\theta)=da_{I,k}\frac{\mathcal{N}_k{}'[a_I]}{\mathcal{N}_k},\qquad\qquad \mathcal{N}_k{}'[a_I]=\int[\mathcal{D}\varphi']_k\exp\left\{-S[a_I+\varphi',0]-\Delta S_k[\varphi',0]\right\}, \tag{6.19}$$

where $a_I+\varphi'$ means $(a_I+\mathcal{Q},c,\bar c,\chi',\bar\chi')$. In (6.19) we dropped the dependence of the zero mode contribution on $\mathcal{Q}$ and $\bar A$, since it is only next to leading order in $1/k$ (see Appendix B B.2). We also used that the contribution from the sector with instanton number -1 is the Hermitian conjugate of the sector with instanton number 1. We concentrate on the sector with instanton number $+1$ and compute

$$\int\prod_{s=1}^{N_f}d\bar b_0^s\,da_0^s\exp\int_x(\bar\eta\,\chi_0+\bar\chi_0\,\eta)=\prod_{s=1}^{N_f}\left(\int_x\bar\eta_s\,\phi_0\right)\left(\int_x\phi_0^+\,\eta_s\right) \tag{6.20}$$

with

$$(\chi_0)^A_{s,\xi}=a_0^s\,\phi^A_{0,\xi},\qquad(\bar\chi_0)^A_{s,\xi}=(\phi_0^+)^A_\xi\,\bar b_0^s,\qquad\int_x\phi_0^+\,\phi_0=1,\qquad d\bar\chi_0\,d\chi_0=\prod_{s=1}^{N_f}d\bar b_0^s\,da_0^s. \tag{6.21}$$

Higher powers of $(\int\bar\eta\,\chi_0)(\int\bar\chi_0\,\eta)$ vanish because of the properties of Grassmann variables. These calculations result in an effective action Γ_k, which is given in terms of an integro-differential equation even in the limit $k\to\infty$.

$$\Gamma_k[\phi,\bar A]\stackrel{k\to\infty}{\longrightarrow}S[\phi,\bar A]+P_k[\phi,\bar A]+O(1/k), \tag{6.22}$$

where the $U_A(1)$ violating term P_k is

$$P_k[\phi, \bar{A}] = \int d\mu_1(\theta) \prod_{s=1}^{N_f} \left(\int_x \bar{\eta}_s \, \phi_0 \right) \left(\int_x \phi_0^+ \, \eta_s \right) + \text{h.c.} \tag{6.23}$$

The sources $\eta, \bar{\eta}$ are given in (6.11) as functional derivatives of Γ_k with respect to $\psi, \bar{\psi}$. Terms which are suppressed by powers of $1/k$ but do not violate the $U_A(1)$ are contained in $O(1/k)$. Although we will see in the next section that P_k is also suppressed by powers of $1/k$, it is not possible to neglect it since it introduces $U_A(1)$-violation.

D. Effective action in the large scale limit

Equation (6.22) is a functional differential equation for Γ_k. In the limit $k \to \infty$ we are able to solve this equation. First we note that for $k \to \infty$ the $U_A(1)$-violating term takes a local form. The explicit calculation is given in Appendix B B.2. As mentioned before, we do not have gauge field zero modes. For quantitative purposes one should work within the valley method (see [164] and references therein). However for our purpose it is enough to estimate of the value of the coupling of the $U_a(1)$-violating term. As we will see later on only instantons with width $\rho \sim 1/k \to 0$ contribute. The qualitative behaviour of the corresponding fermionic zero modes does not change in the presence of the cut-off term. The zero modes have width $\rho \to 0$ and are peaked about the centre of the instanton. It follows that (see Appendix B B.2, (4.6,B.27,B.29))

$$P_k[\psi, \bar{\psi}] = \int_z \Delta[k, \theta] \det_{s,t} \bar{\eta}_s(z) \frac{1-\gamma_5}{2} \eta_t(z) + \text{h.c.} + O(\Delta[k, \theta]/k) \tag{6.24}$$

with

$$\Delta[k, \theta] = \int d\bar{\mu}_1(\theta) \, (2^5 \pi^2 \rho^4)^{N_f} \, a[N, N_f] \ \sim k^{-5N_f+4}. \tag{6.25}$$

P_k does not depend on A, $\bar{A}$ to leading order (see Appendix B B.2), and so the $U_A(1)$-violation is purely fermionic to leading order. Now we concentrate on the measure $d\bar{\mu}_1(\theta)$. The fluctuation fields $a', \chi', \bar{\chi}'$ decouple approximately from the instanton for large scales k (see the discussion about the use of the valley method). This can be used to effectively remove the gauge fixing term for a_I. We have in the limit $k \to \infty$ (see (6.19))

$$\frac{\mathcal{N}_k{}'[a_I]}{\mathcal{N}_k} \sim \frac{1}{\mathcal{N}_k} \int \mathcal{D}\phi' e^{-S_k[\phi', \bar{A}]} \exp\left\{-\Delta S_A[a_I, 0] - S_A[a_I] - S_\theta[a_I]\right\}, \tag{6.26}$$

where the measures do not include the zero (or quasi-zero) modes related to a_I. The integrals in (6.26) become Gaussian for $k \to \infty$. Taking into account the normalisation $\mathcal{N}_k$ they lead to a factor $\rho^{n_0^f - n_0^g}$, where n_0^f, n_0^g are the number of fermionic (f) zero modes and corresponding gauge field (g) modes and ρ is the width of the instanton. The factor $\exp\{-\Delta S_A[a_I,0]\}$ provides an exponential suppression of the zero mode contribution for $\rho \gg 1/k$ due to the infrared regularisation of the gauge field (B.36). This ensures the infrared finiteness of the ρ integration. Therefore we can assume ρ to be of order $1/k$ or smaller. The term

$$\exp\{-S_A[a_I]\} = \exp\left\{-\frac{8\pi^2}{g^2}\right\} \tag{6.27}$$

is well known from instanton calculations (see [164] and references therein). The exponent S_θ of the remaining factor is related to the instanton number $\frac{1}{16\pi^2}\int \mathrm{tr}\, F\tilde{F} = 1$. In the limit $\rho \to 0$ the density $\mathrm{tr}\, F\tilde{F}[a_I(x)]$ serves as a δ-function which is peaked at the centre z of the instanton. Hence we get for $\rho \sim 1/k \to 0$

$$\exp\left\{-\frac{i}{16\pi^2}\int_x \theta(x)\mathrm{tr}\, F\tilde{F}[a_I(x)]\right\} \to \exp\left\{-i\theta(z)\frac{1}{16\pi^2}\int_x \mathrm{tr} F\tilde{F}[a_I]\right\} = \exp\{-i\theta(z)\}. \tag{6.28}$$

Thus the θ-term leads to the following modification of the $\det_{s,t}$-term:

$$\int_z \Delta[k,\theta] \det_{s,t} \bar{\eta}_s \frac{1-\gamma_5}{2}\eta_t = \int_z \Delta[k,0]e^{-i\theta(z)} \det_{s,t} \bar{\eta}_s \frac{1-\gamma_5}{2}\eta_t(1+O(1/k)). \tag{6.29}$$

Anti-instantons have instanton number -1, and so in this case one picks up a factor $\exp\{i\theta(z)\}$. Using these results in (6.22) we end up with

$$\Gamma_k[\phi,\bar{A},\psi] = S[\phi,\bar{A}] + P_k[\psi,\bar{\psi}] + O(1/k) \tag{6.30}$$

with

$$P_k[\psi,\bar{\psi}] = \int_z \Delta[k,\theta] \det_{s,t}\left[-\frac{\delta}{\delta\psi_s}(\Gamma_k + \Delta S_\psi)\frac{1-\gamma_5}{2}\frac{\delta}{\delta\bar{\psi}_t}(\Gamma_k + \Delta S_\psi)\right] + \text{h.c.} \tag{6.31}$$

In (6.31) we have used the explicit dependence of $\eta, \bar{\eta}$ on Γ_k as given in (6.11). The term $O(1/k)$ includes sub-leading orders of $U_A(1)$-conserving contributions and $U_A(1)$-violating contributions. The factor $\Delta[k,\theta]$ provides a suppression of P_k proportional to k^{-5N_f+4}.

