Models without a Higgs

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After last semester's introduction into the Higgs mechanism and into Higgs searches at the LHC I am now following the example of the CERN theory group, covering my back and discussing models which more or less successfully avoid including a fundamental Higgs boson. Such ideas have a long history, for example as technicolor models. After we thought LEP had killed all of them, they have recently surfaced in the context of extra dimensions. This introduction is based on a brief SUPA course, it is as usually full of typos which completely reflect the fact that I am trying to learn the topic while teaching it...

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I. ELECTROWEAK SYMMETRY BREAKING

The usual argument for the existence of a Higgs boson starts from a completely massless Lagrangian of a gauge theory with matter fermions — and the fact that neither gauge–boson nor fermion masses can be simply included without breaking gauge invariance. This is of course correct, but it does not automatically imply the existence of a fundamental scalar Higgs boson. As an introduction to this topic, let us try to give masses to a photon and to fermions and this way break electroweak gauge invariance, but avoiding to postulate a fundamental Higgs boson.

A. Massive photon

As a starting point we choose electrodynamics, *i.e.* a (massless) photon in a locally $U(1)$–symmetric Lagrangian. To its kinetic $F \cdot F$ term we add a photon mass and a real uncharged scalar field without a mass and without a coupling to the photon, but with a scalar–photon mixing term:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^2 f^2 A_\mu^2 - ef A_\mu \partial^\mu \phi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^2 f^2 \left(A_\mu - \frac{1}{ef}\partial_\mu\phi\right)^2 - \frac{1}{2}\left(\partial_\mu\phi\right)^2 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2 f^2 \left(A_\mu - \frac{1}{ef}\partial_\mu\phi\right)^2 \end{aligned} \tag{1}$$

e is the usual electric charge, *i.e.* just a c-number without any specific relevance in this interaction–less Lagrangian, while f is a mass scale describing the photon mass as well as the mixing term. The Lagrangian includes only terms with mass dimension four, if we remember that bosonic fields like A_μ and ϕ have mass dimension one. We can define a simultaneous gauge transformation of both fields in the Lagrangian

$$A_\mu \longrightarrow A_\mu + \frac{1}{ef}\partial\chi \qquad\qquad \phi \longrightarrow \phi + \chi \tag{2}$$

under which the Lagrangian is indeed invariant. Here, χ is a real number. If we now re-define the photon field as $B_\mu = A_\mu - \partial_\mu\phi/(ef)$ we can first compare the two kinetic terms

$$\begin{aligned} F_{\mu\nu}\Big|_B &= \partial_\mu B_\nu - \partial_n u B_\mu = \partial_\mu\left(A_\nu - \frac{1}{ef}\partial_\nu\phi\right) - \partial_\nu\left(A_\mu - \frac{1}{ef}\partial_\mu\phi\right) \\ &= \partial_\mu A_\nu - \partial_n u A_\mu = F_{\mu\nu}\Big|_A \end{aligned} \tag{3}$$

and then rewrite the Lagrangian as

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2 f^2 B_\mu^2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_B^2 B_\mu^2} \tag{4}$$

This Lagrangian effectively describes a massive photon field B_μ, which has absorbed the real scalar ϕ as its additional longitudinal component. Remember that a massless gauge boson A_μ has only two on-shell degrees of freedom, namely left and right–handed polarization, while the massive $B\mu$ has an additional longitudinal polarization degree of freedom. Without any fundamental Higgs boson appearing, the photon has 'eaten' the real scalar field ϕ.

The difference to the usual $SU(2)$ Higgs mechanism is that we have chosen not to introduce a charged $SU(2)$ doublet, so there are no degrees of freedom left after the photon gets is mass. On the other hand, this little trick means that our toy model is not going to well-suited to make $SU(2)$ gauge bosons massive. What is illustrates is only how by introducing a neutral scalar particle without an interaction but with a mixing term we make gauge bosons heavy. This mechanism we will use later again.
What kind of properties does this field ϕ need to have, so that we can use it to provide a photon mass? From the combined gauge transformation we immediately see that any additional purely scalar terms in the Lagrangian (like a scalar potential $V(\phi)$) need to be symmetric under the linear shift $\phi \to \phi + \chi$, not to spoil gauge invariance. This means that we cannot write down polynomial terms ϕ^n, like a mass or a self coupling of ϕ. Similarly, a regular ϕAA interaction would not be possible, either. Only derivative interactions proportional to $\partial\phi$ to any conserved currents are fine. In that case we can absorb the shift by χ into a total derivative in the Lagrangian.

