Little Higgs at Colliders

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You are looking at a script following my next attempt to give an introduction into the phenomenology of little–Higgs models to our SUPA graduate students. Again, assume it is full of mistakes and typos, because I am using these lectures to learn new things myself. If you find things unclear or wrong or stupid, please drop me an email, so I can work on it for my next lecture on it.

Contents

I. ELECTROWEAK SYMMETRY BREAKING

To discuss the motivation of a new–physics model, like the Little–Higgs models, we have to sketch the Standard Model Lagrangian, including mass terms. These introductory comments are particularly nicely presented in Wolfgang Kilian's book, and I will try to follow his conventions. Fermion fields have mass dimension 3/2, so it is easy to add mass terms to the dimension-4 Lagrangian. The only thing we have to make sure is that we combine the left– and right–handed doublet and singlets properly

$$\mathcal{L}_3 \sim -\overline{Q}_L M_Q Q_R - \overline{L}_L M_L L_R + ... \tag{1}$$

Dirac mass terms simply link $SU(2)$ doublet fields for leptons and quarks with right–handed singlets and gives all fermions in the Standard Model masses. In general, these mass terms can be diagonal matrices in generation space, which implies that we might have to rotate the fermion field from an interaction basis into the mass basis where these mass matrices are diagonal. The only problem with these mass terms is that they are not gauge invariant... The interaction of fermions with gauge bosons is most easily written in terms of covariant derivatives. The terms

$$\mathcal{L}_4 \sim \overline{Q}_L i \not{D} Q_L + \overline{Q}_R i \not{D} Q_R + \overline{L}_L i \not{D} L_L + \overline{L}_R i \not{D} L_R - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} ... \tag{2}$$

describe electromagnetic interactions using such a covariant derivative $D_\mu = \partial_\mu + ieqA_\mu$ with the photon field collected in the field–strength tensor $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The same form works for the weak interactions, except that the weak interaction knows about the chirality of the fermion fields, so we have to distinguish $\not{D} \to \not{D}_{L,R}$. The covariant derivatives in terms of the $SU(2)$ basis matrices read

$$\begin{aligned}
D_{L\mu} &= \partial_\mu + ieqA_\mu + ig_Z \left(-qs_W^2 + \frac{\tau^3}{2} \right) + i\frac{g}{\sqrt{2}} \left(\tau^+ W_\mu^+ + \tau^- W_\mu^- \right) \\
D_{R\mu} &= D_{L\mu} \Big|_{\tau \equiv 0} \\
\tau^+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
\tau^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{3}$$

Note that we can write the Pauli matrices as $\tau^{1,2,3}$ as well as $\tau^{+,-,3}$. The latter form of the generators corresponds to the two charged and one neutral vector bosons. While the usual basis is written in terms of complex numbers, the second set of generators reflects the fact that for $SU(2)$ as for any $SU(N)$ we can find a set of real generators in the adjoint representation. When we exchange the two bases we only have to make sure we get the factors $\sqrt{2}$ right

$$\begin{aligned}
&\sqrt{2} \left(\tau^+ W_\mu^+ + \tau^- W_\mu^- \right) = \sqrt{2} \begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 0 & 0 \\ W_\mu^- & 0 \end{pmatrix} \equiv \tau^1 W_\mu^1 + \tau^2 W_\mu^2 = \begin{pmatrix} 0 & W_\mu^1 \\ W_\mu^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iW_\mu^2 \\ iW_\mu^2 & 0 \end{pmatrix} \\
&\iff W_\mu^+ = \frac{1}{\sqrt{2}} \left(W_\mu^1 - iW_\mu^2 \right) \qquad W_\mu^- = \frac{1}{\sqrt{2}} \left(W_\mu^1 + iW_\mu^2 \right)
\end{aligned} \tag{4}$$

The third term in the Standard Model Lagrangian we have to have a close look at is the dimension-2 mass term for gauge bosons which we know as

$$\mathcal{L}_2 \sim M_W^2 W^{+,\mu} W_\mu^- + \frac{1}{2} M_Z^2 Z^\mu Z_\mu. \tag{5}$$

The factor 1/2 in front of the W mass corresponds to the factors $1/sqrt2$ in the $SU(2)$ generators $\tau^\pm$. Of course, in the complete Standard Model Lagrangian there are many additional terms, *e.g.* kinetic terms of all kinds, but they do not affect our discussion of $U(1)_Y$ and $SU(2)_L$ gauge invariance. We know already that the problems with gauge invariance lies in the dimension-2 and dimension-3 mass terms.

Again following Wolfgang's book we write down the local $U(1)_Y$ and $SU(2)_L$ transformations. We start with a slightly complicated–looking way of writing the abelian hypercharge $U(1)$ transformations, making it more obvious how they mix with the neutral component of $SU(2)$ to give the electric charge

$$
\begin{aligned}
V^\dagger(x) = \exp\left(\frac{i}{2}\beta(x)\tau^3\right) \quad &\Leftrightarrow \quad V(x) = \exp\left(-\frac{i}{2}\beta(x)\tau^3\right) \\
\exp(-i\beta q)\exp\left(\frac{i}{2}\beta\tau^3\right) &= \exp\left(-i\beta\frac{\mathbb{1}+\tau^3}{2}\right)\exp\left(\frac{i}{2}\beta\tau^3\right) \qquad q \equiv \frac{y\mathbb{1}+\tau^3}{2} \\
&= \exp\left(-i\frac{\beta}{2}y\mathbb{1} - i\beta\frac{\tau^3}{2} + i\beta\frac{\tau^3}{2}\right) \qquad y_Q = \frac{1}{3} \quad y_L = -1 \\
&= \exp\left(-i\frac{\beta}{2}y\mathbb{1}\right)
\end{aligned}
\tag{6}
$$

The numbers $y_{Q,L}$ are the quark and lepton hypercharges of the $U(1)$ symmetry in the Standard Model. Properly combined with the isospin they give the correct electric charges $q_{Q,L}$. From the manipulations above we see that the combination of $\exp(-i\beta q)$ and $V(x)$ written down in the beginning is proportional to $\exp(\mathbb{1})$ and hence an abelian transformation. When combining the different exponentials a la Baker–Campbell–Hausdorff we have to remember that $\mathbb{1}$ commutes with any matrix, as does $\exp(-i\beta y_Q\mathbb{1}/2)$. Left and right–handed quark and lepton fields transform under the electric–charge $U(1)$ as

$$
\begin{aligned}
L_L \to \exp\left(-i\frac{\beta}{2}y_L\mathbb{1}\right)L_L \qquad & Q_L \to \exp\left(-i\frac{\beta}{2}y_Q\mathbb{1}\right)Q_L \\
L_R \to \exp\left(-i\frac{\beta}{2}q_L\mathbb{1}\right)L_R \qquad & Q_R \to \exp\left(-i\frac{\beta}{2}q_Q\mathbb{1}\right)Q_R
\end{aligned}
\tag{7}
$$

Similarly, we define the local (adjoint) weak $SU(2)$ transformation

$$
U(x) = \exp\left(-i\alpha^a(x)\frac{\tau^a}{2}\right) \qquad a = 1,2,3 \tag{8}
$$

which only transforms the left–handed fermion fields and leaves the right–handed fields untouched

$$
\begin{aligned}
L_L \to UL_L \qquad & Q_L \to UQ_L \\
L_R \to L_R \qquad & Q_R \to Q_R
\end{aligned}
\tag{9}
$$

It is obvious that left–right mass terms are not invariant under this left–handed $SU(2)$ gauge transformation

$$
\overline{Q}_L M_Q Q_R \to_U \overline{Q}_L U^{-1} M_Q Q_R \neq \overline{Q}_L M_Q Q_R \tag{10}
$$

In other words, to write a gauge–invariant Lagrangian for massive fermions (and vector bosons) we have to add something to our minimal Standard Model Lagrangian. Note that this addition does not have to be a fundamental scalar Higgs field, dependent on how picky we are with the properties of our new Lagrangian beyond its gauge invariance.

A. Sigma Model

One way of solving this problem which at this point almost looks like a cheap trick is to introduce an additional field $\Sigma(x)$. Properties like the quantum numbers of Σ will become obvious from it's appearance in the Lagrangian. Obviously, the equation of motion for the Σ field will also have to follow from the way we introduce it in the Lagrangian. We first use it to modify the fermionic mass term and make it gauge invariant under the weak $SU(2)$ transformation

$$
\overline{Q}_L \Sigma M_Q Q_R \to_U \overline{Q}_L U^{-1}\Sigma^{(U)} M_Q Q_R \equiv \overline{Q}_L \Sigma M_Q Q_R \qquad \Longleftrightarrow \qquad \Sigma \to \Sigma^{(U)} = U\Sigma \tag{11}
$$

The first thing we notice about Σ is it mass dimension $m^0 = 1$. The same we can do for the $SU(2)$ transformation V which mixes later on with the hypercharge

$$\begin{aligned}
\overline{Q}_L \Sigma M_Q Q_R \to_V & \overline{Q}_L V \exp(i\beta q)\, \Sigma^{(V)} M_Q \exp(-i\beta q)\, Q_R \\
=& \overline{Q}_L \Sigma^{(V)} V \exp(i\beta q)\, M_Q \exp(-i\beta q)\, Q_R \qquad \text{assuming } M_Q \text{ diagonal} \\
=& \overline{Q}_L \Sigma^{(V)} V M_Q Q_R \\
\equiv& \overline{Q}_L \Sigma V M_Q Q_R \\
\Sigma \to \Sigma^{(V)} = \Sigma V^\dagger \qquad & \Longleftrightarrow \qquad \boxed{\Sigma \to U \Sigma V^\dagger}
\end{aligned} \tag{12}$$

This means for any Σ with this transformation property the $\mathcal{L}_3$ part of the Lagrangian has the required $U(1) \times SU(2)$ symmetry. Note that from the way it transforms Σ is a 2×2 matrix with mass dimension zero. We have shown by construction that including a Σ field in the fermionic mass term indeed gives a $U(1)_Y$ and $SU(2)_L$-invariant Lagrangian, without saying much about possible representations of Σ for example in terms of physical fields

$$\boxed{\mathcal{L}_3 \sim -\overline{Q}_L \Sigma M_Q Q_R - \overline{L}_L \Sigma M_L L_R + \text{h.c.} + ...} \tag{13}$$

To write down a gauge–invariant gauge–boson mass we start with the left–handed covariant derivative

$$\begin{aligned}
D_{L\mu} &= \partial_\mu + ig' \left(q - \frac{\tau^3}{2} \right) B_\mu + igW_\mu^a \frac{\tau^a}{2} \\
&= \partial_\mu + ig' \frac{y}{2} B_\mu + igW_\mu^a \frac{\tau^a}{2}
\end{aligned} \tag{14}$$

We skip the reasoning for this, but whoever is interested can show that the covariant derivative acting on the Σ field in the gauge–symmetric Lagrangian has to be

$$D_\mu \Sigma = \partial_\mu \Sigma - ig' \Sigma B_\mu \frac{\tau^3}{2} + igW_\mu^a \frac{\tau^a}{2} \Sigma \tag{15}$$

Instead of showing how we would have to write a gauge–invariant mass terms for the W and Z bosons we start with a promising ansatz. If we introduce $V_\mu \equiv \Sigma (D_\mu \Sigma)^\dagger$ and $T = \Sigma \tau^3 \Sigma^\dagger$ we can write the boson mass term as

$$\boxed{\mathcal{L}_2 = -\frac{v^2}{4} \, \text{Tr}[V_\mu V^\mu] - \beta' \frac{v^2}{8} \, \text{Tr}[T V_\mu] \, \text{Tr}[T V^\mu]} \tag{16}$$

The trace acts on the 2×2 $SU(2)$ matrices. We will show the specific form soon for the different gauge choices.