The properties of (6.30) lead to an effective action Γ_k, which is well-defined in the limit $k \to \infty$. In addition an explicit expression for Γ_k can be derived. Note that we have used in the derivation that $\Gamma_k \to S + O(1/k)$ is also valid in the instanton sectors. Hence proving the existence of a well-defined limit of Γ_k serves as a self-consistency check. The only source for a diverging contribution is P_k, which is purely fermionic. In the limit $k \to \infty$ we have (see 6.8)

$$\frac{\delta}{\delta\bar{\psi}}\Delta S_\psi \to k\frac{\partial\!\!\!/}{|\partial\!\!\!/|}\psi, -\frac{\delta}{\delta\psi}\Delta S_\psi \to k\bar{\psi}\frac{\partial\!\!\!/}{|\partial\!\!\!/|}. \tag{6.32}$$

Combined with $\Delta[k,\theta]$ these terms are still suppressed by powers of $1/k$ ($N_f > 1$). In addition, (6.30) is inconsistent for

$$\frac{\delta}{\delta\psi}\Gamma_k,\ \frac{\delta}{\delta\bar{\psi}}\Gamma_k \sim k^n, \qquad n \neq 0. \tag{6.33}$$

This follows by using $\Delta[k,\theta] \sim k^{-5N_f+4}$ and (6.32). The only consistent choice in (6.33) is $n = 0$ which also ensures the finiteness of Γ_k in the limit $k \to \infty$. Thus we drop the part of P_k which depends on $\frac{\delta}{\delta\psi}\Gamma_k,\ \frac{\delta}{\delta\bar{\psi}}\Gamma_k$ since it is of sub-leading order. With (2.6) and (6.8) we get

$$\bar{\psi}_s R^\psi \frac{1\pm\gamma_5}{2} R^\psi \psi_t = \bar{\psi}_s {R^\psi}^2 \frac{1\mp\gamma_5}{2}\psi_t \stackrel{k\to\infty}{\longrightarrow} k^2 \bar{\psi}_s \frac{1\mp\gamma_5}{2}\psi_t \tag{6.34}$$

and the final result for the effective action for large scales k is

$$\Gamma_k[\phi,\bar{A}] = S[\phi,\bar{A}] + \int_z \left(\Delta[k,\theta]k^{2N_f} \det_{s,t} \bar{\psi}_s \frac{1+\gamma_5}{2}\psi_t + \Delta^*[k,\theta]k^{2N_f} \det_{s,t} \bar{\psi}_s \frac{1-\gamma_5}{2}\psi_t\right) + O(1/k). \tag{6.35}$$

with $\Delta[k,\theta]k^{2N_f} \sim k^{-3N_f+4}$. The term $O(1/k)$ includes sub-leading orders of $U_A(1)$-conserving contributions of order $1/k$ and $U_A(1)$-violating contributions of order $\Delta[k,\theta]k^{2N_f-1}$.

In (6.35) the contributions of the trivial sector and the sector with instanton number ± 1 are included. However, contributions $P_k(n)$ of sectors with instanton number n, $|n| \geq 2$ are only sub-leading terms in $1/k$. Because of the cut-off term for the gauge field (3.8a), the contributions exhibit a natural size $\approx 1/k \to 0$. This locality is sufficient to allow qualitatively the same arguments as in the derivation of the $U_A(1)$-violating terms in the one instanton sector and end up with powers of the flavor determinant

$$P_k(n) \sim k^{-3n\,N_f+4} \int_z \left(\det_{s,t} \bar{\psi}_s^{A_s} \frac{1-\gamma_5}{2}\psi_t^{B_t}\right)^n \mathcal{T}_n^{A_1 B_1 \cdots A_{nN_f} B_{nN_f}}, \tag{6.36}$$

where $\mathcal{T}_n$ denotes the color structure. These are sub-leading terms.

E. Discussion

We have calculated the fermionic $U_A(1)$-violating terms contributing to the effective action Γ_k in the presence of instantons to leading order in $1/k$. Because of the infrared cut-off term of the gauge field, there are no problems with infrared divergences, and so there are no gauge field zero modes. We have shown that even in the presence of the fermionic cut-off term we have fermionic zero modes for instanton configurations. The integration of the fermionic zero mode sector factorises and we end up with the well known 't Hooft determinant as the first order correction in $1/k$. The coupling $\Delta[k,\theta]$ of the 't Hooft determinant is infrared finite due to the gauge field regularisation.

Further corrections are of sub-leading order in $1/k$. In addition they show the same flavor structure as the 't Hooft determinant. Inclusion of fermionic mass terms is straightforward. The effective action (7.10) with suitable wave function renormalisations, an explicit ghost sector and additional $U_A(1)$-conserving terms may serve as an appropriate input to the exact flow equation in order to study the $U(1)$-problem.

It is possible to calculate $\mu[k,\theta]$ numerically based on the results of Appendix B B.2. The result does not allow quantitative statements, but provides a validity bound of the $1/k$-expansion. If the expansion is still valid at about $k \sim 700$ MeV, it would be possible to take the value of $\mu[k,\theta]$ at $k \sim 700$ MeV as an input for a phenomenological quark-meson model (see [16]). We briefly discuss the approximations used in the numerical calculation and comment on the result. The renormalisation scheme proposed by the framework of flow equations (with appropriate boundary conditions) can be related to the $\overline{\text{MS}}$-scheme [55]. This allows to use in a first approximation the well known results at the one-loop level of instanton calculations. The only new ingredient was the the infrared cut-off in the gauge field sector. Clearly these approximations allow only a rough estimate of the value of $\mu[k,\theta]$. Moreover one can determine the validity-bound of the $1/k$-expansion in the case of QCD. For scales $k \sim 1 - 1.3$ GeV the $1/k$ approximation breaks down, and one has to use the flow equation to extrapolate to lower energies. It is known from instanton calculations, that in this region corrections which are proportional to the gluonic condensate $\langle 1/g^2\, F^2 \rangle$ become important (see [162,164]). Since the flow should be smooth (as opposed to the case of correlation functions connected with phase transitions), one expects fewer problems with the numerical integration of the flow equation for the relevant correlation functions (e.g. $\Delta[k,\theta]$, which is connected to the η'-mass). On the other hand the value of $\Delta[k,\theta]$ in the physical region should be dominated by the contributions collected during the flow, otherwise the value would depend on the initial scale k_0, which has no physical meaning. Therefore the calculation of correlation functions connected with the instanton-induced terms is interesting for two reasons: It is a good check for the computational power of flow equations and it would be a great success to derive the η'-mass quantitatively from first principles.

We have also discussed the leading order corrections due to the θ-term. It leads to an additional phase factor in the 't Hooft determinant breaking CP-invariance in the presence of massive fermions. In the case of massless fermions the θ-angle can be absorbed in a redefinition of the fields. For massive fermions one can calculate the flow of θ. To solve the strong CP-problem, one has to calculate θ in the full quantum theory. First computations with ERG methods indicated a non-trivial flow of θ.

VII. THERMAL FLOWS

A. Introduction

A long standing problem of thermal field theory concerns the non-perturbative resummation of thermodynamical quantities like the thermal pressure. Any perturbative treatment faces problems at some given loop order when massless bosonic excitations are present [89]. This is in particular the case for gauge theories, but qualitatively, this problem is already encountered in scalar theories. Furthermore, even quantities that are perturbatively accessible up to some loop order typically show a very poor convergence behaviour, e.g. the thermal pressure of QCD. The poor convergence of this series suggests that a perturbative expansion may not be the most appropriate scheme even before the non-perturbative magnetic sector is reached. In any case it seems important to find an approach with much better convergence properties than perturbation theory, and which allows us to study as well the non-perturbative magnetic sector of the theory in a well-controlled way.

In [47], a resummation for the thermal pressure of scalar field theories has been proposed, see also [48–50]. Their philosophy is to organise a resummation in form of a mass integral. However, an extension to the entire thermal effective action (of which the pressure is only the constant part) or the inclusion of fermions or gauge fields is still missing.

Here we propose an application of ERG flows to thermal field theory. The aim is to show that, in contrast to the previous chapters which dealt with *quantum* fluctuations, for *thermal* fluctuations a fully gauge invariant flow can be constructed. Within a real time formulation, this has been done in [41,45,46,147].[9] Here, we elaborate on the alternative resummation scheme for integrating-out thermal fluctuations proposed in [111]. We construct a Wilsonian flow which is well-defined both in the IR and in the UV. In particular, the flow respects *gauge invariance* for arbitrary scale k. Using the imaginary time formalism the thermal flow equation for a pure (non-)Abelian gauge theory in four dimensions is given by

$$\frac{\partial \Delta\Gamma_{k,T}[A]}{\partial k^2} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left\{ T\sum_n \left[\frac{\delta^2\Gamma_{k,T}[A]}{\delta A_\mu^a\,\delta A_\mu^a}+k^2\right]^{-1} - \int \frac{p_0}{2\pi}\left[\frac{\delta^2\Gamma_{k,0}[A]}{\delta A_\mu^a\,\delta A_\mu^a}+k^2\right]^{-1}\right\}, \tag{7.1}$$

and relates the zero temperature effective action $\Gamma_{k,0}[A]$ with the thermal one $\Gamma_{k,T}[A] = \Gamma_{k,0}[A] + \Delta\Gamma_{k,T}[A]$. The r.h.s. in (7.1) is well-defined in the UV due to the subtraction, and in the IR due to the regulator, which is introduced as an effective mass term $\sim k$. The generalisation to matter fields is straightforward (see below).