B. Fermion masses and chiral symmetry

Giving a mass to a fermion without a Higgs boson is a little more involved. We start by splitting a Dirac fermion, *i.e.* a 4-spinor, into its left-handed and right-handed projections

$$\psi_L = \frac{1\!\!1 - \gamma_5}{2}\,\psi \equiv P_L \psi \qquad\qquad \psi_R = \frac{1\!\!1 + \gamma_5}{2}\,\psi \equiv P_R \psi \tag{5}$$

where $P_{L,R}$ are projectors in the 4×4 Dirac space. The kinetic term of the Dirac fermion can be rewritten as

$$\begin{aligned}
\mathcal{L} \supset \overline{\psi}\, i\, \partial\!\!\!/ \psi &= \overline{\psi}\, i\, \partial\!\!\!/ \,(P_L + P_R)\, \psi \\
&= \overline{\psi}\, i\, \partial\!\!\!/ \left(P_L^2 + P_R^2\right) \psi \\
&= i\overline{\psi}\, \left(P_R\, \partial\!\!\!/ P_L + P_L\, \partial\!\!\!/ P_R\right) \psi \qquad &&\text{with} \qquad \{\gamma_5, \gamma_\mu\} = 0 \\
&= i(\overline{P_L\psi})\, \partial\!\!\!/ (P_L\psi) + i(\overline{P_R\psi})\, \partial\!\!\!/ (P_R\psi) \qquad &&\text{with} \qquad \overline{\psi} = \psi^\dagger \gamma^0 \\
&= \overline{\psi}_L\, i\, \partial\!\!\!/ \psi_L + \overline{\psi}_R\, i\, \partial\!\!\!/ \psi_R
\end{aligned} \tag{6}$$

Under a global so-called chiral symmetry transformation $U(1)_L \times U(1)_R$ which independently transforms the two chiralities $\phi_{L,R}$

$$\boxed{\psi_L \longrightarrow e^{-i\theta} \psi_L} \qquad\qquad \boxed{\psi_R \longrightarrow e^{-i\omega} \psi_R} \tag{7}$$

this Lagrangian is symmetric. Obviously, we can combine these two parts of the chiral transformation into different basis elements, constructing a vector-type and an axial-vector-type combination:

$$\begin{aligned}
\psi_L &\longrightarrow e^{-i\theta} \psi_L \qquad\qquad & \psi_L &\longrightarrow e^{-i\theta} \psi_L \\
\psi_R &\longrightarrow e^{-i\theta} \psi_R \qquad\qquad & \psi_R &\longrightarrow e^{+i\theta} \psi_R
\end{aligned} \tag{8}$$

A gauge-invariant Lagrangian under one definition of the chiral symmetry will always be invariant under the other. The same way we can now rewrite a Dirac mass in terms of the two chiralities

$$\begin{aligned}
\mathcal{L} \supset m\, \overline{\psi}\psi &= m\, \overline{\psi}\, \left(P_L^2 + P_R^2\right) \psi \\
&= m\, (\overline{P_R\psi})(P_L\psi) + m\, (\overline{P_L\psi})(P_R\psi) \\
&= m\, \left(\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R\right)
\end{aligned} \tag{9}$$

and immediately notice that the $U(1)_L \times U(1)_R$ symmetry is broken and only its vector combination $\theta = \omega$ remains. The question arises — can we write down a fermion mass while keeping the chiral symmetry intact, and without introducing an additional fundamental Higgs boson.