The problems in our Σ–field model are additional terms of mass dimension 4 we can write down using the (dimensionless) field Σ and which are gauge invariant. For such terms we have to find a selection rule or symmetry which only allows the Σ terms in the Lagrangian which we need to include massive fields. Without the trace we can construct terms which are forbidden by gauge invariance

$$\Sigma^\dagger \Sigma \to \; (U \Sigma V^\dagger)^\dagger (U \Sigma V^\dagger) = V \Sigma^\dagger U^\dagger U \Sigma V^\dagger = V \Sigma^\dagger \Sigma V^\dagger \neq \Sigma^\dagger \Sigma \tag{17}$$

On the other hand, $\text{Tr}(\Sigma^\dagger \Sigma) = \text{Tr}(V \Sigma^\dagger \Sigma V^\dagger) = \text{Tr}[\Sigma^\dagger \Sigma]$ is gauge invariant, which allows the additional potential terms (terms with no derivatives)

$$\boxed{\mathcal{L}_\Sigma = -\frac{\mu^2 v^2}{4} \text{Tr}(\Sigma^\dagger \Sigma) + \frac{\lambda v^4}{16} \left(\text{Tr}(\Sigma^\dagger \Sigma) \right)^2} \tag{18}$$

with properly chosen prefactors μ, v, λ. The factors μ and v have mass dimension one while λ has mass dimension zero. To give mass to the gauge bosons we have to assume that $\text{Tr}(\Sigma^\dagger \Sigma)$ assumes a finite value after we deal properly with the field Σ. The simplest way to achieve this is to generally assume

$$\Sigma(x) = \mathbb{1} \tag{19}$$

This assumption is called unitary gauge. In this gauge the covariant derivative again becomes

$$D_\mu \Sigma = igW_\mu^a \frac{\tau^a}{2} - ig' B_\mu \frac{\tau^3}{2} \tag{20}$$

Moreover, we can simply compute the auxiliary field V_μ in unitary gauge

$$\begin{aligned} V_\mu &= -igW_\mu^a \frac{\tau^a}{2} + ig' B_\mu \frac{\tau^3}{2} \\ &= -igW_\mu^+ \frac{\tau^+}{\sqrt{2}} - igW_\mu^- \frac{\tau^-}{\sqrt{2}} - igW_\mu^3 \frac{\tau^3}{2} + ig' B_\mu \frac{\tau^3}{2} \\ &= -i\frac{g}{\sqrt{2}} \left(W_\mu^+ \tau^+ - W_\mu^- \tau^- \right) - ig_Z Z_\mu \frac{\tau^3}{2} \qquad \text{with } Z_\mu = c_W W_\mu^3 - s_W B_\mu \text{ and } g_Z = \frac{g}{c_W}, g' = \frac{s_W}{c_W} g \end{aligned} \tag{21}$$

This field gives for the first of the two terms in the gauge–boson mass Lagrangian

$$\begin{aligned} \mathrm{Tr}[V_\mu V^\mu] &= -2\frac{g^2}{2} W_\mu^+ W_\mu^- \mathrm{Tr}(\tau^+\tau^-) - \frac{g_z^2}{4} Z_\mu Z_\mu \mathrm{Tr}(\tau_3^2) \\ &= -g^2 W_\mu^+ W_\mu^- - \frac{g_z^2}{2} Z_\mu Z_\mu \end{aligned} \tag{22}$$

The second term proportional to β' better is similarly simple in unitary gauge

$$\begin{aligned} T &= \Sigma \tau^3 \Sigma^\dagger = \tau^3 \\ \Rightarrow \mathrm{Tr}(TV_\mu) &= \mathrm{Tr}\left(-ig_Z Z_\mu \frac{\tau_3^2}{2} \right) = -ig_Z Z_\mu \frac{\mathrm{Tr}(\mathbb{1})}{2} = -ig_Z Z_\mu \\ \Rightarrow \mathrm{Tr}(TV_\mu)\, \mathrm{Tr}(TV^\mu) &= -g_Z^2 Z_\mu Z^\mu \end{aligned} \tag{23}$$

Combining both terms gives the gauge boson masses

$$\begin{aligned} \mathcal{L}_2 &= -\frac{v^2}{4} \left(-g^2 W_\mu^+ W^{-\mu} - \frac{g_Z^2}{2} Z_\mu Z^\mu \right) - \beta' \frac{v^2}{8} \left(-g_Z^2 Z_\mu Z^\mu \right) \\ &= \frac{v^2 g^2}{4} W_\mu^+ W^{-\mu} + \frac{v^2 g_Z^2}{8} Z_\mu Z^\mu + \beta' \frac{v^2 g_z^2}{8} Z_\mu Z^\mu \\ &= \frac{v^2 g^2}{4} W_\mu^+ W^{-\mu} + \frac{v^2 g_Z^2}{8} (1+\beta') Z_\mu Z^\mu \end{aligned} \tag{24}$$

Identifying the masses and assuming the universality of neutral and charged current interactions ($\beta' = 0$) we find

$$M_W = \frac{gv}{2} \qquad\qquad M_Z = \frac{g_Z v}{2}. \tag{25}$$

This scale choice for $\Sigma(x)$ is not the only one possible. The weakest assumption to obtain finite gauge–boson masses would be $\langle \mathrm{Tr}(\Sigma^\dagger(x)\Sigma(x)) \rangle \neq 0$ in the vacuum. In the canonical normalization we write

$$\frac{1}{2} \langle \mathrm{Tr}(\Sigma^\dagger(x)\Sigma(x)) \rangle = 1 \qquad \forall x \tag{26}$$

which can also be fulfilled through

$$\Sigma^\dagger(x)\Sigma(x) = \mathbb{1} \qquad \forall x \tag{27}$$

This means $\Sigma(x)$ is now a unitary matrix which like any 2×2 unitary matrix can be expressed in terms of the Pauli matrices

$$\boxed{\Sigma(x) = \exp\left(\frac{-i}{v} \vec{w}(x) \right)} \qquad \text{with} \quad \vec{w}(x) = w^a(x) \tau^a. \tag{28}$$

Note that $\vec{w}(x)$ has mass dimension one, so it can be a physical scalar field. The normalization scale v is given by the energy scale of our Lagrangian. For reason which will be obvious in a few seconds, $\vec{w}(x)$ is called the non–linear representation of the symmetry related Σ field. Using the commutation properties of the Pauli matrices We can expand Σ as

$$\begin{aligned}\Sigma &= \mathbb{1} - \frac{i}{v}\vec{w} + \frac{1}{2}\frac{(-1)}{v^2} w^a\tau^a w^b\tau^b + \frac{1}{6}\frac{i}{v^3} w^a\tau^a w^b\tau^b w^c\tau^c \\ &= \mathbb{1} - \frac{i}{v}\vec{w} - \frac{1}{2v^2} w^a w^a \mathbb{1} + \frac{i}{6v^3} w^a w^a \vec{w} \\ &= \left(1 - \frac{1}{2v^2} w^a w^a \pm ...\right)\mathbb{1} - \frac{i}{v}\left(1 - \frac{1}{6v^2} w^a w^a \pm ...\right)\vec{w}\end{aligned} \tag{29}$$

From this expression we can for example read off the Feynman rules.

Obviously, a third way of expressing a unitary field Σ in terms of the Pauli matrices is the properly normalized linear representation

$$\boxed{\Sigma(x) = \frac{1}{\sqrt{1 + \frac{w^a w^a}{v^2}}}\left(1 - \frac{i}{v}\vec{w}(x)\right)} \tag{30}$$

The different ways of writing the Σ field in terms of the Pauli matrices cannot have any impact on the physics. However, the three forms of $\Sigma(x)$ we briefly discussed (unitary gauge Σ=1, exponential and linear representation) have different Feynman rules and Green's functions, and for a given problem one or the other might be the most efficient to use in computations or proofs. For example in electroweak calculations, the proof of renormalizability was first formulated in unitary gauge. Loop calculations might be more efficient in the Feynman gauge, because of the simplified propagator structure, while some QCD processes benefit from an explicit projection on the physical external gluons. Modern tree–level helicity amplitudes are usually computed in the unitary gauge, etc. Each of these techniques clearly have their strengths and weaknesses.

For example from the introductions to supersymmetry and extra dimensions in recent semesters we know that if we do not introduce something new, the Standard Model with gauge-bosons masses violates unitarity, most notably in $WW \to WW$ scattering. This argument can even be used to fix all the Higgs couplings, the only remaining free parameter is the Higgs mass, because unitarity arguments always affect the high–energy (*i.e.* massless) limit of the theory. In other words, our Σ model can only be viewed as an effective theory unless we give the new field a physical meaning. To extend the simple Σ model we can allow for fluctuations of $\mathrm{Tr}(\Sigma^\dagger\Sigma)$ around the vacuum value $\Sigma^\dagger\Sigma = 1$ and parameterize the new degrees of freedom as a physical field

$$\boxed{\Sigma \to \left(1 + \frac{H}{v}\right)\Sigma} \tag{31}$$

which means for our usual trace

$$\frac{1}{2}\,\mathrm{Tr}(\Sigma^\dagger\Sigma) = \left(1 + \frac{H}{v}\right)^2 \tag{32}$$

The non-dynamic limit is again $\Sigma^\dagger\Sigma = 1 \iff H = 0$. Interpreting the fluctuations around the non-trivial vacuum as a physical Higgs field is really nothing but the usual Higgs mechanism (named after one of the University of Edinburgh's most famous sons), except that the static limit has a proper definition as an effective gauge–invariant theory, the Σ model. This way, the Higgs field does not have to be fundamental, but could just be one step in a ladder built out of effective theories. The potential terms $\mathcal{L}_\Sigma$ produce a potential for the new Higgs field H

$$\mathcal{L}_2 = -\frac{\mu^2 v^2}{2}\left(1 + \frac{H}{v}\right)^2 + \frac{\lambda v^4}{2}\left(1 + \frac{H}{v}\right)^4 + ... \tag{33}$$

The dots stand for higher–dimensional terms which might or might not be there, just like in the Standard Model. Some of them are not forbidden by any symmetry, but they are not realized at tree level in the Standard Model. In the static limit we have to recover the vacuum condition $\mathrm{Tr}(\Sigma^\dagger(x)\Sigma(x))/2 = 1$, so there $H = 0$ and hence $\mathcal{L}_2 = 0$ means $\mu^2 = \lambda v^2$.

Just as for the Σ field alone we can move from the simple unitary gauge to a different (linear) representation of the Σ field including a physical Higgs scalar

$$\Sigma \to \left(1+\frac{H}{v}\right)\mathbb{1} - \frac{i}{v}\vec{w} = \mathbb{1} + \frac{1}{v}\begin{pmatrix} H-iw^3 & -i\sqrt{2}w^+ \\ -i\sqrt{2}w^- & H+iw^3 \end{pmatrix} = \mathbb{1} + \frac{1}{v}(\tilde{\Phi}\Phi) \tag{34}$$

The last step is just another way to write the 2×2 matrix in terms of the two doublets

$$\tilde{\Phi} = \begin{pmatrix} H-iw^3 \\ -i\sqrt{2}w^- \end{pmatrix} \qquad \Phi = \begin{pmatrix} -i\sqrt{2}w^+ \\ H+iw^3 \end{pmatrix} \tag{35}$$

These two doublets give mass to up–type and down–type fermions.

Instead of deriving both relevant doublets from one physical Higgs doublet Φ and $\tilde{\Phi}$ we can include two sigma fields in the fermion–mass terms

$$\mathcal{L}_3 \sim -\overline{Q}_L M_{Qu}\Sigma_u \frac{1+\tau^3}{2} Q_R - \overline{Q}_L M_{Qd}\Sigma_d \frac{1-\tau^3}{2} Q_R + ... \tag{36}$$

and in the gauge–boson mass terms

$$\mathcal{L}_2 = \frac{v_u^2}{2}\,\mathrm{Tr}\left[(D_\mu\Sigma_u)^\dagger D^\mu\Sigma_u\right] + \frac{v_d^2}{2}\,\mathrm{Tr}\left[(D_\mu\Sigma_d)^\dagger D^\mu\Sigma_d\right] \tag{37}$$

Each of the two Σ fields we can express in the usual linear representation

$$\Sigma_j = \mathbb{1} + \frac{1}{v_j}\Phi_j^0 - \frac{i}{v_j}\vec{\Phi}_j \qquad\qquad i=u,d \quad \vec{\Phi}_j = \Phi_j^a\tau^a. \tag{38}$$

From the gauge–boson masses we know that

$$v_u^2 + v_d^2 = v^2 \qquad\Longleftrightarrow\qquad v_u = v\sin\beta \quad v_d = v\cos\beta \tag{39}$$

which means that the longitudinal vector bosons are

$$\vec{w} = \cos\beta\,\vec{\Phi}_u + \sin\beta\,\vec{\Phi}_d \tag{40}$$

This two–Higgs doublet model is for example the minimal choice in supersymmetric extensions of the Standard Model. But type-II two–Higgs doublet models where one Higgs doublet gives mass to up-type and another one to down–type fermions are much more general than that.

B. Custodial Symmetry

From the discussion in the last section we have seen that electroweak symmetry breaking with a simple sigma field or Higgs doublet links the couplings of neutral and charged currents firmly to the masses of the W and Z bosons. After the precision measurements at LEP this link has turned into a seriously strong constraint on all kind of new–physics models. As a matter of fact, this constraint is responsible for the almost death of (technicolor) models which describe the Higgs boson as a bound state under a new QCD-like interaction.