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[9] There, gauge invariance is a subtle issue as gauge transformations have to vanish to all orders at $\pm T$. It is not completely clear whether this is not too strong a property

Our flow equation is motivated by the following two observations. We note in the first place that a mass-like regulator is not acceptable for integrating-out *quantum* fluctuations. However, it is, for reasons that shall be detailed below, a viable regulator for *thermal* fluctuations. The second observation concerns Wilsonian flows for gauge theories in axial gauges [107,109–111,119]. Gauge invariance for physical Green functions is controlled via modified Ward Identities (mWI). In an axial gauge, they reduce to the standard WI for *any* scale k, if a mass-like regulator is employed. Although of very limited use in the generic case, this is precisely the missing piece to construct a flow for $\Delta\Gamma_{k,T}[A]$, which respects gauge invariance for all scales.

B. Wilsonian approach

To start with, we discuss the standard Wilsonian approach to quantum field theories. All the physically relevant information can be obtained from the (regularised) partition function. It reads

$$\exp W_{k,T}[J] = \int \mathcal{D}\phi \exp\Big(- S_{k,T}[\phi] + \mathrm{Tr}\, J\phi\Big) \tag{7.2}$$

In the imaginary time formalism, the trace stands for

$$\mathrm{Tr} = T \sum_{\phi} (-1)^{2s_\phi} \sum_{n} \int \frac{d^3p}{(2\pi)^3} \,, \tag{7.3}$$

and the implicit replacements $p_0 = 2\pi n T$ for bosonic and $p_0 = \pi(2n+1)T$ for fermionic fields are understood, n labelling the Matsubara frequencies $n = \{0, \pm 1, \pm 2, ...\}$. The sum $\sum_\phi$ runs over all possible fields and their indices, $\phi = (A, \psi, \bar\psi, \varphi)$, $\phi^* = (A, -\bar\psi, \psi, \varphi^*)$, s_ϕ is the spin of ϕ and J stands for the corresponding sources.[10] The term $S_{k,T}[\phi] = S + \Delta_{k,T}S$ contains the (gauge-fixed) classical action $S[\phi]$ and a quadratic regulator term $\Delta_{k,T}S[\phi]$, given by

$$\Delta_{k,T}S[\phi] = \frac{1}{2}\mathrm{Tr}\,\Big(\phi^*(-p)\; R^\phi_k(p)\; \phi(p)\Big). \tag{7.4}$$

(7.4) introduces a coarse-graining via the operator $R^\phi_k(p)$. The flow of (7.2) related to an infinitesimal change of $t = \ln k/\Lambda$ (with Λ being some fixed UV scale) is

$$\partial_t \exp W_{k,T}[J] = \frac{1}{2} \int \mathcal{D}\phi\, \mathrm{Tr}\,\Big(\phi^*\, \partial_t R^\phi_k\, \phi\Big) \exp\Big(- S_{k,T}[\phi] + \mathrm{Tr}\; J\phi\Big)\,. \tag{7.5}$$

Performing a Legendre transformation leads to the coarse-grained effective action $\Gamma_{k,T}[\phi]$,

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[10] In a slight abuse of notation we will also refer with Tr to traces that involve only one field.

$$\Gamma_{k,T}[\phi] = \mathrm{Tr}\, J\phi - W_{k,T}[J] - \Delta_{k,T} S[\phi] - \mathcal{C}_{k,T}, \phi = \frac{\delta}{\delta J} W_{k,T}[J] \ . \tag{7.6}$$

Note that the constant $\mathcal{C}_k$ is usually not mentioned when one is interested in field independent quantities. However as we shall see later in the discussion of the thermal pressure we have to take it into account. It is straightforward to obtain the flow equation for $\Gamma_{k,T}[\phi]$ by using (7.6):

$$\partial_t \Gamma_{k,T}[\phi] = \frac{1}{2} \mathrm{Tr} \left\{ G^{\phi\phi^*}_{k,T}[\phi] \, \frac{\partial R^\phi_k}{\partial t} \right\} - \partial_t \mathcal{C}_{k,T} \tag{7.7}$$

with

$$G^{\phi\phi^*}_{k,T}[\phi] = \left(\frac{\delta^2 \Gamma_{k,T}[\phi]}{\delta\phi\delta\phi^*} + R^\phi_k \right)^{-1} \tag{7.8}$$

denoting the full (field-dependent) regularised propagator of ϕ. We recapitulate the requirements on a regulator function $R^\phi_k(p)$. The general requirements on the trace-class operator R_k are that

(i) it has a non-vanishing limit for $p^2 \to 0$, typically $R^\phi_k \to k^2$ for bosons, and $R^\phi_k \to k$ for fermions. This precisely ensures the IR finiteness of the propagator at non-vanishing k even for vanishing momentum p.

(ii) it vanishes in the limit $k \to 0$, and for $p^2 \gg k^2$. The latter condition ensures that large momentum fluctuation have efficiently been integrated-out whereas the first condition guarantees that any dependence on R^ϕ_k drops out in the limit $k \to 0$.

(iii) R^ϕ_k diverges like Λ^2 for bosons, and like $R^\phi_k \to \Lambda$ for fermions, when $k \to \infty$ (or $k \to \Lambda$ with Λ being some UV scale much larger than the relevant physical scales). Thus, the saddle point approximation to (2.7) becomes exact and $\Gamma_{k\to\Lambda}$ reduces to the (gauge-fixed) classical action S.

These conditions guarantee that Γ_k has the limits

$$\lim_{k\to\infty} \Gamma_{k,T}[\phi] = S[\phi] \tag{7.9}$$

$$\lim_{k\to 0} \Gamma_{k,T}[\phi] = \Gamma_T[\phi] \ . \tag{7.10}$$

The flow equation (7.7) connects the classical action $S[\phi]$ with the full quantum effective action $\Gamma_T[\phi]$ at temperature T. Note that the limits (7.9) and (7.10) are strictly speaking only valid with a suitable choice of $\mathcal{C}_k$.

C. Mass-like regulators

In the following we want to understand when it might be appropriate to employ a mass-like regulator function. The aim behind it is to find a reliable, but still sufficiently simple and manageable formulation of the flow equation. The first observation is that the flow equation indeed is simplified for a mass-like regulator given by

$$\begin{aligned} &\text{bosons:} \quad R_k(p) \sim k^2 \\ &\text{fermions:} \; R_k(p) \sim k \end{aligned} \tag{7.11}$$

or variants of it.[11] The key characteristic of R_k^ϕ in (7.11) is that it does not depend on momenta. This is why a mass-like regulator has often been used to perform preliminary computations regarding the qualitative behaviour of the theory under investigation.

Formally speaking, (7.11) is a viable IR regulator in the sense of condition (i), and it allows as well to reach the UV initial condition, due to condition (iii). However, the choice (7.11) violates condition (ii), which is one of the basic requirements for a Wilsonian cut-off. The operator $\partial_t R_k^\phi$ appearing in (2.12) is *neither* peaked about $p^2 \approx k^2$, *nor* does it lead to a sufficient suppression of high momentum modes. The flow equation (2.12) receives contributions from the high momentum region for *any* value of k. An immediate consequence of this is that an additional UV regularisation is required, as the flow equation (2.12) is no longer well-defined for large loop momenta. Equivalently, one might say that a mass term regulator leads to a break-down of the Wilsonian picture, since it is no longer related to an integrating-out of momentum degrees of freedom. Rather, it corresponds to a flow within the space of massive theories. This makes it a rather questionable choice in the general situation, e.g. for $T = 0$ theories.

Apart from these more formal objections one should mention that a mass-like regulator seems not adequate for numerical solutions of (2.12). At every iterative step a d-dimensional momentum integral has to be performed in (2.12) over a non-trivial function which is *not* strongly peaked about some momentum region. This is, numerically, a quite tedious problem. For this reason most of the sophisticated numerical investigations are based on *non-local* regulator functions (like the sharp cut-off, exponential or algebraic ones). It has also been observed that *approximate* solutions to the flow equation (expansions in powers of the field, derivative expansions) show a rather poor convergence behaviour, when (7.11) is used.

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[11]Sometimes it is convenient to multiply the mass term with a (momentum-independent) wave function renormalisation, $R_k^\phi = Z_k^\phi \, k^2$ (and analogous for fermions). The same reasoning applies.