We remember that the Lagrangian for the gauge-boson masses involves two terms, both symmetric under $SU(2)\times U(1)$ and hence allowed in the electroweak Standard Model

$$\mathcal{L}_2 = -\frac{v^2}{4}\,\mathrm{Tr}[V_\mu V^\mu] - \beta'\,\frac{v^2}{8}\,\mathrm{Tr}[TV_\mu]\,\mathrm{Tr}[TV^\mu] \tag{41}$$

In unitary gauge we actually computed the mass terms coming from $\mathrm{Tr}[V_\mu V^\mu]$, which gave M_W and M_Z proportional to $g\equiv g_W$ and g_Z. Their relative size can be expressed in terms of the weak mixing angle θ_w, together with the assumption that G_F or g universally govern charged current ($W^\pm$) and neutral-current (W^3) interactions. This relations at tree level is simply

$$\frac{M_W^2}{M_Z^2} = c_w^2. \tag{42}$$

A free parameter ρ breaking this relation can be introduced as a shift

$$\boxed{g_Z^2 \to g_Z^2 \cdot \rho} \qquad\qquad m_Z \to m_Z \cdot \sqrt{\rho}\,, \tag{43}$$

which from measurements it is very strongly constrained to be unity. In $\mathcal{L}_2$ the Z-mass term proportional to β' precisely predicts the deviation $\rho = 1+\beta' \neq 1$. To bring our Lagrangian into agreement with measurements we better find a reason to constrain β' to zero, and the $SU(2) \times U(1)$ gauge symmetry unfortunately does not do the job.

Looking ahead, we will find that $\rho = 1$ is violated in the Standard Model, for example by the difference in up-type and down-type quark masses $m_b \neq m_t$. Which means we are looking for an approximate symmetry of the entire Standard Model, but in particular a good symmetry in the $SU(2)$ gauge sector. There is one possibility...
We can replace the $SU(2)_L \times U(1)_Y$ symmetry with a larger symmetry $SU(2)_L \times SU(2)_R$, which obviously would have to act like

$$\begin{aligned}
&\Sigma \to U\Sigma V^\dagger \qquad U \in SU(2)_L \qquad V \in SU(2)_R \\
&\mathrm{Tr}(\Sigma^\dagger\Sigma) \to \mathrm{Tr}\left[V\Sigma^\dagger U^\dagger U\Sigma V^\dagger\right] = \mathrm{Tr}[\Sigma^\dagger\Sigma] \qquad \text{(because of circular trace)}
\end{aligned} \tag{44}$$

From the definition of the covariant derivative $D_\mu\Sigma$ including a simple τ^3 we can already guess that the complete group $SU(2)_R$ will not allow B-field interactions which are proportional to $s_W \sim \sqrt{1/4}$. It also does not allow $\beta' \neq 0$, but it does allow all terms in the Higgs potential $\mathcal{L}_\Sigma$. Giving the Σ field a finite vacuum expectation value Σ field changes the picture: in the minimal (non–Higgs) version and in the unitary gauge the Σ field now reduces to $\mathbb{1}$, which for the combined $SU(2)$ transformations means

$$\langle\Sigma\rangle \to \langle U\Sigma V^\dagger\rangle = \langle U\mathbb{1}V^\dagger\rangle = UV^\dagger \equiv \mathbb{1} \tag{45}$$

The last step, *i.e.* the symmetry requirement for the Lagrangian can only be satisfied if we require $U = V$. In other words, the vacuum expectation value for Σ or for the Higgs field breaks $SU(2)_L \times SU(2)_R$ to the diagonal subgroup $SU(2)_{L+R}$. The technical term is precisely defined this way — the two $SU(2)$ symmetries reduce to one remaining symmetry which can be written as $U = V$. In the extended symmetry group the ρ parameter is indeed protected to be $\rho = 1$, while under only the diagonal symmetry group we can accommodate a general ρ.

Leading corrections to the ρ parameter come from Higgs loops in the case $g' \neq 0$

$$\Delta\rho \sim -\frac{11 G_F M_Z^2 s_W^2}{24\sqrt{2}\pi^2} \log\frac{m_h^2}{M_Z^2}. \tag{46}$$

Others come from virtual bottoms and tops in the W and Z self energies

$$\begin{aligned}
\Delta\rho &\sim \frac{3G_F}{8\sqrt{2}\pi^2}\left(m_t^2 + m_b^2 - 2\frac{m_t^2 m_b^2}{m_t^2 - m_b^2}\log\frac{m_t^2}{m_b^2}\right) \\
&\sim \frac{3G_F}{8\sqrt{2}\pi^2}\left(2m_b^2 + \delta - 2\frac{(m_b^2+\delta)m_b^2}{\delta}\log\left(1+\frac{\delta}{m_b^2}\right)\right) \qquad\qquad m_t^2 = m_b^2 + \delta \\
&= \frac{3G_F}{8\sqrt{2}\pi^2}\left(2m_b^2 + \delta - 2\left(\frac{m_b^4}{\delta} + m_b^2\right)\left(\frac{\delta}{m_b^2} - \frac{\delta^2}{2m_b^4} + \mathcal{O}(\delta^3)\right)\right) \\
&= \frac{3G_F}{8\sqrt{2}\pi^2}\left(2m_b^2 + \delta - 2m_b^2 + 2\frac{\delta}{2} - 2\delta + \mathcal{O}(\delta^2)\right) \\
&= \frac{3G_F}{8\sqrt{2}\pi^2}\,\mathcal{O}(\delta^2)
\end{aligned} \tag{47}$$

and indeed vanish for $m_t = m_b$.

The obvious next question is: how do physical modes, which we introduce in the parameterization of the Σ field $\Sigma(x) = \exp(-i\vec{w}/v)$ and which we will describe in more detail in the next section transform under these two different $SU(2)$ symmetries?
Clearly, under the usual $SU(2)_L$ we still find $\Sigma \to U \cdot \Sigma$, the way we actually introduced U earlier. We can write U in terms of the $SU(2)$ generators as $U = \exp(-i\alpha \cdot \tau/2)$. In general, we denote $\vec{w} = w^a\tau^a = w \cdot \tau$ and $\vec{\alpha} = \alpha \cdot \tau$ in

terms of the Pauli matrices. We can read off the transformation properties of $\vec{w}$ from

$$\begin{aligned} U\Sigma &= e^{-i(\alpha\cdot\tau)/2}\, e^{-i(w\cdot\tau)/v} \\ &= e^{-i(\alpha\cdot\tau)/2-i(w\cdot\tau)/v}\, e^{-\frac{i}{2}[\alpha\cdot\tau)/2,(w\cdot\tau)/v]} \\ &= e^{-i(\alpha\cdot\tau)v/2+(w\cdot\tau))/v} \\ &= e^{-i(w'\cdot\tau)/v} \end{aligned} \tag{48}$$

In the second line we have used the Baker-Campbell-Hausdorff formula $e^A e^B = e^{A+B} e^{[A,B]/2}$ which for the Pauli matrices becomes

$$\begin{aligned} [\tau_i,\tau_j] = 2i\varepsilon_{ijk}\tau_k \quad &\Rightarrow \quad (\vec{\alpha}\cdot\vec{\tau})(\vec{w}\cdot\vec{\tau}) = \vec{\alpha}\cdot\vec{w} + i\vec{\tau}\,(\vec{\alpha}\times\vec{w}) \\ &\Rightarrow \quad [(\vec{\alpha}\cdot\vec{\tau}),(\vec{w}\cdot\vec{\tau})] = 2i\vec{\tau}\,(\vec{\alpha}\times\vec{w}) \end{aligned} \tag{49}$$

From the symmetry requirement $U\Sigma \equiv \Sigma$ we find the transformation property for the physical modes in Σ

$$\boxed{w_a \to w'_a = w_a + \frac{v}{2}\alpha_a} \tag{50}$$

This is a non-linear transformation, in the sense that w'_a is not proportional to w_a. Note that we have derived this shift–symmetry operation only for infinitesimal transformations, so for general transformations we might end up with higher terms in α. The crucial conclusion is the same, though: these modes in Σ shift under the $SU(2)$ transformation, their transformation is not linear. When we construct a symmetric Lagrangian this non–linear transformation forbids mass terms, gauge interactions, Yukawa couplings, and quadratic potential terms for these modes in Σ. Only derivative terms like the kinetic term and derivative couplings are allowed unter the $SU(2)$ symmetry.

Similarly, we can evaluate the transformation of these physical modes under the custodial symmetry group $SU(2)_{L+R}$ and find the linear transformation

$$\boxed{w_a \to w'_a = w_a - \varepsilon_{abc}\alpha_b w_c} \tag{51}$$

In other words, when we transform the physical modes corresponding to the symmetry generators in Σ by the good symmetry $SU(2)_L$ we find a non–linear transformation, while the approximate symmetry $SU(2)_{L+R}$ leads to a linear transformation. A linear transformation for example of a scalar means that we can write a potential for this particle which is symmetric under $SU(2)_{L+R}$ transformations.

This leads us to the definition of Goldstone modes: if we have a global symmetry group which is spontaneously broken into a smaller symmetry group, the broken generators of the original group correspond to physical Goldstone modes. These modes transform non–linearly under the larger group and linearly under the smaller group. If our symmetry groups are gauge groups, Goldstone modes are absorbed into the broken gauge bosons to make them massive. If this spontaneous symmetry breaking involves a vacuum expectation value f, the mass of the heavy gauge bosons which eat the Goldstone modes is of the order f.

A little more tailored towards our later use, we see that because of their non–linear transformation property, Goldstone bosons cannot form a potential symmetric under the original group, so they have to for example be massless. This does not change if we break the original symmetry group spontaneously — potential terms are still forbidden. However, if we also break the larger symmetry group explicitely, for example through a coupling g, potential terms can now occur. They will be proportional to g and proportional to f and can be induced for example through loop effects. In the presence of explicit symmetry breaking the Goldstone modes are called pseudo–Goldstone modes.

II. LITTLE–HIGGS MECHANISM

Until now we have not talked about any physics beyond the Standard Model. As a matter of fact, we have mostly talked about a watered–down version of the electroweak Standard Model, namely the Σ model. However, first of all it is good to know that we can actually write down a perfectly fine Lagrangian for the electroweak gauge theory including finite W and Z boson masses without introducing a Higgs field, if we are happy with an effective–theory approach. And secondly, the starting point of little–Higgs theories is the attempt to make the Higgs boson a pseudo–Goldstone mode under some broken global symmetry to explain its small mass, and it is a good idea to review this mechanism before diving into the exciting new physics.

A. Some Goldstone bosons

In the following, we will track the behavior of different degrees of freedom under $SU(N)$ transformations. We can start with a simple $U(1)$ transformation of a complex scalar field, *i.e.* with two degrees of freedom. For this scalar field $\phi(x)$ we assume a potential $V = V(\phi^*\phi)$ and a global $U(1)$ symmetry transformation $\phi \to e^{i\alpha}\phi$. After expanding the scalar field around its (real) vacuum we find a massive radial mode $r(x)$, with its mass given by the form of the potential around the vacuum. The transformation of the scalar field in terms of these two modes reads

$$\phi \to e^{i\alpha}\phi = e^{i\alpha}\,\frac{v + r(x)}{2}\,e^{i\theta(x)/v} = \frac{v + r(x)}{2}\,e^{i(\theta(x)+v\cdot\alpha)/v} \tag{52}$$

Just as before, we find a non–linear shift of the massless mode in the scalar field: $\theta \to \theta + v\cdot\alpha$. This means $\theta(x)$ has to stay massless, protected by the $U(1)$ symmetry. Only derivative couplings of θ are allowed in a $U(1)$-symmetric Lagrangian.