D. Mass-like regulator for thermal fluctuations

Although (7.11) is not viable for *quantum* fluctuations, we shall argue that it is viable for *thermal* ones. Instead of computing the flow for $\Gamma_{k,T}[\phi]$ as in (2.12), we propose to compute the flow for the *difference* $\Delta\Gamma_{k,T}[\phi] = \Gamma_{k,T}[\phi] - \Gamma_{k,0}[\phi]$.[12] The main point is that the large momenta fluctuations (not sufficiently controlled by (7.11), introducing UV divergences to the flow equation) have nothing to do with the heat bath. Therefore, subtracting the zero temperature quantities will render the flow equation (7.15) finite and well-defined.

Thus the flow of $\Delta\Gamma_{k,T}[\phi]$ involves a projection on the thermal fluctuations. With mass-like regulators R_k^ϕ, it is given by

$$\partial_t \Delta\Gamma_{k,T}[\phi] = \sum_\phi (-)^{2s_\phi} R_k^\phi \int \frac{d^3p}{(2\pi)^3} \left\{ T \sum_n G_{k,T}^{\phi\phi^*}[\phi] - \int \frac{dp_0}{2\pi} G_{k,0}^{\phi\phi^*}[\phi] \right\} - (\partial_t \mathcal{C}_{k,T} - \partial_t \mathcal{C}_{k,0}) \ . \quad (7.13)$$

Note, that that the momentum-independent regulator $\partial_t R_k \sim R_k$ acts only as a multiplicative constant, because of (7.11). In this case, the suppression of large momenta does not come from $\partial_t R_k^\phi$, but from the cancellation between the propagator terms. For large internal momenta, the Matsubara sum can be replaced by an integral, thereby cancelling the $T = 0$ contribution.

Thus, one may read (7.13) as a Wilsonian flow for thermal fluctuations: At the starting point $k = \Lambda$ (Λ being some large UV scale) all fluctuations are suppressed and (7.13) vanishes. For any $k < \Lambda$, the flow of $\Gamma_{k,T}[\phi]$ would receive contributions for all momenta. In contrast, the difference $\Delta\Gamma_{k,T}[\phi]$ is sensitive only to thermal fluctuations, which are peaked in the infrared region and naturally decay in the UV region. It follows that the integrand in (7.13) is peaked about $p^2 \approx k^2$. In other words, condition (ii) is effectively guaranteed even in the case of a mass-like regulator by the very nature of the temperature fluctuations. This amounts to the fact that the mass-like regulator seems to be a reasonable choice even for numerical applications in thermal field theories.

More generally, we can introduce the flow for $\Delta\Gamma_{k,T}[\phi]$ and arbitrary regulators. Let us define

$$H_{k,T}^{\phi\phi^*}(\Omega) = T \sum_n \Omega\, G_{k,T}^{\phi\phi^*}[\phi], \qquad \lim_{T\to 0} H_{k,T}^{\phi\phi^*}(\Omega) = \int \frac{dp_0}{2\pi} \Omega\, G_{k,0}^{\phi\phi^*}[\phi] \quad (7.14)$$

[12]The functional $\Gamma_{k,T}[\phi]$ provides a map

$$\Gamma_{k,T} : \left[\mathcal{S}_T[\phi] = \{\phi : [-1/T, 1/T] \times \mathbb{R}^3 \to \mathbb{C} \text{ with } \phi[-T] = (-)^{2s_\phi}\phi[T]\}\right] \to \mathbb{R}. \quad (7.12)$$

The difference $\Delta\Gamma_{k,T}[\phi]$ is properly defined for fields ϕ such that $\Delta\Gamma_{k,T}[\phi]$ provides a map $\mathcal{S}_T[\phi] \to \mathbb{R}$. Even for $\phi \in \mathcal{S}_0[\phi]$ the difference $\Gamma_{k,T}[\phi] - \Gamma_{k,0}[\phi]$ is well-defined but has no physical meaning.

which is the discretised one-dimensional integral over the integrand in (2.12) if the arbitrary operator Ω is given by $\Omega = \partial_t R_k^\phi$. In terms of (7.14), the flow equation for $\Delta\Gamma_{k,T}[\phi]$ reads

$$\partial_t \Delta\Gamma_{k,T}[\phi] = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \sum_\phi (-)^{2s_\phi} \left\{ H_{k,T}^{\phi\phi^*}(\partial_t R_k^\phi) - H_{k,0}^{\phi\phi^*}(\partial_t R_k^\phi) \right\} - (\partial_t \mathcal{C}_{k,T} - \partial_t \mathcal{C}_{k,0}) \ . \tag{7.15}$$

Let us comment on the initial condition to (7.13) [and (7.15)]. In contrast to the flow (7.7) with the limits (7.9) and (7.10), the new flow equation (7.13) [resp. (7.15)] has the limits

$$\lim_{k\to\infty} \Delta\Gamma_{k,T}[\phi] = 0 \tag{7.16}$$

$$\lim_{k\to 0} \Delta\Gamma_{k,T}[\phi] = \Gamma_T[\phi] - \Gamma_0[\phi] \ . \tag{7.17}$$

The boundary condition (7.16) looks rather simple. The flow equation (7.13) needs in addition, however, the knowledge of the massive $T = 0$ quantum theories. This point is qualitatively shared by other recent proposals [45–47]. It seems likely to find a good approximation for the issues under investigation, since (7.13) is eventually projecting-out thermal fluctuations. Those should not be too sensitive to the details of the quantum effective action at $T = 0$. Moreover we deal with a situation were the original fields are still sensible degrees of freedom. Thus, a perturbatively resummed quantum effective action should be a good starting point. Here we are actually taking advantage of the fact that for a mass-like regulator the flow is describing a path in the set of massive vector boson theories rather than a Wilsonian integrating-out.

E. Gauge invariance

We will now discuss the requirement of gauge invariance and its implications. The first question to raise concerns the gauge fixing. In [107], we discussed the various advantages of an axial gauge fixing given by

$$S_{\rm gf}[A] = \frac{1}{2}{\rm Tr}\ n_\mu A_\mu^a \ \frac{1}{\xi n^2}\ n_\nu A_\nu^a. \tag{7.18}$$

Note that we have an additional Lorentz vector n_μ at our disposal, due to the presence of a heat bath, which makes the axial gauge fixing quite natural. It has the further advantage that both ghost fields (which decouple completely) and possible Gribov copies are absent. We will also make use of some results obtained in [107,109–111,119]. There we have showed that the spurious propagator singularities of perturbation theory are naturally absent in a Wilsonian approach. Furthermore, the gauge fixing parameter ξ with mass dimension -2 has a non-perturbative fixed point at $\xi = 0$. This singles out the $nA = 0$ gauges and tremendously simplifies the problem of gauge invariance, because it allows for a momentum independent choice of ξ.

Gauge invariance for physical Green functions corresponds to the requirement of a *modified* Ward Identity (mWI) to hold. We recall the generator of gauge transformations $\mathfrak{g}_\omega$:

$$\mathfrak{g}_\omega(A\,,\,\phi_{\rm m}) = ([D\,,\,\omega],[\phi_{\rm m}\,,\,\omega]), \tag{7.19}$$

where $\phi_{\rm m}$ stand for the matter fields. For momentum independent gauge fixing parameter, the mWI related to the flow (2.12) reads

$$\mathfrak{g}_\omega\Gamma_{k,T}[\phi] = \frac{1}{n^2\xi}{\rm Tr}\; n_\mu\partial_\mu\omega\; n_\nu A_\nu + \frac{g}{2}\sum_\phi(-)^{2s_\phi}\int\frac{d^3p}{(2\pi)^3}\, H^{\phi\phi^*}_{k,T}([\omega,R^\phi_k]) \tag{7.20}$$

The two terms on the r.h.s. are remnants from the gauge fixing and the coarse-graining, respectively. The mWI (7.20) has been shown in [107] to be compatible with (2.12), e.g. a solution to (7.20) at some scale $k=\Lambda$ remains a solution for $k<\Lambda$ if $\Gamma_{k,T}[\phi]$ is integrated according to the flow equation. In particular, the terms containing R^ϕ_k do vanish for $k\to 0$, thereby ensuring gauge invariance for physical Green functions.

The mWI related to the flow (7.15), that is for $\Delta\Gamma_{k,T}[\phi]$, follows from (7.20) as

$$\mathfrak{g}_\omega\Delta\Gamma_{k,T}[\phi] = \frac{g}{2}\sum_\phi(-)^{2s_\phi}\int\frac{d^3p}{(2\pi)^3}\left\{H^{\phi\phi^*}_{k,T}([\omega,R^\phi_k]) - H^{\phi\phi^*}_{k,0}([\omega,R^\phi_k])\right\} \tag{7.21}$$

The compatibility of (7.21) with (7.15) is an immediate consequence of the compatibility of (7.20) with (7.7). The linear term related to the gauge fixing [the second term in (7.20)] cancels for $\phi,\alpha\in\mathcal{S}_T[\phi]$. The interpretation of this condition is that we are only looking at fields ϕ *at* temperature T and corresponding gauge transformations, which also implies -up to modifications for topologically non-trivial configurations- that α has to be periodic. Apart from this simplification, the same reasoning as for (7.20) above applies.