Unfortunately, we now have to move to the non–abelian case, where we will have to write tons of matrices and any lecturer is bound to get things wrong on the blackboard. First, we can break the global (ungauged) gauge group $SU(N) \to SU(N-1)$ and look at the Goldstone modes associated with the reduced number of degrees of freedom in the symmetry group. We expect

$$(N^2-1)^2 - ((N-1)^2 - 1) = 2N - 1 \tag{53}$$

generators which are not anymore associated with the reduced symmetry group. Think for example of a basis for $SU(3)$ and $SU(2)$, the Gell-Mann and the Pauli matrices. They are traceless hermitian (and unitary) matrices, and generators of the Lie groups $SU(N)$ with $N = 2,3$. For $SU(2)$ the three Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{54}$$

with strictly speaking: $SU(2) = \text{span}\{i\sigma^k\}$. The corresponding 8 Gell-Mann matrices can be written in terms of the three Pauli matrices and the remaining degrees of freedom

$$\begin{aligned}
&\lambda^1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix} = \begin{pmatrix} \sigma^1 & & 0\\ & & 0\\ 0&0&0 \end{pmatrix} \quad
\lambda^2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix} = \begin{pmatrix} \sigma^2 & & 0\\ & & 0\\ 0&0&0 \end{pmatrix} \quad
\lambda^3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix} = \begin{pmatrix} \sigma^3 & & 0\\ & & 0\\ 0&0&0 \end{pmatrix} \\
&\lambda^4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} = \begin{pmatrix} 0 & & 1\\ & & 0\\ 1&0&0 \end{pmatrix} \quad
\lambda^5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix} = \begin{pmatrix} 0 & & -i\\ & & 0\\ i&0&0 \end{pmatrix} \quad
\text{combined to complex} \quad \begin{pmatrix} 0 & & w_1\\ & & 0\\ w_1^*&0&0 \end{pmatrix} \\
&\lambda^6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix} = \begin{pmatrix} 0 & & 0\\ & & 1\\ 0&1&0 \end{pmatrix} \quad
\lambda^7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix} = \begin{pmatrix} 0 & & 0\\ & & -i\\ 0&i&0 \end{pmatrix} \quad
\text{combined to complex} \quad \begin{pmatrix} 0 & & 0\\ & & w_2\\ 0&w_2^*&0 \end{pmatrix} \\
&\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} \mathbb{1} & & 0\\ & & 0\\ 0&0&-2 \end{pmatrix}
\end{aligned} \tag{55}$$

We can arrange all generators of $SU(3)$ which are not generators of $SU(2)$ in the outside column and row of the 3×3 matrix

$$\boxed{U_N \sim \begin{pmatrix} SU(2) & & w_1 \\ & & w_2 \\ w_1^* & w_2^* & w_0 \end{pmatrix}} \tag{56}$$

The entry w_0 is fixed by the requirement that U_N has to be traceless when we add $\mathbb{1}$ to the $SU(2)$ matrices in the top–left corner. If, as they were introduced in the Σ model, the Goldstone modes describe modes of a system around its broken ground state with a symmetry–breaking scale v, we can collect them in a vector–shaped field ϕ for general $SU(N) \to SU(N-1)$ breaking as

$$\phi = \exp\left\{-\frac{i}{v}\begin{pmatrix} SU(N-1) & & & w_1 \\ & & & \cdots \\ & & & w_{N-1} \\ w_1^* & \cdots & w_{N-1}^* & w_0 \end{pmatrix}\right\}\begin{pmatrix} 0 \\ \cdots \\ 0 \\ v \end{pmatrix} \equiv e^{-i\vec{w}\cdot\vec{\tau}/v}\phi_0 \tag{57}$$

This notation has the advantage that we can write ϕ and ϕ_0 as columns, *i.e.* as fields symmetric under $SU(N)$ or $SU(N-1)$ in the fundamental representation. The vector ϕ then is defined such that its upper $N-1$ component are symmetric under the smaller symmetry group $SU(N-1)$. In the first–order term in the Taylor series in $1/v$ the mass scale v drops out between the exponent and ϕ_0.

Another example for a global symmetry group more similar to our old custodial $SU(2)_{L+R}$ would be $\underline{SU(N) \times SU(N) \to SU(N)}$. The number of Goldstone bosons associated with the broken generators is

$$2(N^2-1) - (N^2-1) = N^2 - 1 \tag{58}$$

Unfortunately, they are not as easily written in matrix form as those of the two gauge groups $SU(N) \to SU(N-1)$. The gauge transformations we know from before: $\phi \to L\phi R^\dagger$. The symmetry–breaking ground state of the combined scalar field is $\langle\phi\rangle \equiv \phi_0 \equiv v\mathbb{1}_N$: it is invariant under the diagonal subgroup where we identify the two $SU(2)$ transformations to a simpler $\phi_0 \to U\phi_0 U^\dagger$. The remaining (axial) generators are broken and turn into Goldstone bosons collected in $\phi = \exp(-i(\vec{w}\cdot\vec{\tau})/v)\phi_0 = v\exp(-i(\vec{w}\cdot\vec{\tau})/v)\mathbb{1}$. The matrices $(\vec{w}\cdot\vec{\tau})$ are traceless hermitian matrices with (N^2-1) degrees of freedom, *i.e.* independent entries.

From our simple examples $SU(2)_L$, $SU(2)_{L+R}$ and $U(1)$ we already have a good idea how to compute the transformation of the Goldstone bosons under broken and unbroken symmetry transformations. We repeat the argument for $SU(N) \to SU(N-1)$, starting with the transformation properties of the scalar field ϕ. This scalar field can be parameterized as $\phi \equiv \exp(-i(\vec{w}\cdot\vec{\tau})/v)\phi_0$ with the generators $\vec{\tau}$ including the broken subgroup $SU(N)/SU(N-1)$. Under the unbroken symmetry group $SU(N-1)$ represented as an $(N\times N)$ matrix the scalar field transform as

$$\begin{aligned}
\phi \to U_{N-1}\,\phi &= U_{N-1}\; e^{-i\vec{w}\cdot\vec{\tau}/v}\;\phi_0 \\
&= U_{N-1}\; e^{-i\vec{w}\cdot\vec{\tau}/v}\; U^\dagger_{N-1} U_{N-1}\phi_0 \\
&= U_{N-1}\; e^{-i\vec{w}\cdot\vec{\tau}/v}\; U^\dagger_{N-1}\phi_0 \qquad\qquad (\phi_0 \text{ invariant under } U_{N-1}, \text{ but not } U_N\;)
\end{aligned} \tag{59}$$

This relation will give us the transformation properties for the Goldstones. We can rewrite the $(N\times N)$ matrix acting on the leading term in ϕ

$$\begin{aligned}
&U_{N-1} = \begin{pmatrix} \hat{U}_{N-1} & 0 \\ 0 & 1 \end{pmatrix} \\
\Rightarrow\quad & U_{N-1}\; e^{-i\vec{w}\cdot\vec{\tau}/v}\; U^\dagger_{N-1} \sim -\frac{i}{v}\begin{pmatrix} \hat{U}_{N-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \vec{w} \\ \vec{w}^\dagger & 0 \end{pmatrix}\begin{pmatrix} \hat{U}_{N-1} & 0 \\ 0 & 1 \end{pmatrix} = -\frac{i}{v}\begin{pmatrix} 0 & \hat{U}_{N-1}\vec{w} \\ (\hat{U}_{N-1}\vec{w})^\dagger & 0 \end{pmatrix}
\end{aligned} \tag{60}$$

This means the Goldstones transform as $\vec{w} \to \hat{U}_{N-1}\vec{w}$. However, this transformations with $\hat{U}_{N-1}$ from the left is just the usual symmetry transformation for vectors in the fundamental representation of $SU(N-1)$. In the $SU(N-1)$ symmetric Lagrangian we can write any terms for the Goldstones we can write for other states in the fundamental representation.
To compute the more interesting transformation properties under $SU(N)$ we need the fact, that a $SU(N)$ transformation can be written as a product of an $SU(N)/SU(N-1)$ transformation times a $SU(N-1)$ transformation. This means

$$\begin{aligned}
\phi \to U_N\phi &= U_N U_*(\vec{w})\phi_0 \qquad\qquad \text{with the } SU(N)/SU(N-1) \text{ transformation } U_*(\vec{w}), \text{ so } U_N = U_* U_{N-1} \\
&= U_*(\vec{\alpha})U_{N-1}U_*(\vec{w})\phi_0 \\
&= U_*(\vec{\alpha})U_{N-1}U_*(\vec{w})U^\dagger_{N-1}U_{N-1}\phi_0 \\
&= U_*(\vec{\alpha})U_{N-1}U_*(\vec{w})U^\dagger_{N-1}\phi_0
\end{aligned} \tag{61}$$

The combination $U_{N-1}U_*(\vec{w})U^\dagger_{N-1}$ is just what we found above, while the additional $U_*(\alpha) = \exp(-i(\vec{\alpha}\cdot\tau)/2)$ will produce the same behavior we saw in the $SU(2)$ and $U(1)$ cases: if we write out the infinitesimal transformations we find $\vec{w} \to \vec{w}' = \vec{w} + \vec{\alpha}/2$, which forbids Goldstone masses and other potential terms in the Lagrangian and only allows derivative interactions. The Goldstone Lagrangian of mass dimension four with a global $SU(N)$ symmetry will therefore be of the general form

$$\boxed{\mathcal{L} = |\partial_\mu\phi|^2 + \mathcal{O}(\partial^4) + \text{const}} \tag{62}$$

Any mass scale in this spontaneously broken Goldstone Lagrangian is given by the vacuum expectation value f. Constants can for example arise from the gauge–invariant combination $\phi^\dagger\phi = \phi_0^\dagger\phi_0 = f^2$. Note that here we switch from v to f for the same thing, namely the scale responsible for breaking the larger symmetry group, to be consistent with Martin's review. Similarly, we will switch from $-\vec{w}$ to $\vec{\pi}$ for the Goldstones in ϕ.

B. Protecting the Higgs mass

For example from the lecture on supersymmetry or the lecture on extra dimensions you might remember that one of the puzzles of high–energy physics is the question why the Higgs is so light. From general field–theoretical considerations any fundamental scalar should acquire a loop–induced mass of the order of the cutoff of the theory. Clearly, the LEP precision measurements point too a Higgs mass much below the Planck or the unification scales. One way to explain a small Higgs mass would be to make the Higgs a pseudo–Goldstone of a symmetry which is broken at a mass scale around the weak scale. Compared to this scale the Higgs mass has to be small because of the larger symmetry group, which means the Higgs mass cannot diverge quadratically at large energy scales.

This idea has been around for a long time, but for decades people did not know how to construct such a symmetry. Before we solve this problem via the little–Higgs mechanism, let us unsuccessfully start constructing a symmetry which protects the Higgs mass from quadratic divergences at one loop using a global $SU(3)$ as the broken symmetry including $SU(2)_L$. Everything we need to know for this construction we can read off from the general $SU(N) \to SU(N-1)$ case. The $SU(3) \to SU(2)$ Goldstone modes written in the usual matrix form are

$$\vec{\pi} = \begin{pmatrix} SU(2) & h \\ h^\dagger & \eta \end{pmatrix} \tag{63}$$

We of course assume that the $SU(2)_L$ doublet among the $SU(3)$ Goldstones which can acquire a mass once we break $SU(3)$ is the Higgs doublet of the Standard Model. Again, note that to be in agreement with Martin Schmaltz's review we now denote the Goldstone fields as π instead of $\vec{w}$. The additional field η is an $SU(2)$ singlet and can be ignored for now — we will discuss it briefly at the very end of the lecture. To translate the degrees of freedom from the matrix $\vec{\pi}$ to the fields h we are interested in we write the usual matrix representation of the Goldstones with a symmetry–breaking scale f

$$\begin{aligned}
\phi &= \exp\left[\frac{i}{f}\begin{pmatrix} 0_{2\times 2} & h \\ h^\dagger & 0\end{pmatrix}\right]\begin{pmatrix} 0_2 \\ f\end{pmatrix} \\
&= \left(\mathbb{1} + \frac{i}{f}\begin{pmatrix} 0 & h \\ h^\dagger & 0\end{pmatrix} - \frac{1}{2f^2}\begin{pmatrix} 0 & h \\ h^\dagger & 0\end{pmatrix}\begin{pmatrix} 0 & h \\ h^\dagger & 0\end{pmatrix}\right)\begin{pmatrix} 0 \\ f\end{pmatrix} \qquad\qquad h = (h_1, h_2) \\
&= \begin{pmatrix} 0 \\ f\end{pmatrix} + \begin{pmatrix} ih \\ 0\end{pmatrix} - \frac{1}{2f^2}\begin{pmatrix} 0 \\ h^\dagger h f\end{pmatrix} \\
&= \begin{pmatrix} 0 \\ f\end{pmatrix} + \begin{pmatrix} ih \\ -h^\dagger h/(2f)\end{pmatrix}
\end{aligned} \tag{64}$$

Note that only in the first line we indicate which of the zeros in the 3×3 matrix is a 2×2 sub-matrix. This is easy to keep track of if we remember that the Higgs field h is a doublet, while $h^\dagger h$ is a scalar number. This transformation allows us to rewrite the kinetic term as a function of h

$$\begin{aligned}
|\partial_\mu \phi|^2 &= (\partial_\mu \phi^*)_i (\partial^\mu \phi)_i = (-i\partial_\mu h)_i (i\partial^\mu h^*)_i + \frac{1}{4f^2}(\partial_\mu h^\dagger h)_i (\partial^\mu h^\dagger h)_i \\
&= |\partial_\mu h|^2 + \frac{1}{4f^2}(\partial_\mu \sum_j h_j^* h_j)_i (\partial^\mu \sum_j h_j^* h_j)_i \\
&= |\partial_\mu h|^2 + \frac{1}{4f^2}\left(\sum_j (\partial_\mu h_j^*) h_j + \sum_j h_j^* (\partial_\mu h_j)\right)_i \left(\sum_j (\partial^\mu h_j^*) h_j + \sum_j h_j^* (\partial^\mu h_j)\right)_i \\
&= |\partial_\mu h|^2 + \frac{1}{4f^2}\, 4\, |\partial_\mu h|^2\, h^\dagger h \\
&= |\partial_\mu h|^2 \left(1 + \frac{h^\dagger h}{f^2}\right)
\end{aligned} \tag{65}$$

The second term in the parentheses looks like a kinetic term, so it is fine in the Goldstone Lagrangian. However, it includes an additional factor $h^\dagger h$, which corresponds to an outgoing and an incoming Higgs field and which we should have a close look at. These two fields can be linked, giving a one-loop graph which diverges as

$$\int^\Lambda \frac{d^4 q}{(2\pi)^4}\, \frac{1}{q^2} \sim \frac{\Lambda^2}{(4\pi)^2} \tag{66}$$

Comparing the two terms in the parentheses above there is an upper limit to the size of the $h^\dagger h$ term, where this term dominates our theory. This means that our effective theory should will only be valid as long as

$$\boxed{\frac{\Lambda^2}{(4\pi)^2 f^2} \lesssim 1} \tag{67}$$

In other words, the massless Higgs boson has additional high-dimensional Lagrangian terms which become strong for energy scales around $\Lambda \sim 4\pi f$. Above this scale, our effective theory will not be useful.