With a mass-like regulator, however, we can go a step further. For a regulator as in (7.11), the right hand side in (7.21) vanishes since then $[\omega,R_k]=0$. All coarse-graining dependence of (7.21) drops out for arbitrary scale k, and not only in the limit $k\to 0$. This is an immediate consequence of R_k being momentum independent and the axial gauge fixing [109–111,119], and reduces (7.20) to the standard Ward Identity in the presence of an axial gauge fixing.[13] It follows, that

$$\mathfrak{g}\,\Delta\Gamma_{k,T}[\phi] = 0 \tag{7.22}$$

and we end up with the statement that (7.13) corresponds to a gauge invariant thermal Wilsonian renormalisation group for $\Delta\Gamma_{k,T}[\phi]$ valid at any scale k.

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[13]This has recently been observed as well in [166–168].

F. Conclusions

In summary, the following picture has emerged. The Wilsonian flow equation (2.12), equipped with the mWI (7.20), allows the consistent computation of IR quantities starting with the bare action in the UV. Gauge invariance is ensured for $k \to 0$, while a gauge invariant implementation for all scales k -in the Wilsonian sense- fails due to the poor performance of a mass-like regulator.

The important new result is that the Wilsonian flow (7.15), instead, stays well-defined even for a mass-like regulator (7.13). Combined with the axial gauge fixing, it allows for a *gauge invariant* implementation for all scales k. The difference to the flow (2.12) stems now from the initial condition, which is no longer the bare action, but the $T = 0$ quantum effective action (or some approximation to it). Our approach can also be seen as the extension of [47] to gauge theories. It is straightforward to implement these ideas even in a real-time formulation.

A number of interesting projects are now waiting in line. On might compute the gluonic pressure of thermal QCD along the lines proposed in [47–50]. This approach might as well shed new light on the problem of the magnetic mass. We hope to report on these matters in future.

VIII. SUMMARY

We have reviewed various aspects of the ERG approach to non-Abelian gauge theories. The results where discussed in length at the end of every chapter. Here we want to summarise the main results of this review.

The introduction of an infra-red cut-off term quadratic in the fields leads to non-linear additional constraints for the theory, summarised in modified Ward-Takahashi and BRST identities. This is the price to pay for a method with convincing flexibility in practical computations. In applications of ERG flows to the infra-red sector of QCD one has to resort to truncations as the full problem is too hard to attack. Quantitative results obtained in truncations have to be evaluated in view of the reliability of the truncation. These reliability checks have been a key issue here. We concentrated on background field methods as they are a favourable method for ERG flows: the flow equation has a one loop structure and allows the use of heat-kernel methods even at a non-perturbative level. So far, this was done at the cost of the approximation that the difference between background field and fluctuation field is neglected. Even though this is not mandatory, the computations get far more complicated when the difference is taken into account. If this approximation is dropped completely, we are fully back to the problem of dealing with mWI/mBRST. Still, these identities have to be evaluated only for vanishing fluctuations, where all quantities are gauge invariant, e.g. the full field dependent propagator entering the flow equation transforms as a tensor under gauge transformations. These observations were put to work in the background field gauge as well as in general axial gauges.

It is believed that for the physics of the infrared regime of QCD topologically non-trivial configurations play an important rôle. Here, we computed instanton-induced terms in the effective action in the presence of an infra-red regulator. These terms are important for the explanation of chiral symmetry breaking. More generally, the computation examplifes how in general to deal with topology in this framework. In particular this includes topological defects important in most confinement scenarios.

Finally, the conceptual insights reported here were used to construct gauge invariant thermal flows on the basis of an axial gauge formulations of gauge theories. With the methods evaluated here non-perturbative computations at zero and at finite temperature are in reach.

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A. APPENDICES TO ANALYTIC METHODS IN AXIAL GAUGES

A.1. Evaluation of the one loop effective action

The calculation of the last term in (5.24) is a bit more involved. Note that the following argument is valid for $m \geq -1$, $m > -1$ is of importance for the evaluation of (5.24), $m = -1$ will be used in Appendix A A.2. We first convert the factor $\tau^{m+1/2}$ appearing in the expansion of the heat kernel using $\tau^{1/2+m} = (-1)^{m+1} \frac{\tau}{\sqrt{\pi}} \int dz \partial_{z^2}^{m+1} e^{-\tau z^2}$. We further conclude that

$$\begin{aligned} B_m^{nD} &= \frac{1}{4\pi} \int dp_n \, dz \frac{(p_n^2 - \partial_\tau)\partial_t r(p_n^2 - \partial_\tau)}{p_n^2 + (p_n^2 - \partial_\tau) r(p_n^2 - \partial_\tau)} \tau^{m+1} e^{-\tau z^2}|_{\tau=0} \\ &= \frac{(-1)^{m+1}}{4\pi} \int dp_n \, dz \, \partial_{z^2}^{m+1} \frac{\partial_t r(p_n^2 - \partial_\tau)}{p_n^2 + (p_n^2 - \partial_\tau) r(p_n^2 - \partial_\tau)} (p_n^2 - \partial_\tau) e^{-\tau z^2} \Bigg|_{\tau=0} \\ &= \frac{(-1)^{m+1}}{4\pi} \int dp_n dz \partial_{z^2}^{m+1} \frac{\partial_t r(z^2 + p_n^2)}{\frac{p_n^2}{z^2+p_n^2} + r(z^2 + p_n^2)}, \end{aligned} \tag{A.1}$$

The expression in (A.1) can be conveniently rewritten as

$$\begin{aligned} B_m^{nD} &= \frac{(-1)^{m+1}}{8\pi} \int_0^\infty dx \int_0^{2\pi} d\phi \left(\partial_x - \frac{1}{x}\alpha\partial_\alpha\right)^{m+1} \frac{\partial_t r(x)}{\alpha \sin^2\phi + r(x)} \Bigg|_{\alpha=1} \\ &= \frac{(-1)^{m+1}}{4} \int_0^\infty dx \left(\partial_x - \frac{1}{x}\alpha\partial_\alpha\right)^{m+1} \frac{\partial_t r(x)}{\sqrt{r(x)}\sqrt{r(x)+\alpha}} \Bigg|_{\alpha=1}. \end{aligned} \tag{A.2}$$

where $x = z^2 + p_n^2$ and $\sin^2\phi = p_n^2/(z^2 + p_n^2)$. It is simple to see that $-(1/x)\alpha\partial_\alpha$ is a representation of ∂_{z^2} on $\sin^2\phi = p_n^2/(z^2 + p_n^2)$ and ∂_x a representation of ∂_{z^2} on functions of x only. The expression in (A.2) is finite for all $m \geq 0$. Evidently it falls of for $x \to \infty$. For the behaviour at $x = 0$ the following identity is helpful:

$$\left(\partial_x - \frac{1}{x}\alpha\partial_\alpha\right)^{m+1} = \sum_{i=0}^{m+1} (-1)^{m+1-i} \begin{pmatrix} m+1 \\ i \end{pmatrix} \partial_x^i \left(\frac{\alpha}{x}\right)^{m+1-i} \partial_\alpha^{m+1-i}, \tag{A.3}$$

(A.3) guarantees that the integrand in (A.2) only contains terms of the form

$$\partial_x^i \left(\frac{\dot r}{\sqrt{r}\sqrt{1+r}} (x + xr)^{i-m-1} \right) \tag{A.4}$$

with $i = 0, ..., m+1$. For $x \to 0$ one has to use that $\partial_t r \to 2n^A r$ and $r \to (k^2/x)^{n^A}$. The terms of integrand in (A.2) as displayed in (A.4) are finite for $x = 0$.

We are particularly interested in B_0^{nD} relevant for the coefficient of S_A in the one loop effective action (5.24). With (A.2) it follows

$$B_0^{nD} = -\frac{1}{4}\int_0^\infty dx\,\left(\partial_x - \frac{1}{x}\alpha\partial_\alpha\right)\frac{\partial_t r(x)}{\sqrt{r(x)}\sqrt{r(x)+\alpha}}\Bigg|_{\alpha=1}$$

$$= -\frac{1}{4}\left(\frac{\partial_t r(x)}{\sqrt{r(x)}\sqrt{1+r(x)}} - 2\frac{\sqrt{r(x)}}{\sqrt{1+r(x)}}\right)_{x=0}^{x=\infty} = -\frac{1}{2}(1-n^A), \tag{A.5}$$

where we have used $\partial_t r(x) = -2x\partial_x r(x)$ and the limits $\partial_t r(x\to 0) \to 2n^A r$, $r(x\to 0)\to (k^2/x)^{n^A}$ and $r(x\to\infty)=0$.