After we now know how the kinetic term for the massless pseudo–Goldstone–Higgs doublet looks we next have to generate a coupling to the $SU(2)$ gauge bosons and see what happens with the Higgs mass. Of course, from the discussion of Goldstones and pseudo–Goldstones we know that we will not be able to generate the mass or a potential term we want, but it is constructive to see the problems which will arise.

First attempt: We can simply add $g\,(\vec{W}^\mu \vec{\tau})$ in the covariant derivative of the Goldstone. Or in other words, we gauge the $SU(2)$ subgroup of the global $SU(3)$. This automatically creates a 4-point coupling of the kind $|g\vec{W}_\mu h|^2$. Like before, the two W bosons coupling to the Higgs propagator can be linked to a loop and generate a one–loop mass term of the kind

$$\mathcal{L} \subset g^2 \frac{\Lambda^2}{(4\pi)^2}\, h^\dagger h \tag{68}$$

This term is a quadratically divergent Higgs mass. Which means that our operator breaks the shift symmetry $SU(3)$ into $SU(2)$ and at the same time introduces the same kind of mass for which spoils the Standard Model.

Second attempt: We can write the same interaction as in the first attempt in terms of the triplet ϕ, where we simply leave the third entry in the gauge–boson matrix empty

$$\left| g \begin{pmatrix} \vec{W}_\mu \tau & 0 \\ 0 & 0 \end{pmatrix} \phi \right|^2 \tag{69}$$

We can again square this relevant interaction term contributing to the Higgs mass and find (in a suitable $SU(2)$ basis)

$$\phi^\dagger \begin{pmatrix} g\,\mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g\,\mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \phi = \phi^\dagger \begin{pmatrix} g^2\,\mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \phi = g^2\; h^\dagger h\; \mathbb{1} \tag{70}$$

which means that the mass terms now read

$$\mathcal{L} \subset g^2 \frac{\Lambda^2}{(4\pi)^2}\, \phi^\dagger \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \phi = g^2\, \frac{\Lambda^2}{(4\pi)^2}\, h^\dagger h \tag{71}$$

This is precisely what we had before. And it is not surprising, because we really only wrote the same thing in a different notation, using $\phi^\dagger\phi$ instead of $h^\dagger h$ and adding zeros into the gauge–boson matrix which in turn acts as a projector onto the $h^\dagger h$ part.

Third attempt: Learning from the previous cases we can instead add a proper covariant derivative not only including the W fields in $SU(2)$, but also the degrees of freedom of the complete $SU(3)$. Closing all of them into loops we obtain again in a proper basis

$$\mathcal{L} \subset g^2 \frac{\Lambda^2}{(4\pi)^2}\, \phi^\dagger \mathbb{1}_3 \phi = g^2 \frac{\Lambda^2}{(4\pi)^2} |\phi_0|^2 = g^2 \frac{\Lambda^2}{(4\pi)^2}\, f^2 \tag{72}$$

There is indeed no Higgs–mass contribution, because our $SU(3)$ gauge bosons ate the Goldstones altogether. This is simple an effect of including a complete set of $SU(3)$ gauge bosons of freedom, where there are no Goldstone degrees of freedom left for the Higgs.
On the other hand, so this attempt brings us closer to solving Higgs–Goldstone problem. The problem we are stuck in is that either we include only the $SU(2)$ covariant derivative and find quadratic divergences in the Higgs mass or we include the $SU(3)$ covariant derivative and turn the Higgs into a Goldstone mode which gives a mass of scale f to these gauge bosons.

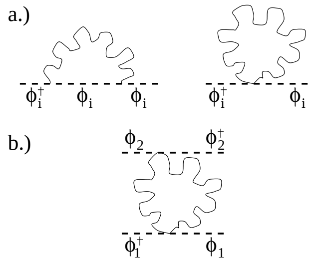

FIG. 1: Feynman diagrams contributing to the Higgs mass in little–Higgs models. This beautiful picture is stolen from Martin Schmaltz's review article.

Correct attempt: We obviously have to come up with something better than the usual set of Goldstones from $SU(3)$ breaking. Digesting the unsuccessful attempts we can see a way out: we should use two independent sets of $SU(3)$ generators. These we break to our $SU(2)_L$ gauge group through a combination of spontaneous and explicit breaking. Because of this mixing we will get pseudo–Goldstones which make the $SU(3)$ gauge bosons heavy while the uneaten Goldstones which can form our Higgs. Note that this requires us to only include one set of $SU(3)$ gauge bosons for two $SU(3)$ symmetries, so in a way only one of them will be gauged. Naively, we have $8+8-3=13$ Goldstones degrees of freedom to distribute. However, we have have to be careful not to double count three of them in the case where we identify both $SU(2)$ fractions of the two original sets of $SU(3)$ generators, in which case we are down to ten Goldstone modes. The art will be to arrange the spontaneous and hard symmetry breakings into a workable model.
First, we write each of the set of $SU(3)$ generators into the usual matrices and identify the relevant degrees of freedom in the Goldstone matrix which we hope will become the Higgs

$$\phi_j = \exp\left(\frac{i}{f} \begin{pmatrix} 0_{2\times 2} & h_j \\ h_j^\dagger & 0 \end{pmatrix} \right) \begin{pmatrix} 0 \\ f \end{pmatrix} \qquad j=1,2 \tag{73}$$

For simplicity we here set the vevs equal $f_1 \equiv f_2 \equiv f$. At one loop, each of them couples to the set of $SU(3)$ gauge bosons with the usual $SU(3)$ covariant derivative

$$\mathcal{L} \subset |D_\mu \phi_1|^2 + |D_\mu \phi_2|^2 \subset g_1^2\, |W_\mu \phi_1|^2 + g_2^2\, |W_\mu \phi_2|^2 \tag{74}$$

These terms can be linked to loop diagrams of the kind shown in Fig. 1(a). From our attempt number three we we know that for universal couplings g_j they read

$$\frac{1}{(4\pi)^2}\, \Lambda^2\, \left(g_1^2\, \phi_1^\dagger \phi_1 + g_2^2\, \phi_2^\dagger \phi_2 \right) = \frac{g^2}{(4\pi)^2}\, \Lambda^2\, 2f^2 \tag{75}$$

However, these are not the only terms we can write down with two sets of Goldstones. For example, we can write diagrams like the one in Fig. 1(b), coupling ϕ_1 to ϕ_2 directly through a gauge-boson loop. Counting the powers in momentum we can guess its contribution to the Lagrangian to be of the kind

$$\frac{g_1^2 g_2^2}{(4\pi)^2}\, \log \frac{\Lambda^2}{\mu^2}\, |\phi_1^\dagger \phi_2|^2 \tag{76}$$

The combination $\phi_1^\dagger \phi_2$ is a scalar and not a matrix and is gauge invariant only under the diagonal $SU(3)$ subgroup of $[SU(3)]^2$. And last but not least, it is not a simple mass term for the ϕ_j, nor is it quadratically divergent, so we simply accept its existence.
In the next step, we have to translate its form into a Lagrangian term in the Higgs fields h_j and see if it gives us a mass term. Its form suggests a reorganization of the h_j, to treat them more symmetrically; if we shift them such that

$h_j \to k \pm h$ we find to leading order (neglecting commutators)

$$\begin{aligned}
\phi_1^\dagger \phi_2 &= \left[e^{\frac{i}{f}\vec{k}\vec{\tau}} e^{+\frac{i}{f}\vec{h}\vec{\tau}} \begin{pmatrix} 0 \\ f \end{pmatrix} \right]^\dagger \left[e^{\frac{i}{f}\vec{k}\vec{\tau}} e^{-\frac{i}{f}\vec{h}\vec{\tau}} \begin{pmatrix} 0 \\ f \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & f \end{pmatrix} e^{-\frac{i}{f}\vec{h}\vec{\tau}} e^{-\frac{i}{f}\vec{k}\vec{\tau}} e^{+\frac{i}{f}\vec{k}\vec{\tau}} e^{-\frac{i}{f}\vec{h}\vec{\tau}} \begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= \begin{pmatrix} 0 & f \end{pmatrix} e^{-\frac{2i}{f}\vec{h}\vec{\tau}} \begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= \begin{pmatrix} 0 & f \end{pmatrix} \left[1\!\!1 - \frac{2i}{f} \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} + \frac{1}{2} \left(\frac{2i}{f} \right)^2 \begin{pmatrix} hh^\dagger & 0 \\ 0 & h^\dagger h \end{pmatrix} + ... \right] \begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= f^2 1\!\!1 - \frac{2}{f^2} \begin{pmatrix} 0 & f \end{pmatrix} \begin{pmatrix} 0 \\ h^\dagger h f \end{pmatrix} + ... \\
&= f^2 1\!\!1 - 2h^\dagger h + ... \qquad \Rightarrow \qquad \boxed{|\phi_1^\dagger \phi_2|^2 \sim f^4 1\!\!1 - 4f^2 h^\dagger h + ...}
\end{aligned} \tag{77}$$

The Goldstone modes k are $SU(3)$ rotations common to ϕ_1 and ϕ_2 and lead to massive longitudinal $SU(3)$ gauge bosons when we break the $SU(3)$ symmetry spontaneously.
Because of the combination of the spontaneous symmetry breaking of the two $SU(3)$ symmetries and the explicit breaking to the diagonal $SU(3)$ the pseudo–Goldstone field h develops a mass and general potential terms of the kind $|\phi_1^\dagger \phi_2|$. For example its mass term just combining the two above formulae reads

$$\mathcal{L} \subset -\frac{g_1^2 g_2^2 f^2}{(2\pi)^2} \log \frac{\Lambda^2}{\mu^2} h^\dagger h \tag{78}$$

To summarize, of the two Goldstones $h_1 = k + h$ and $h_2 = k - h$ we use $k = (h_1 + h_2)/2$ to make the gauge bosons of the broken $SU(3)$ heavy. The remaining Goldstones $h = (h_1 - h_2)/2$ are pseudo–Goldstone bosons which can develop a mass and a potential with a mass scale f at which we break $SU(3) \to SU(2)$. Comparing this mass term to the Standard–Model mass scales, we expect or hope for f values which give us

$$\boxed{M_{\text{weak}} \sim \frac{g^2 f}{2\pi}} \tag{79}$$

The mechanism described above is called collective symmetry breaking. It is a convoluted way of spontaneously and explicitely breaking a global symmetry into a gauged subgroup (here $SU(3)_{\text{diag}}$) and then down to our $SU(2)_L$. Part of the Goldstones from the original global symmetry group will make the additional gauge bosons heavy, with a mass scale f. The remaining Goldstones turn into pseudo–Goldstones because of the explicit breaking of the global symmetry. The reason why this symmetry breaking is called 'collective' is that we need to break two symmetries explicitely to produce mass and potential terms for the pseudo–Goldstone. Only breaking one of them leaves the respective other one as a global symmetry under which the Higgs fields transforms non–linearly. This way we ensure that the Higgs mass and potential terms have a squared g^2 suppression compared to f. As a side remark we notice that while this gives us a suppression of g^2 instead of g, we do not collect additional factors $1/(4\pi)$, because we are still looking at one–loop diagrams.
Looking back, we now have a scale interval where our little–Higgs effective theory does exactly what it is supposed to do: below $g^2 f/(2\pi)$ we have the Standard Model with it usual Higgs mass. Above $4\pi f$ we have a strongly interacting UV completion which we are ignoring at this point, because it might be a mess. In between, there is an energy range $g^2 f/(2\pi), ..., 4\pi f$ where we can compute effects of the new physics using the little–Higgs theory.

III. LITTLE–HIGGS MODELS

From the last chapters we now know how to generally build models which protect the SM Higgs mass from quadratic divergences at one loop: we pick a global symmetry of which we gauge only a part. Then we break it spontaneously to our $SU(2)_L$ at a scale f and at the same time break it explicitely via gauge or Yukawa couplings. Part of the complete

set of Goldstones will make the additional gauge bosons heavy and the remaining pseudo–Goldstones include the SM Higgs sector and protect its low mass.
Because the original global symmetry group is explicitely broken via collective symmetry breaking, the Higgs will develop mass and potential terms governed by the scale f, but doubly loop suppressed (via gauge–boson or fermion loops). It will come as a surprise that this scheme can be realized in many different ways. In the following, we will discuss two realizations, one starting from a global $[SU(3)]^2$ and the other starting from a global $SU(5)$ symmetry.

A. The simplest little Higgs

The smallest useful extension of $SU(2)_L$ is $SU(3)$ as discussed before and as Weinberg pointed out decades ago. To protect the Higgs mass a single broken $SU(3)$ symmetry is not sufficient. We instead need a more complex setup, so we postulate a global $[SU(3)]^2$ symmetry and break it in steps down to $SU(2)_L$. We can then express all mass scales in terms of the symmetry–breaking scale f. Starting from the UV the basic structure of our model is

– for $E > 4\pi f$ we know our effective theory in E/f breaks down, so our theory is strongly interacting and/or needs a UV completion.

– below that, the effective Lagrangian obeys a $[SU(3)]^2$ symmetry transformation $U_j \quad (j = 1, 2)$ with two gauge couplings g_j and two Yukawa couplings λ_j. They couple to one set of $SU(3)$ gauge bosons, which contains three $SU(2)$ gauge bosons, plus complex $W'_\pm, W'_0$ with hypercharge 1/2 and a singlet Z'.

– through loop effects gauge and Yukawa couplings explicitely break $[SU(3)]^2 \to SU(3)_{\text{diag}}$. The related pseudo Goldstones give masses of the order $g\, f$ to the heavy $SU(3)$ gauge bosons.

– the other five broken generators of $[SU(3)]^2$ become Goldstones h, η including the Higgs. Terms like $\phi_1^\dagger\phi_2$ give rise to Higgs masses around $g^4 f^2/(2\pi)^2 \equiv M^2_{\text{weak}}$. Fermion loops also lead to a Higgs potential through $\phi_1^\dagger\phi_2 = f^2 - 2h^\dagger h + 2(h^\dagger h/(3f^2)$ which breaks $SU(3)_{\text{diag}} \to SU(2)_L$.

– to introduce hypercharge $U(1)_Y$ we have to postulate another $U(1)_X$, which includes a heavy gauge boson mixing with the $SU(3)/SU(2)$ and the $SU(2)$ gauge bosons, to produce γ, Z, Z'. This will be a problem, because this way we lose the custodial $SU(2)$ which is experimentally so well confirmed.

Because we will definitely need it later, we first compute the one helpful $SU(3)$-invariant term in the Lagrangian after rotating away the eaten Goldstones and to an order higher in $1/f$ than before

$$\phi_1^\dagger\phi_2 = \begin{pmatrix} 0 & f \end{pmatrix} \exp\begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} = f^2 - 2h^\dagger h + \frac{2}{3f^2}(h^\dagger h)^2 + \mathcal{O}\left(\frac{1}{f^4}\right) \tag{80}$$

Note that we omit the 8th generators of $SU(3)$, diag$(-1, -1, 2)$, and its corresponding Goldstone η and will dicuss it's physics at the end of the lecture. Moreover, we assume $f_1 = f_2 = f$. We will see that such terms can be loop–induced by gauge–boson or top loops, but we can always write them in terms of this combination $\phi_1^\dagger\phi_2$.

The $SU(3)$ gauge interactions sketched in the last sections now include terms like

$$\boxed{\mathcal{L} \supset \left|g_1 A_\mu \phi_1\right|^2 + \left|g_2 A_\mu \phi_2\right|^2} \tag{81}$$

To study their behavior we can for example set $g_2 = 0$, so that both terms are symmetric under both the two $SU(3)$ symmetries

$$\phi_1 \to U_1\phi_1 \qquad A_\mu \to U_1^\dagger A_\mu U_1 \qquad\qquad \phi_2 \to U_2\phi_2 \qquad A_\mu \to U_2^\dagger A_\mu U_2 \tag{82}$$

Switching on g_1 and g_2 in parallel then breaks this $[SU(3)]^2$ symmetry to a single diagonal symmetry $SU(3)_{\text{diag}}$

$$\phi_2 \to U\phi_2 \qquad \phi_1 \to U\phi_1 \qquad A_\mu \to U^\dagger A_\mu U \tag{83}$$

As we showed in the last section, the $SU(3)$–gauge–boson loops contribute to the Higgs mass as

$$\mathcal{L} \supset \frac{g_1^2 g_2^2}{(4\pi)^2}\, |\phi_1^\dagger\phi_2|^2\, \log\frac{\Lambda^2}{\mu^2} \sim -\frac{g_1^2 g_2^2}{(2\pi)^2}\, f^2\, \log\frac{\Lambda^2}{\mu^2}\, h^\dagger h + \mathcal{O}\left(h^4\right) \tag{84}$$

FIG. 2: Top–quark contributions to the Higgs mass from top loops. Note that the two–point diagram (left) involves a Standard–Model top quark, while the one–point diagram (right) exists only for the heavy top quark.

For a weak-scale Higgs $m_H \sim M_{\text{weak}}$ and $SU(2)$-type gauge couplings g_j, this means $f \sim$ TeV, which in turn means that our theory will break down around ~ 10 TeV.

Next, we remember that until now we only dealt with the gauge sector leading to quadratic divergences in the Higgs mass. We obviously need to extend the fermion sector, which otherwise creates quadratic divergences for the Higgs mass proportional to the top Yukawa. So we enlarge the $SU(2)$ heavy–quark doublet Q to an $SU(3)$ triplet $\Psi = (t, b, T) \equiv (Q, T)$. The Yukawa couplings look like $\lambda_j \phi_j^\dagger \Psi t_j^c$, in analogy to the Standard Model, but with two right–handed top singlets t_j^c which will combine to the Standard–Model and to the heavy right–handed tops. We can compute

$$\begin{aligned}
\phi_j^\dagger \Psi &= (0 f) \exp\left[\mp\frac{i}{f}\begin{pmatrix} & h \\ h^\dagger & \end{pmatrix}\right]\begin{pmatrix} Q \\ T \end{pmatrix} \\
&= (0 f)\left[\mathbb{1} \mp \frac{i}{f}\begin{pmatrix} & h \\ h^\dagger & \end{pmatrix} - \frac{1}{2f^2}\begin{pmatrix} hh^\dagger & \\ & h^\dagger h \end{pmatrix} + \dots\right]\begin{pmatrix} Q \\ T \end{pmatrix} \\
&= (0 f)\left[\begin{pmatrix} Q \\ T \end{pmatrix} \mp \frac{i}{f}\begin{pmatrix} hT \\ h^\dagger Q \end{pmatrix} - \frac{1}{2f^2}\begin{pmatrix} hh^\dagger Q \\ h^\dagger h T \end{pmatrix} + \dots\right] \\
&= fT \mp i h^\dagger Q - \frac{1}{2f}\, h^\dagger h T + \dots
\end{aligned} \tag{85}$$

Combining them gives assuming the simplification $\lambda_1 = \lambda_2 = \lambda$

$$\mathcal{L} \supset \lambda f \, \left(1 - \frac{1}{2f^2}\, h^\dagger h\right) T T^c + \lambda \, h^\dagger Q t^c + \dots \tag{86}$$

where we define the SM top quark as $t_2^c - t_1^2 = -i\sqrt{2} t^c$ and where its orthogonal partner $t_1^c + t_2^c = \sqrt{2} T^c$ appears in the T–mass term λf.

Both top quarks contribute to the Higgs mass as shown in Fig. 2. Note the factor 2 in the $T\bar{T}hh$ coupling from the two permutations of the Higgs fields. The scalar integrals involved we know, generally omitting a factor $1/(4\pi)^2$: $B(0; m, m)\big|_{\text{UV}} \sim (\Lambda/m)^2$. Adding two fermion propagators with mass m_t and two couplings alters the behavior of the Standard–Model diagram to $-i^4 \, \lambda^2 \, \Lambda^2 = -\lambda^2 \, \Lambda^2$. The second diagram starts from a scalar UV-divergent $A(m_T)\big|_{\text{UV}} \sim \Lambda^2$. Adding one fermion line and the 4-point coupling $-\lambda/f$ yields $-i^2 \lambda/f \, m_T \, \Lambda^2 = +\lambda^2 \Lambda^2$. From this hand–waving estimate we get an idea how these two top quarks cancel each other's quadratic divergence for the Higgs mass.
If we do this calculation more carefully, we find that indeed, for an $SU(3)$-invariant regulator, the two diagrams cancel. Actually, just like in supersymmetry, only the quadratic divergences cancel, and terms proportional to $\log m_t/m_T$ remain.

Note that again switching off $\lambda_2 = 0$ the Yukawa couplings are symmetric under both $\phi_j \to U_j \phi_j$, $\Psi \to U_j \Psi$. Just as for the gauge couplings, having $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ breaks $[SU(3)]^2 \to SU(3)_{\text{diag}}$ as the symmetry of the Yukawa part of the Lagrangian.
Strictly speaking, we could keep the two λ_j separated and would find

$$\begin{aligned}
m_T &= \sqrt{\lambda_1^2 f_1^2 + \lambda_2^2 f_2^2} \sim \max\nolimits_j(\lambda_j f_j) \\
\lambda_t &= \lambda_1 \lambda_2 \, \frac{1}{m_T} \, \sqrt{f_1^2 + f_2^2}
\end{aligned} \tag{87}$$

Writing down the SM with a protected light Higgs mass requires us to break both groups, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$,. This makes the Higgs a pseudo–Goldstone and allows only contributions proportional to $\lambda_1\lambda_2$ in the Higgs potential (and the Higgs mass). Strictly speaking, we could even show that only terms proportional to $\lambda_1^2\lambda_2^2$ appear, and terms with four Yukawa couplings never lead to quadratic divergences.

The remaining big mystery in this model is the Higgs potential, and in particular the relation between mass and quartic coupling. We can compare the relative sizes of the mass and self coupling which we get from the fermion loops

$$|\phi_1^\dagger \phi_2|^2 = f^2 - 4f^2(h^\dagger h) + \frac{14}{3}\,(h^\dagger h)^2 + \ldots \equiv -m^2 h^\dagger h + \lambda (h^\dagger h)^2$$
$$\Longrightarrow \qquad \boxed{\left|\frac{m^2}{\lambda}\right| \sim \frac{12}{14}\,f^2 \sim \mathcal{O}\left(\text{TeV}^2\right)} \qquad \text{while} \qquad \left|\frac{m^2}{\lambda}\right|_{\text{SM}} = 2v^2 \tag{88}$$

In other words, compared to the Standard Model, the mass is too large in comparison to the quartic coupling. There is no easy cure to this, so we resort to ad–hoc introducing a μ parameter with the proper sign

$$\mathcal{L} \supset \mu^2 \phi_1^\dagger \phi_2 = \mu^2 \left(f^2 - 2\,h^\dagger h + \mathcal{O}\left(\frac{1}{f^2}\right)\right) \tag{89}$$

Roughly $\mu \sim M_{\text{weak}}$ brings the Higgs mass to the correct value. Note that such a term also breaks the $U(1)$ symmetry linked to the 8th $SU(3)$ generators and gives η a mass of the order M_{weak}.