A.2. $\bar{A}$-Derivatives

For the calculation of (5.32) the following identity is useful:

$$\mathrm{Tr}\left(\frac{\delta}{\delta A_\mu^a}\mathcal{O}\right)e^{\tau\mathcal{O}} = \frac{1}{\tau}\mathrm{Tr}\frac{\delta}{\delta A_\mu^a}e^{\tau\mathcal{O}}, \tag{A.1}$$

where we need (A.1) for $\mathcal{O} = D^2$ and $\mathcal{O} = -D_T$. Now we proceed in calculating the first term in (5.32) by using a similar line of arguments as in the calculation of (5.24) and in Appendix A A.1. We make use of the representation of $\tau^{-1} = \int_0^\infty dz \exp -\tau z$ and arrive at

$$\frac{1}{2}\mathrm{Tr}\,\partial_t\left(\frac{R_k'[D_T]}{D_T + R_k[D_T]}\frac{\delta D_T}{\delta A_\mu^a}\right) = \frac{1}{2}\mathrm{Tr}\,\partial_t\left(\frac{R_k'(-\partial_\tau)}{-\partial_\tau + R_k[-\partial_\tau]}\frac{1}{\tau}\frac{\delta}{\delta A_\mu^a}K_{-D_T}(\tau)\right)_{\tau=0}$$

$$= \frac{1}{2}\int_0^\infty\frac{dx}{x}\partial_t\left(\frac{R_k'[x]}{1+r[x]}\right)\frac{Ng^2}{16\pi^2}\frac{20}{3}\frac{\delta}{\delta A_\mu^a}\left(S_A[A]+O[g]\right)$$

$$= -\frac{Ng^2}{16\pi^2}\frac{20}{3}(1-n^A)\frac{\delta}{\delta A_\mu^a}\left(S_A[A]+O[g]\right). \tag{A.2}$$

Note that ∂_t acts as $-2x\partial_x$ on functions which solely depend on x/k^2. The term $R'/(1+r)$ is such a function. The second term can be calculated in the same way leading to

$$\frac{1}{4}\mathrm{Tr}\;\partial_t\left\{\frac{-R_k'[D^2]}{-D^2+R_k[-D^2]}\frac{\delta}{\delta A_\mu^a}D^2\right\} = \frac{1}{4}\int_0^\infty\frac{dx}{x}\;\partial_t\left(\frac{R_k'[x]}{1+r[x]}\right)\frac{Ng^2}{16\pi^2}\frac{4}{3}\frac{\delta}{\delta A_\mu^a}\left(S_A[A]+O[g]\right)$$

$$= -\frac{Ng^2}{16\pi^2}\frac{2}{3}(1-n^A)\frac{\delta}{\delta A_\mu^a}\left(S_A[A]+O[g]\right). \tag{A.3}$$

The calculation of the last term in (5.32) is a bit more involved, but boils down to the same structure as for the other terms. Along the lines of Appendix A A.1 it follows that this term can be written as

$$\frac{1}{8}\mathrm{Tr}\,\partial_t\left\{\frac{-R_k'[-D^2]}{(-nD)^2+R_k[-D^2]}\frac{\delta D^2}{\delta A_\mu^a}\right\} = \frac{1}{8}\mathrm{Tr}\;\partial_t\left\{\int dp_n\frac{R_k'[p_n^2-\partial_\tau]}{p_n^2+R_k[p_n^2-\partial_\tau]}\frac{\tau^{-1/2}}{\sqrt{\pi}}\frac{\delta}{\delta A_\mu^a}K_{D^2}(\tau)\right\}_{\tau=0}$$

$$= -\frac{1}{8}\int_0^\infty\frac{dx}{x}\partial_t\frac{R_k'}{\sqrt{r}\sqrt{1+r}}\frac{Ng^2}{16\pi^2}\frac{4}{3}\frac{\delta}{\delta A_\mu^a}\left(S_A[A]+O[g]\right),$$

$$= \frac{Ng^2}{16\pi^2}\frac{1}{3}(1-n^A)\frac{\delta}{\delta A_\mu^a}\left(S_A[A]+O[g]\right) \tag{A.4}$$

Note that when rewriting the left hand side of (A.4) as a total derivative w.r.t. A this also includes a term which stems from $\frac{\delta}{\delta A}(nD)^2$. This, however, vanishes because it is odd in p_n.

B. APPENDICES FOR $U(1)$-PROBLEM

B.1. Zero modes of the regularised Dirac operator

In this appendix we discuss the behaviour of the regularised fermionic functional integral. We are interested in

$$Z_{\psi,k}[a,\bar{A},\bar{\eta},\eta] = \frac{1}{\mathcal{N}_k}\int d\bar{\chi}\, d\chi \exp\int_x \left\{-\bar{\chi}(\not{D}(a+\bar{A}) + R^{\psi}[\not{D}(\bar{A})])\chi + \bar{\eta}\,\chi + \bar{\chi}\,\eta\right\}, \tag{B.1}$$

with the normalisation given by $\mathcal{N}_k = Z_{\psi,k}[0,0,0,0]$. The Dirac operator $\not{D}[a+\bar{A}]$ has one zero eigenvalue (non-degenerate) for a configuration $a+\bar{A}$ with instanton number ± 1. First we discuss the ultraviolet regularisation of (B.1). For this purpose we concentrate on

$$Z_{\psi,k}[a,\bar{A},0,0] = \left(\frac{\det_\Lambda\left(\not{D}(a+\bar{A}) + R^{\psi}[\not{D}(\bar{A})]\right)}{\det_\Lambda\left(\not{D}(0) + R^{\psi}[\not{D}(0)]\right)}\right)^{N_f}. \tag{B.2}$$

The subscript Λ is related to the fact that an ultraviolet regularisation of the determinants is needed. An appropriate regularisation in (B.2) would be the ζ-function regularisation. More generally, high momenta should be suppressed in a gauge-invariant way. These conditions are satisfied by the regularisations $g_\Lambda[\not{D}(a+\bar{A}) + R^{\psi}[\not{D}(\bar{A})]]$ of the Dirac operator with the properties

$$g_\Lambda[0] = 0, \qquad g_\Lambda[x] \stackrel{x^2 \gg \Lambda^2}{\longrightarrow} 0, \qquad \{g_\Lambda[x],\gamma_5\} = 0 \ \text{ if } \ \{x,\gamma_5\} = 0. \tag{B.3}$$

An explicit g_Λ satisfying (B.3) is

$$g_\Lambda[x] = x e^{-x^2/\Lambda^2}. \tag{B.4}$$

g_Λ does not influence the infrared behaviour of the Dirac operator. In particular it vanishes for zero modes. Hence we use $\not{D}(a+\bar{A}) + R^{\psi}[\not{D}(\bar{A})]$ for the discussion of the zero mode. $\not{D}(a+\bar{A}) + R^{\psi}[\not{D}(\bar{A})]$ is not invertible on the one-dimensional subspace of the zero mode χ_0 of $\not{D}(a+\bar{A})$, i.e. acting with the inverse of $\not{D}(a+\bar{A}) + R^{\psi}[\not{D}(\bar{A})]$ on χ_0 does not lead to a square-integrable function. This indicates the existence of a zero mode. To prove the existence of a zero mode for the infrared regularised Dirac operator we introduce

$$H_t = \not{D}(a+\bar{A}) + tR^{\psi}[\not{D}(\bar{A})]. \tag{B.5}$$

This operator is the usual Dirac operator for $t=0$ and the infra-red regularised Dirac operator for $t=1$. Now we concentrate on the evaluation of the zero eigenvalue for $t \in [0,1]$. We are dealing with the eigenfunctions $\psi_n(t)$ of H_t with

$$H_t\psi_n(t) = E_n(t)\psi_n(t). \tag{B.6}$$

Since R^ψ is a compact operator, the normalisability of $\psi_n(t)$ is guaranteed for every t. Moreover, the Taylor series in t of E_n, ψ_n are convergent. In the one instanton sector there is one eigenvector ψ_0 with

$$H_0\psi_0(0) = 0. \tag{B.7}$$

$E_0(0) = 0$. We prove by induction that all derivatives of E_0,

$$E_0^{(n)}[t] = \partial_t^n E_0(t), \tag{B.8}$$

are vanishing at $t = 0$. This leads to $E_0(t) = 0$ for $t \in [0,1]$. We start with $E_0^{(0)} = 0$ and assume that $E_0^{(m)} = 0$ for all $m \leq n-1$. It follows that

$$\partial_t^m \left[H_t\psi_0(t)\right]_{t=0} = 0 \quad \forall m \leq n-1 \tag{B.9}$$

or

$$\not{D}(a+\bar{A})\psi_0^{(m)}(0) = -mR^\psi[\not{D}(\bar{A})]\psi_0^{(m-1)}(0) \quad \forall m \leq n-1, \psi_0^{(m)}(t) = \partial_t^m\psi_0(t). \tag{B.10}$$