To summarize, we have analyzed the particle spectrum in the μ-model or Schmaltz model or simple–group model, which is necessary to avoid quadratic divergences in the Higgs mass at one loop. In this model we start from a global symmetry group $[SU(3)]^2$. These two symmetries we break spontaneously into a $SU(2)$ each, freeing up 10 Goldstone modes corresponding to the $[SU(3)/SU(2)]^2$ degrees of freedom.
At the same time, gauge and Yukawa interactions break $[SU(3)]^2$ to the diagonal, now gauged, subgroup $SU(3)_{\text{diag}}$ which is the one which is really spontaneously broken by a vev f. This means that half of these 10 Goldstone modes are going to be absorbed into massive $SU(3)/SU(2)$ gauge bosons. The other five now pseudo–Goldstones can develop a mass and a potential, but each term has to be proportional to both of the gauge (or Yukawa) couplings. As a check we can switch off one of the two gauge couplings: now we have two exact $SU(3)$ symmetries, one of which is gauged spontaneously broken, and acquires heavy gauge–boson masses of the scale f, while the other one is exact, *i.e.* protecting its Goldstones from acquiring a mass at all.
Apart from the Standard–Model particles and a light protected Higgs we find the particle spectrum

$$\begin{aligned}
SU(3) \text{ gauge bosons } W'^{\pm}, W'^{0} \quad &\text{with} \quad m_{W'} = \frac{g^2 f^2}{2} \\
\text{singlet } Z' \quad &\text{with} \quad m_{Z'} = g^2 f^2 \frac{2}{3-t^2} \qquad (t = \tan\theta_w) \\
\text{heavy top } T \quad &\text{with} \quad m_T = \sqrt{2}\lambda_t f \\
\text{Standard Model } Z \quad &\text{with} \quad m_Z = \frac{g^2 v^2}{4}\,(1+t^2) \qquad \text{etc...}
\end{aligned} \tag{90}$$

To avoid extending this particle content and correcting for the mass–quartic ratio in the Higgs potential we in addition need a tree–level parameter $\mu^2 \phi_1^\dagger \phi_2$.

B. The littlest Higgs

Combining what we know about sigma models and collective symmetry breaking we can construct another particularly economic little–Higgs model. In the μ model we write two sets of Goldstones in the fundamental representations of $SU(3)$, which are partly gauged and then broken to our $SU(2)_L$ via the high–scale vev f. It is crucial to have two distinct $SU(3)$ gauge groups (and gauge couplings) to forbid one–loop quadratically divergent Higgs self energies. The same trick we can play with two Yukawas, so that a Higgs potential is proportional to $g_1^2 g_2^2$ or to $\lambda_1^2\lambda_2^2$.

This time, we want to embed two gauge symmetries which overlap by the Standard Model Higgs doublet into one matrix field Σ: in other words, we write a matrix–valued Σ field which includes two copies of $SU(2)$ which are broken to the $SU(2)_L$ and which at the same time includes a pseudo–Goldstone–Higgs doublet. Two $SU(2)$ generators inside a 5×5 matrix could look like

$$Q_1^a = \frac{1}{2} \begin{pmatrix} -\sigma^{a*} & 0_{2x3} \\ 0_{3x2} & 0_{3x3} \end{pmatrix} \qquad Q_2^a = \frac{1}{2} \begin{pmatrix} 0_{3x3} & 0_{2x3} \\ 0_{3x2} & \sigma^a \end{pmatrix} \tag{91}$$

The Goldstone modes in the Σ field should include the Higgs doublet in a form which means that neither of the sets of $SU(2)$ generators include it. This means that when we break $SU(5)$ into one of the $SU(2)$ subgroups the Higgs will always stay a (pseudo–) Goldstone, which is the idea of collective symmetry breaking

$$\Sigma = e^{2i(\pi\cdot\hat{T})/f} \langle\Sigma\rangle \qquad \pi\cdot\hat{T} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} & h^* & \\ h^\dagger & & h^\dagger \\ & h & \end{pmatrix} \tag{92}$$

If the global $SU(5)$ symmetry is broken by $\langle\Sigma\rangle$, this will allow the h doublet to develop a potential, *i.e.* a mass and a self coupling. Note that $\hat{T}$ is indeed hermitian and traceless, so $\exp(2i(\pi\cdot\hat{T})/f)$ is a unitary transformation. The Standard–Model $SU(2)_L$ generators Q^a or the Higgs should of course not be affected by $\langle\Sigma\rangle$, because otherwise they would acquire masses of the order f. Therefore, we write

$$\langle\Sigma\rangle = \begin{pmatrix} & & 1\!\!1_{2\times2} \\ & 1 & \\ 1\!\!1_{2\times2} & & \end{pmatrix} \tag{93}$$

This vev obviously breaks our global $SU(5)$ symmetry, written in the adjoint representation. $SU(5)$ has $(N^2-1)=24$ generators $U=\exp(i\theta\cdot T)$ under which the Σ field transforms as $\Sigma \to U\Sigma U^T$. What remains after $\langle\Sigma\rangle$ is an $SO(5)$ symmetry, generated by the antisymmetric tensor with $(4+3+2+1)=10$ entries. We can use the transformation of Σ to derive the commutation properties of the <u>10 unbroken generators $\overline{T}$</u> and the <u>14 broken generators $\hat{T}$</u>. For the broken generators we find

$$\Sigma = e^{i(\pi\cdot\hat{T})/f} \langle\Sigma\rangle\, e^{i(\pi\cdot\hat{T}^T)/f} = e^{2i(\pi\cdot\hat{T})/f}\langle\Sigma\rangle\,, \tag{94}$$

or in other words $\langle\Sigma\rangle\,\hat{T}^T = \hat{T}\,\langle\Sigma\rangle$. For the remaining unbroken, good generators we require

$$\Sigma = e^{i(\pi\cdot\overline{T})/f} \langle\Sigma\rangle\, e^{i(\pi\cdot\overline{T}^T)/f} = \langle\Sigma\rangle \tag{95}$$

which translates into $\langle\Sigma\rangle\,\overline{T}^T = -\overline{T}\,\langle\Sigma\rangle$. We can explicitely compute the commutators for the sum of hermitian $SU(2)$ generators $Q^a = Q_1^a + Q_2^a$, to check that they are indeed not broken

$$Q\,\langle\Sigma\rangle = \frac{1}{2}\begin{pmatrix} & -\sigma^* \\ \sigma & \end{pmatrix} \qquad \langle\Sigma\rangle\,Q = \frac{1}{2}\begin{pmatrix} & \sigma^* \\ -\sigma & \end{pmatrix} \qquad \Rightarrow \qquad Q\,\langle\Sigma\rangle = -\langle\Sigma\rangle\,Q^T \tag{96}$$

So the generators Q^a, which we plan to make the generators of our Standard–Model $SU(2)$ gauge group are indeed part of the unbroken set of $SU(5)$ generators $\overline{T}$. The corresponding $U(1)$ generators are the diagonals $\text{diag}(-3,-3,2,2,2)/10$ and $\text{diag}(-2,-2,-2,3,3)/10$.

To compute the spectrum of the littlest Higgs model which breaks $SU(5) \to SO(5)$, we start by writing down the complete set of Goldstones associated with the broken generators in Σ, filling in the remaining 2×2 matrices and the diagonal generator

$$\boxed{\pi\cdot\hat{T} = \begin{pmatrix} \chi_{2\times2} & h^*/\sqrt{2} & \phi^\dagger_{2\times2} \\ h^T/\sqrt{2} & 0 & h^\dagger/\sqrt{2} \\ \phi_{2\times2} & h/\sqrt{2} & \chi^T_{2\times2} \end{pmatrix} + \frac{\eta}{2\sqrt{5}} \begin{pmatrix} 1\!\!1_{2\times2} & & \\ & -4 & \\ & & 1\!\!1_{2\times2} \end{pmatrix}} \tag{97}$$

The form is given by requirement $\langle\Sigma\rangle\,\hat{T}^T = \hat{T}\,\langle\Sigma\rangle$, which links opposite corners of $\pi\cdot\hat{T}$. The $SU(2)$ generators in χ form a hermitian traceless 2×2 matrix, but the combination of χ and χ^T in the opposite corners (instead of $-\sigma^*$ and σ or equivalently $-\sigma^T$ and σ) makes χ part of the broken subset $\hat{T}$. The remaining 2×2 matrix of generators ϕ is not traceless, but complex symmetric. The complex doublet h is hopefully the Standard–Model Higgs doublet, and η is the usual real singlet. Together, these field indeed correspond to the $3+6+4+1=14$ Goldstone degrees of freedom.

Unless something else happens (like collective symmetry breaking) the fields linked to the broken generators $(\pi \cdot \hat{T})$ can either turn into gauge–boson mass terms of the order f or stay massless. In particular, χ will make a set of $SU(2)$ gauge bosons $W'^{\pm}, W'^{0}$ heavy, where η corresponds to the B' field. We can mix the two $SU(2)$ groups described by χ (broken with mass scale f) and σ (unbroken with mass scale v) to the Standard–Model $SU(2)_L$.
For the littlest Higgs collective symmetry breaking occurs just the same way as in the μ model, namely through gauge couplings. The two sets of $SU(2)$ generators Q_j are linked once we remember that the particular combination Q_1+Q_2 is part of the unbroken set of $SU(5)$ generators.

$$D_\mu \Sigma = \partial_\mu \Sigma - i \sum_{j=1,2} g_j (W_{j\mu} Q_j) \Sigma - i \sum_j g_j \Sigma (W_{j\mu} Q_j^T) - i \sum_j g'_j (B_{j\mu} Y_j) \Sigma - i \sum_j g'_j \Sigma (B_{j\mu} Y_j) \tag{98}$$

In other words, the vev $\langle \Sigma \rangle$ again breaks this symmetry $[SU(2) \times U(1)]^2$ to the diagonal $SU(2)_L \times U(1)_Y$ at a scale f. Defining a set of $SU(2)$ and $U(1)$ mixing angles $\tan \Psi^{(\prime)} = g_2^{(\prime)}/g_1^{(\prime)}$ we can write the set of gauge bosons in terms of the Standard–Model and a heavy set of $SU(2) \times U(1)$ bosons

$$\begin{pmatrix} W_H^a \\ W_{SM}^a \end{pmatrix} = \begin{pmatrix} -\cos \Psi & \sin \Psi \\ \sin \Psi & \cos \Psi \end{pmatrix} \begin{pmatrix} W_1^a \\ W_2^a \end{pmatrix} \qquad \begin{pmatrix} B_H \\ B_{SM} \end{pmatrix} = \begin{pmatrix} -\cos \Psi' & \sin \Psi' \\ \sin \Psi' & \cos \Psi' \end{pmatrix} \begin{pmatrix} B_1^a \\ B_2^a \end{pmatrix} \tag{99}$$

As mentioned above the heavy gauge bosons acquire masses though the Goldstones χ

$$M_{W_H} = \frac{gf}{\sin 2\Psi} \qquad M_{B_H} = \frac{g' f}{\sqrt{5} \sin 2\Psi'} \tag{100}$$

The littlest–Higgs model works for quadratic divergences just like the $SU(3)$ model. Each of the two sets of generators $\{Q_j^a, Y_j\}$ corresponds to a 3×3 matrix of Goldstones in the respective other corner in Σ after breaking $SU(5)$ to $SU(2)$. So if we break the global symmetry down to one of the two $SU(2)$ groups the Higgs doublet will be a broken generator of the global $SU(5)$ and therefore remain massless. If we remove one set of gauge couplings $g_j^{(\prime)} = 0$, we indeed find a global $SU(3)$ symmetry which protects the Higgs from quadratic divergences proportional to the respective other $g_k^{(\prime)}$.

Protecting the Higgs mass from top loop works also similarly to the $SU(3)$ model. We extend the $SU(2)_L$ quark doublet to the triplet $\Psi = (b, -t, T)$ and add a right-handed singlet t'^c. Because we expect mixing between the two top singlets which will give us the Standard–Model and a heavy top quark we write two general Yukawa couplings involving the Σ field (just like we write Yukawa couplings in the usual Σ model)

$$\mathcal{L} \supset \lambda_1 \, f \, \epsilon_{ijk} \Psi_i \, \Sigma_{j4} \Sigma_{k5} \, t_1^c + \lambda_2 \, f \, T t_2^c \tag{101}$$

The Σ-field triplets we take from the 2×3 upper-right corner of the Goldstone matrix

$$\sigma_{jm} = \begin{pmatrix} \phi^\dagger \\ h^\dagger/\sqrt{2} \end{pmatrix} \qquad j = 1,2,3 \qquad m = 4,5 \tag{102}$$

If we set $\lambda_2 = 0$ this Yukawa coupling is symmetric under this $SU(3)$ symmetry, because it is the anti-symmetric combination of three triplets. Again, contributions to the Higgs mass therefore have to be proportional to $\lambda_1^2 \lambda_2^2$ and quadratic divergences are forbidden at one loop.
The two heavy quarks mix to the SM top quark and an additional heavy top

$$t_R = \frac{\lambda_2 t_1 - \lambda_1 t_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \qquad T_R = \frac{\lambda_1 t_1 + \lambda_2 t_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \qquad m_T = \sqrt{\lambda_1^2 + \lambda_2^2} f \tag{103}$$

where we as before have assumed $f = f_1 = f_2$. The actual top–Higgs coupling are given in terms of λ_j

$$\lambda_{ttH} \equiv \lambda_t = \frac{\sqrt{2} \lambda_1 \lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \qquad \lambda_{TTHH} \equiv -\frac{\lambda_T}{\sqrt{2} f} = \frac{-\lambda_1^2}{f \sqrt{\lambda_1^2 + \lambda_2^2}} \tag{104}$$

and ensure that the leading divergences in the Standard–Model two–point diagram and the heavy–top one–point diagram cancel.