As an intermediate result we prove that the $\psi_0^{(m)}(0)$ are chirality eigenstates with the same chirality as $\psi_0(0)$ for all $m \leq n-1$. We deduce from (B.10)

$$\gamma_5\psi_0^{(m)} = -m\mathcal{P}\frac{1}{\not{D}(a+\bar{A})}\mathcal{P}R^\psi[\not{D}(\bar{A})]\gamma_5\psi_0^{(m-1)}(0) + \gamma_5(1-\mathcal{P})\psi_0^{(m)}, \tag{B.11}$$

where $\mathcal{P}$ is the projector on the space of non-zero modes of $\not{D}(a+\bar{A})$ and we have used (see (6.4,6.8))

$$\{R^\psi[\not{D}(\bar{A})], \gamma_5\} = \{\not{D}(a+\bar{A}), \gamma_5\} = [\mathcal{P}, \gamma_5] = 0, \mathcal{P} = \left(\not{D}\frac{1}{\not{D}^2}\not{D}\right)(a+\bar{A}). \tag{B.12}$$

However $\psi_0^{(0)}(0)$ is a chirality eigenstate, $\gamma_5\psi_0(0) = \pm\psi_0(0)$. Furthermore, $(1-\mathcal{P})\psi_0^{(m)}(0)$ is proportional to $\psi_0(0)$. Thus it follows from (B.11) that $\psi_0^{(m)}(0)$ has the same chirality as $\psi_0(0)$, if $\psi_0^{(m-1)}(0)$ has this property. Starting iteratively with $m = 1$, the claimed chirality properties follow for all $m \leq n-1$.

With this result and (B.9,B.12) we prove $E_0^{(n)}(0) = 0$:

$$\begin{aligned} E_0^{(n)}(0) =& \partial_t^n\langle\psi_0(t), H_t\psi_0(t)\rangle_{t=0} \\ \text{eq. } (B.9) \to \quad =& n\langle\psi_0(t), R^\psi[\not{D}(\bar{A})]\partial_t^{n-1}\psi_0(t)\rangle_{t=0} \\ \text{chirality of } \psi_0^{(n-1)}(0) \to \quad =& n\langle\gamma_5\psi_0(0), R^\psi[\not{D}(\bar{A})]\gamma_5\psi_0^{(n-1)}(0)\rangle \\ \text{eq. } (B.12) \to \quad =& -n\langle\psi_0(0), R^\psi[\not{D}(\bar{A})]\psi_0^{(n-1)}(0)\rangle \\ =& 0. \end{aligned} \tag{B.13}$$

Therefore the $E_0^{(n)}(0)$ vanish for all $n \in \mathbb{N}$ which leads to $E_0(t) = 0$, $t \in [0,1]$. This proves that $\psi_0(t)$ is a zero mode for all t, in particular for $t = 1$.

With these preliminaries we can easily factorise the fermionic zero mode contribution as in the case without regularisation. It follows for topologically non-trivial configurations $a + \bar{A}$

$$Z_{\psi,k}[a,\bar{A},\bar{\eta},\eta] = \frac{1}{\mathcal{N}_k}\int d\bar{\chi}'\, d\chi' \exp\Big\{\int_x \bar{\chi}'\left(\not{D}(a+\bar{A}) + R^\psi[\not{D}(\bar{A})]\right)\chi' \tag{B.14}$$

$$+\int_x (\bar{\eta}'\,\chi' + \bar{\chi}'\,\eta')\Big\}\int d\bar{\chi}_0\, d\chi_0 \exp\int_x (\bar{\eta}\,\chi_0 + \bar{\chi}_0\,\eta) \tag{B.15}$$

with

$$\left(\not{D}(a+\bar{A}) + R^\psi[\not{D}(\bar{A})]\right)\chi_0 = 0. \tag{B.16}$$

B.2. Zero mode contribution to leading order of $1/k$

We recall the expression for P_k (see (6.22))

$$P_k[\phi,\bar{A}] = \int d\mu_1(\theta)\prod_{s=1}^{N_f}\int_x \bar{\eta}_s\,\phi_0\int_x \phi_0^+\,\eta_s + \text{h.c.} \tag{B.17}$$

We shall argue that P_k depends only to sub-leading order in $1/k$ on $A,\bar{A}$. For that purpose we concentrate on the zero mode equation (B.16) with a purely topological configuration $a = a_I$. In the limit $k \to \infty$ only instantons $a_I(x,\rho)$ with width $\rho \sim 1/k$ contribute to (B.17) due to the infrared regularisation of the gauge field present in $d\mu_1$ (see Appendix B B.3). Note that $a_I(x,\rho) = a_I(x/\rho,1)/\rho$ (see e.g. (B.31),(B.34)) and $R_k^\psi[\not{\partial}_x] = R_{k\rho}^\psi[\not{\partial}_{x/\rho}]/\rho$ (see (6.8)). Hence after multiplying (B.16) with $\rho \sim 1/k \to 0$ and scaling $x \to \rho x$ we conclude that the fermionic zero mode depends on $A,\bar{A}$ only to sub-leading order. Thus P_k is $A,\bar{A}$-independent to leading order. The ghosts are irrelevant for the present questions. We write

$$P_k[\phi,\bar{A}] = P_k[\psi,\bar{\psi}] + O(1/k). \tag{B.18}$$

P_k is non-local. In the limit $k \to \infty$ it is possible to write it as a sum of a local contribution and terms which are suppressed by powers of $1/k$. In this limit we also calculate the normalisation of P_k.

The measure $d\mu_1$ contains integrations over collective coordinates. The interesting collective coordinates are the centre of the instanton z, the width ρ and the global gauge rotations g. The explicit derivation of the ρ,z dependence of $d\mu_1$ is done in Appendix B B.3. Moreover the instanton a_I and the fluctuations a' decouple in the limit $k \to \infty$ which can be used to effectively remove the gauge fixing term for a_I (see derivation of (6.26)). Hence $d\mu_1$ also includes a measure dg_k of local gauge degrees of freedom in this limit. Note that the cut-off term for a_I singles out those local gauge degrees of freedom dependent on large momenta. However, this will not effect the following arguments.

We will use well-known results from instanton calculations. For details we refer the reader to the literature [171,172,164]. The normalised fermionic zero mode is given by

$$\phi^A_{0,\xi}(x;z,\rho) = \frac{\sqrt{2}}{\pi}\frac{\rho}{((x-z)^2+\rho^2)^{\frac{3}{2}}}u^A_\xi, \qquad \sum_A u^A \times \bar{u}^A = \frac{1-\gamma_5}{2}, \qquad \|\phi_0\| = 1 \tag{B.19}$$

and gauge transformations $g(x)\phi_0(x;z,\rho)$ of (B.19), where $g(x)$ could be either $g_k(x)$ or a global gauge rotation g. With $\rho \sim 1/k \to 0$ we write

$$\int_x \bar{\eta}_s\, \phi_0 = \frac{\sqrt{2}}{\pi}\int_x \frac{\rho}{((x-z)^2+\rho^2)^{\frac{3}{2}}}\bar{\eta}_s(x)\, u = \frac{\sqrt{2}}{\pi}\int_x \frac{\rho}{(x^2+\rho^2)^{\frac{3}{2}}}\bar{\eta}_s(x+z)\, u. \tag{B.20}$$

We are interested in the limit $\rho \to 0$. Therefore we calculate (B.20) in an expansion about $\rho = 0$. The coefficient related to the power ρ^0 vanishes. The coefficient proportional to ρ is determined by scaling (B.20) with ρ^{-1}

$$\lim_{\rho\to 0}\frac{1}{\rho}\frac{\sqrt{2}}{\pi}\int_x \frac{\rho}{(x^2+\rho^2)^{\frac{3}{2}}}\bar{\eta}_s(x+z)\, u = \frac{\sqrt{2}}{\pi}\int_x \left(\frac{1}{x^2}\right)^{\frac{3}{2}}\bar{\eta}_s(x+z)\, u. \tag{B.21}$$

The term of order ρ^2 is calculated by subtracting (B.21) times ρ from (B.20). It follows that

$$\lim_{\rho\to 0}\frac{1}{\rho}\frac{\sqrt{2}}{\pi}\int_x \left[\frac{1}{(x^2+\rho^2)^{\frac{3}{2}}} - \left(\frac{1}{x^2}\right)^{\frac{3}{2}}\right]\bar{\eta}(x+z)_s\, u = -2^{5/2}\pi\bar{\eta}_s(z)\, u \tag{B.22}$$