An interesting question would be: can we distinguish little–Higgs models for example by relating the parameters in the top sector. After all, the construction of the μ model and the littlest–Higgs model are quite different. Using these expressions above we can write the heavy top mass in the littlest–Higgs model in term of the Yukawas $\lambda_{t,T}$ (modulo factor $\sqrt{2}$?)

$$m_T = f \, \frac{\lambda_t^2 + \lambda_T^2}{\sqrt{2}\lambda_T} \tag{105}$$

In contrast, in the $SU(3)$ model we saw ($f = f_1 = f_2$)

$$\lambda_T = \frac{\lambda_t}{2f} = \frac{\lambda_1 \lambda_2 \, f}{m_T} \qquad\qquad m_T = f \, \sqrt{\lambda_1^2 + \lambda_2^2} \tag{106}$$

So the relation between m_T and the $HHTT$ coupling are indeed different.

The heavy spectrum of the littlest Higgs model is

$$\begin{aligned} SU(2) \times U(1) \text{ gauge bosons } B', W'^{\pm}, Z' \quad &\text{with} \quad m_{B',W',Z'} = \mathcal{O}(f) \\ \text{Higgs triplet } \phi = \begin{pmatrix} \phi^{++} & \phi^+\sqrt{2} \\ \phi^+\sqrt{2} & \phi^0 \end{pmatrix} \quad &\text{with} \quad m_\phi = \mathcal{O}(f) \\ \text{heavy top } T \quad &\text{with} \quad m_T = \mathcal{O}(f) \end{aligned} \tag{107}$$

As described earlier in the lecture, from B' and ϕ we expect serious violation of custodial $SU(2)$. Electroweak precision data forces us to choose f unusually large in the little–Higgs model. On the other hand, a Higgs triplet with a doubly charged Higgs boson has a smoking–gun signature at the LHC, namely its production in weak–boson fusion: $uu \to ddW^+W^+ \to ddH^{++}$.

In contrast to the μ model, we now do not need a μ term, though. One–loop effects lead to a Coleman–Weinberg potential (which is nothing but a general quartic potential of a massive charged scalar in a gauge theory) with the relative mass scales

$$\frac{m_h^2}{\lambda} \sim \left(\frac{m}{g}\right)^2 \sim (2m)^2 < f^2 \tag{108}$$

after integrating out the heavy ϕ fields.

C. T parity

Looking at the tree–level violation of the custodial $SU(2)$ symmetry (*i.e.* $\rho \neq 1$) and at the benefits of a weakly interacting and stable dark–matter candidate it would be great to introduce a Z_2 symmetry which allows only two heavy little-Higgs particles per vertex. In other words, we would like to define a quantum number with one value for all weak–scale Standard–Model particles and another value for all particles with masses around f. Such a parity will be called T parity.

For the littlest Higgs, we would like to separate the additional heavy $SU(2)$ doublet from our Standard–Model gauge bosons. Assuming $g_1^{(\prime)} = g_2^{(\prime)}$ the Lagrangian involving $D_\mu \Sigma$ is symmetric under the exchange of the two $[SU(2) \times U(1)]$ groups. The eigenstates we can choose as

$$W_\pm = \frac{W_1 \pm W_2}{\sqrt{2}} \qquad\qquad B_\pm = \frac{B_1 \pm B_2}{\sqrt{2}} \tag{109}$$

where W_+, B_+ are Standard–Model gauge bosons, while W_-, B_- are heavy. Exchanging the indices (1↔2) is an even transformation for W_+, while it is odd for W_-, just as we want. Taking into account all broken generators, we apply a factor $(-)$ to each heavy field, while leaving h unchanged. In proper matrix notation we postulate a symmetry Ω which acts on the broken generators for example in the littlest Higgs model

$$\begin{aligned}
\pi\cdot\hat{T} &= \begin{pmatrix} \chi & h^* & \phi^\dagger \\ h^T & 0 & h^\dagger \\ \phi & h & \chi^T \end{pmatrix} + \eta \begin{pmatrix} 1\!\!1 & & \\ & -4 & \\ & & 1\!\!1 \end{pmatrix} \\
&\to^{\Omega} - \begin{pmatrix} 1\!\!1 & & \\ & -1 & \\ & & 1\!\!1 \end{pmatrix} \left[\begin{pmatrix} \chi & h^* & \phi^\dagger \\ h^T & 0 & h^\dagger \\ \phi & h & \chi^T \end{pmatrix} + \eta \begin{pmatrix} 1\!\!1 & & \\ & -4 & \\ & & 1\!\!1 \end{pmatrix} \right] \begin{pmatrix} 1\!\!1 & & \\ & -1 & \\ & & 1\!\!1 \end{pmatrix} \\
&= - \begin{pmatrix} 1\!\!1 & & \\ & -1 & \\ & & 1\!\!1 \end{pmatrix} \left[\begin{pmatrix} \chi & -h^* & \phi^\dagger \\ h^T & 0 & h^\dagger \\ \phi & -h & \chi^T \end{pmatrix} + \eta \begin{pmatrix} 1\!\!1 & & \\ & 4 & \\ & & 1\!\!1 \end{pmatrix} \right] \\
&= - \begin{pmatrix} \chi & -h^* & \phi^\dagger \\ -h^T & 0 & -h^\dagger \\ \phi & -h & \chi^T \end{pmatrix} - \eta \begin{pmatrix} 1\!\!1 & & \\ & -4 & \\ & & 1\!\!1 \end{pmatrix} \\
&= \begin{pmatrix} -\chi & h^* & -\phi^\dagger \\ h^T & 0 & h^\dagger \\ -\phi & h & -\chi^T \end{pmatrix} + (-\eta) \begin{pmatrix} 1\!\!1 & & \\ & -4 & \\ & & 1\!\!1 \end{pmatrix}
\end{aligned} \tag{110}$$

This symmetry work perfectly for the additional gauge bosons, including the heavy scalars ϕ. A problem arises when we assign such a quantum number to the heavy tops. Usually, we expand the $SU(2)_L$ doublet to a triplet under Ω and split the fermions into one set transforming under each $[SU(2)\times U(1)]_j$. At this point, we now have to introduce additional fermions and all hell breaks loose, even though the model by definition agrees better with current electroweak precision constraints.

One final remark concerning such a T parity. Recently (hep-ph/0701044) Chris and Richard Hill have shown that such a discrete parity if naively implemented is broken by anomalies, *i.e.* it is not stable after quantum corrections. Obviously, such considerations affect arguments over large time scales, like the formation of dark matter. On the other hand, I am not sure if our model–building friends will get around this problem using a fancier realization of the T parity. Let's wait and see...

IV. PSEUDO–AXIONS

Remember that until now we have always neglected the additional diagonal generator of our global symmetry group. In the μ model we saw that is acquires a mass through the μ term $\mu^2\phi_1^\dagger\phi_2$

$$m_\eta = \left(\frac{f_1}{f_2}+\frac{f_2}{f_1}\right)^{1/2}\mu \gtrsim \sqrt{2}\,\mu \sim M_{\rm weak} \tag{111}$$

In the littlest Higgs model, in contrast, the same Goldstone mode is eaten by the additional $U(1)_Y$ gauge field, the heavy photon with a mass $m_{B'}\sim f$. Both of these cases are in a sense clever constructions, to avoid the general problem that after breaking a global symmetry group to a lower–rank group, we will typically find diagonal generators which correspond to massless singlet scalars in the low–energy effective theory. Such scalars turn out to be similar to so-called axions.

Fermion coupling: Goldstones we know are protected from becoming massive by to their non–linear shift symmetry $\eta\to\eta+f\cdot\alpha$. This symmetry of course has to be respected by their scalar and pseudo–scalar couplings to fermions, which are of the general form

$$\begin{aligned}
\mathcal{L} &\supset g_S\overline{\Psi}1\!\!1\Psi\;\eta + g_P\overline{\Psi}\gamma^5\Psi\;\eta \qquad\qquad \gamma^0 = \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \\
&= g_S\Psi^\dagger\gamma^0\Psi\;\eta + g_P\Psi^\dagger\gamma^0\gamma^5\Psi\;\eta \\
&= g_S\left(\overline{\Psi}_L\Psi_R+\overline{\Psi}_R\Psi_L\right)\;\eta + g_P\left(\overline{\Psi}_L\Psi_R-\overline{\Psi}_R\Psi_L\right)\;\eta \\
&\to g_S\left(\overline{\Psi}_L\Psi_R+\overline{\Psi}_R\Psi_L\right)\;(\eta+f\cdot\alpha) + g_P\left(\overline{\Psi}_L\Psi_R-\overline{\Psi}_R\Psi_L\right)\;(\eta+f\cdot\alpha)
\end{aligned} \tag{112}$$

The first term is obviously not symmetric for $\alpha \neq 0$, so the global symmetry requires $g_S = 0$. The second term we could compute and find that it is actually allowed... So our diagonal generators η or pseudo–axions couple to fermions like pseudo-scalars. Note, however, that the η coupling to fermions does not include an f in the numerator when we write it in terms of the Σ field, so the $t\bar{t}\eta$ coupling will be suppressed by v/f.

Gauge–boson coupling: We can write down operators like $\eta W^+ W^-$, but they are be forbidden at tree level if η is a pseudo scalar. This is, by the way, the same for the heavy MSSM pseudo scalar A^0. Just as in the MSSM, η could couple to gauge bosons via $\eta W_{\mu\nu} \widetilde{W}^{\mu\nu}$, but this CP–odd combination is of mass dimension 5 and therefore loop suppressed as $\frac{v}{f}$. If will for example be induced by heavy top loops.

Mixing with Higgs: Potential terms like $\eta^2 h^2$ are allowed in the Lagrangian. However, they introduce a quadratic divergence in m_H when we link the two η fields to a loop. At one loop we find $\Delta m_h^2 \sim (\Lambda/4\pi)^2 \sim f^2$, which is precisely what we build little–Higgs models to avoid. As usually, $\Delta m_h^2 \sim v^2 \sim m_\eta^2$ is acceptable, which simply corresponds to a mandatory factor $\mathcal{O}(v/f)$ in front of the $\eta^2 h^2$ term.

Signatures for η are similar to the heavy pseudoscalar A^0 in the MSSM; if it is really light, we can see $h \to \eta\eta$ decays, otherwise we rely on production cross sections suppressed by (v^2/f^2) with subsequent decays to Standard–Model gauge bosons or fermions, similar to Higgs signatures. The CP properties of such scalars we can determine either from jet correlations in weak–boson–fusion production or from lepton–correlations in decays to $ZZ \to 4\ell$.

Acknowledgments: I would like to thank all the people who have tried to explain little–Higgs models to me, in particular Martin Schmaltz. You might have noticed that basically this entire set of note is based on his review. And as usual I would like to thank Maria Ubiali who produced this writeup out of my of unreadable collection of hand–written notes.

Literature: Little–Higgs models have the great advantage that at least I find the original papers very readable. Nevertheless, there are also a few very good review articles on the market...

– for the basics there is Wolfgang Kilian's great book on electroweak symmetry breaking, a very brief and yet complete introduction into the sigma model and strongly interacting theories. It is ridiculously expensive, though.

– there is the usual incredibly useful writeup which for example these lectures are based on. It is Martin Schmaltz' and David Tucker–Smith's review article (hep-ph/0502182). Obviously, it focusses on the μ model or so–called Schmaltz model.

– another equally useful review is Maxim Perelstein's hep-ph/0512128, which starts from the littlest Higgs instead.

– my chapter on T parity is unfortunately very brief. But don't worry, there are very readable papers by Ian Low and collaborators or more phenomenologically by the Cornell group.

– similarly, my chapter on pseudo–axions is too short. You can have a look for example at hep-ph/0411213, in particular for a phenomenological analysis of this general feature of little–Higgs models.

– the collider phenomenology of little–Higgs models you can find in the standard reference hep-ph/0301040. It also includes lots of Feynman rules for those of you who want to calculate for example LHC cross sections.