The final result for the contribution of the fermionic zero mode of an instanton with width $\rho \sim 1/k$ and centre z is

$$\int_x \bar{\eta} g^{-1}\phi_0 = \rho\frac{\sqrt{2}}{\pi}\int_x \left(\frac{1}{x^2}\right)^{\frac{3}{2}}\bar{\eta}_s(x+z)g^{-1}(x+z)\, u - \rho^2 2^{5/2}\pi\bar{\eta}_s(z)g^{-1}(z)\, u + O(\rho^3), \tag{B.23}$$

Contributions to P_k dependent on the first (non-local) term on the right hand side of (B.23) vanish because of the integration over local gauge degrees of freedom g_k present in $d\mu_1$. For the evaluation of the second term on the right hand side of (B.23) we concentrate on the integration over global gauge rotations. We get from (B.17) by using (B.23)

$$P_k[\psi,\bar{\psi}] = \int d^4z \int d\bar{\mu}_1(\theta)(2^5\pi^2\rho^4)^{N_f}\int dg \prod_{s=1}^{N_f}\Big(\bar{\eta}^s(z)g^{-1}u\Big)\Big(\bar{u}\, g\eta^s(z)\Big) + \text{h.c.} + O(1/k), \tag{B.24}$$

where dg is the measure of global gauge rotations and d^4z is the measure of the centre of the instanton:

$$d\mu_1(\theta) = d\bar{\mu}_1(\theta)\, dg\, d^4z, \qquad\qquad \int dg = 1. \tag{B.25}$$

For the evaluation of the g-integration in $SU(N)$ we use [42]

$$\int dg \prod_{i=1}^{N_f} g^{-1}_{A_i\bar{A}_i} g_{\bar{B}_i B_i} = a[N,N_f] \left(\sum_\sigma \prod_{i=1}^{N_f} \delta_{A_i B_{\sigma(i)}} \delta_{\bar{A}_i \bar{B}_{\sigma(i)}} + \frac{1}{N} \mathcal{O}_{A_1 \bar{B}_1 \cdots \bar{A}_{N_f} \bar{B}_{N_f}} \right), \tag{B.26}$$

where σ are the permutations of $(1, ..., N_f)$. The tensor $\mathcal{O}$ is suppressed by $1/N$ and only consists of products of Kronecker deltas $\delta_{A_i B_j} \delta_{\bar{A}_n \bar{B}_m}$. Both $a[N,N_f]$ and $\mathcal{O}$ are complicated functions of N, N_f. With (B.19,B.26) we get

$$\int dg \prod_{s=1}^{N_f} \bar{\eta}_s g\, u\bar{u}\, g^{-1} \eta_s \stackrel{k\to\infty}{\longrightarrow} a[N,N_f] \det_{s,t} \bar{\eta}_s^{A_s} \frac{1-\gamma_5}{2} \eta_t^{B_t} \left(\delta^{A_s B_t} + \frac{1}{N} \mathcal{U}^{A_1 B_1 \cdots A_{N_f} B_{N_f}} \right). \tag{B.27}$$

The tensor $\mathcal{U}$ is related to $\mathcal{O}$ and involves only products of Kronecker deltas $\delta^{A_i B_j}$. However from now on we drop the term dependent on $\mathcal{U}$. This is a suitable approximation within a $1/N$-expansion since it carries the same flavor structure as the leading term but is suppressed by $1/N$. Note however that this is done more for the sake of convenience and the tensor structure can be added without changing the conclusions of the present paper. Moreover even though tedious the calculation of $\mathcal{U}$ is straightforward. This leads to

$$P_k[\psi,\bar{\psi}] \stackrel{k\to\infty,\ N\gg 1}{\longrightarrow} \int d^4z \int d\bar{\mu}_1(\theta)\, (2^5\pi^2\rho^4)^{N_f} a[N,N_f] \det_{s,t} \bar{\eta}_s^A(z) \frac{1-\gamma_5}{2} \eta_t^A(z) + \text{h.c.} \tag{B.28}$$

Now we are able to give a final expression for P_k

$$P_k[\psi,\bar{\psi}] = \int_z \Delta[k,\theta] \det_{s,t} \bar{\eta}_s(z) \frac{1-\gamma_5}{2} \eta_t(z) + O(\Delta[k,\theta]/k) \tag{B.29}$$

with

$$\Delta[k,\theta] = \int d\bar{\mu}_1(\theta)\, (2^5\pi^2\rho^4)^{N_f} a[N,N_f] \ \sim k^{-5N_f+4}. \tag{B.30}$$

The gauge field cut-off ensures the finiteness of $\Delta[k,\theta]$. Then the k-dependence follows by dimensional arguments.

B.3. Properties of the gauge field regularisation

In this section we examine the properties of the cut-off term for the gauge field in the 1 instanton sector. A general instanton is given by

$$A^a_{I,\mu}(x;z,\rho) = \eta^a_{\mu\nu}\frac{(x-z)_\nu}{(x-z)^2+\rho^2} \tag{B.31}$$

and global gauge rotations of $A^a_{I,\mu}(x;z,\rho)$. Here $\eta^a_{\mu\nu}$ are the 't Hooft symbols [171]. In order to stay in contact with [53,54] we first discuss this approach where the background field is missing. In this case the field a consists of both, the instanton (B.31) and the fluctuations about the instanton. The configurations (B.31) are not square-integrable because of their infrared behaviour. To see this, let us recall the cut-off term for the gauge field (see (3.8a) and (2.6)) for vanishing background field $\bar{A} = 0$. Then the regulator is just $R^{\mathcal{Q}}[0] = R_k(p^2) = p^2 r^{\mathcal{Q}}(p^2)\mathbb{1}$. The region of large x corresponds to small momenta. With $R_k(p^2) \propto k^2$ in the limit $p^2/k^2 \to 0$ the cut-off term is bounded from below by

$$\Delta S_A[a,0] \leq C[a]\frac{1}{2}\int_x a\, R_k(p^2)\, a \tag{B.32}$$

with $C[a] > 0$. In other words, if a configuration a is ultraviolet finite but is not square-integrable due to infrared divergences, then $\Delta S_A[a,0]$ diverges. These configurations have zero measure in the path integral, since

$$\exp\{-\Delta S_A[a,0]\} = 0. \tag{B.33}$$

We conclude that only configurations which decrease faster than $1/x^2$ can contribute to the infrared regularised path integral. This reflects the fact that within this particular approach [53,54] the cut-off term introduces trivial (infrared) boundary-conditions. Thus the infrared cut-off term (without background field dependence) introduces a constraint on the class of gauge fixings, i.e. allowing only for those compatible with trivial infrared behaviour. Instantons in the singular gauge satisfy this condition (see for example [164]). They are given by

$$a^a_{I,\mu}(x;z,\rho) = \eta^a_{\mu\nu}\frac{(x-z)_\nu}{(x-z)^2}\frac{\rho^2}{(x-z)^2+\rho^2}. \tag{B.34}$$

These configurations are square-integrable, and so $\Delta S_A[a,\bar{A}]$ is finite. We write explicitly for an instanton $a_I(x,z,\rho)$ with centre z and width ρ

$$\Delta S_A[a_I(x,z,\rho),0] = \frac{1}{2}\int_{\tilde{x}} a_I(\tilde{x},0,1)\, R^A_{k\rho}[-\partial^2_{\tilde{x}}]\, a_I(\tilde{x},0,1), \tag{B.35}$$

where we have used the translation invariance of the cut-off term for $D_T(0)$ and have changed the variable x to $\tilde{x} = (x+z)/\rho$. Using the limit in (B.32) we get

$$\exp\{-\Delta S_A[a_I(x,z,\rho),0]\} \stackrel{k\rho\gg 1}{\longrightarrow} \exp\left\{-(k\rho)^2\frac{1}{2}\int_{\tilde{x}} a_I(\tilde{x},0,1)\, a_I(\tilde{x},0,1)\right\} \sim e^{-\#(k\rho)^2}. \tag{B.36}$$

Hence in the limit $k \to \infty$ only instantons with width $\rho \sim k$ contribute. This result extends easily to the more general case with a non-vanishing background field $\bar{A}$. Moreover, the constraint on the class of gauge fixings is related entirely to the introduction of trivial infrared boundary conditions by choosing $\bar{A} = 0$. Following the background field approach to instantons [171,172] one chooses the background field $\bar{A}$ as the configuration (B.31). In this case the field a consists of fluctuations about the instanton which are square-integrable by definition (after a complete gauge fixing). This leads immediately to (B.36).

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