Just like in the Standard Model we first introduce a complex scalar field Φ with a Yukawa coupling to the fermions:

$$\mathcal{L} \supset \overline{\psi}\, i\, \partial\!\!\!/ \psi - g\, \left(\overline{\psi}_L \psi_R \Phi + \overline{\psi}_R \psi_L \Phi^*\right) + |\partial_\mu \Phi|^2 - V(|\Phi|) \tag{10}$$

If the scalar field transforms under the $U(1)_L \times U(1)_R$ chiral symmetry as

$$\Phi \longrightarrow e^{-i(\theta - \omega)}\, \Phi \tag{11}$$

the Yukawa couplings as well as the kinetic and the potential terms for Φ are gauge invariant. As usual, we now spontaneously break the chiral symmetry by introducing a potential for Φ with a nontrivial (*i.e.* $\Phi \neq 0$) minimum:

$$V = -M^2 |\Phi|^2 + \frac{\lambda}{2} |\Phi|^4 = -\frac{\lambda}{2} v^2 |\Phi|^2 + \frac{\lambda}{2} |\Phi|^4 = \frac{\lambda}{2} \left(|\Phi|^2 - \frac{v^2}{2}\right)^2 + \text{const} \qquad \text{with} \qquad \langle \Phi \rangle \equiv \frac{v}{\sqrt{2}} = \frac{M}{\sqrt{\lambda}} \tag{12}$$

Note that there are definitions with a factor λ and those with $\lambda/2$ around. I am here sticking to the conventions in the technicolor review. We can define the two on-shell degrees of freedom of a complex scalar (c-number) as

$\sqrt{2}\Phi = (v+h(x))\ \exp(i\phi(x)/f)$, again with a dimensionful constant f compensating the mass dimension of the scalar field in the exponent. The Φ-dependent part of the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} \supset & \frac{1}{2}(\partial h)^2 + \frac{M^2}{2}(v+h)^2 - \frac{\lambda}{8}(v+h)^4 + \frac{1}{2}(v+h)^2 \left|\frac{\partial\phi}{f}\right|^2 \\
= & \frac{1}{2}(\partial h)^2 + \frac{M^2}{2}\left(h + \frac{\sqrt{2}M}{\sqrt{\lambda}}\right)^2 - \frac{\lambda}{8}\left(h + \frac{\sqrt{2}M}{\sqrt{\lambda}}\right)^4 + \frac{1}{2}(v+h)^2 \left|\frac{\partial\phi}{f}\right|^2 \\
= & \frac{1}{2}(\partial h)^2 + \frac{M^2}{2}h^2 + M^2 h\frac{\sqrt{2}M}{\sqrt{\lambda}} - \frac{\lambda}{8}h^4 - \frac{\lambda}{2}h^3\frac{\sqrt{2}M}{\sqrt{\lambda}} - \frac{3\lambda}{4}h^2\frac{2M^2}{\lambda} - \frac{\lambda}{2}h\frac{2M^2}{\lambda}\frac{\sqrt{2}M}{\sqrt{\lambda}} \\
& + \frac{1}{2}(v+h)^2 \left|\frac{\partial\phi}{f}\right|^2 + \text{const.} \\
= & \frac{1}{2}(\partial h)^2 - \frac{M^2}{2}h^2 - \sqrt{\frac{\lambda}{2}}Mh^3 - \frac{\lambda}{8}h^4 + \frac{1}{2}(v+h)^2 \left|\frac{\partial\phi}{f}\right|^2 + \text{const.}
\end{aligned} \tag{13}$$

Again, the field ϕ has no mass or coupling and only appears as $(\partial\phi)$.
The Higgs field h has a mass M and a self coupling λ. However, in our calculation we have only made use of the finite combination $v = \sqrt{2}M/\sqrt{\lambda}$. As long as v stays finite we can take the combined limit $M \to \infty$ and $\lambda \to \infty$. This way, all terms proportional to $h^n (n = 2, 3, 4)$ become very large. In contrast, after Fourier-transforming we know that the kinetic term $(\partial h)^2$ will give contributions o f the order of the typical momentum or energy scale E we are probing in a given process. If we make M and with it $\sqrt{\lambda}$ much larger than that, $M \gg E$, we can neglect the kinetic term for the Higgs field in the Lagrangian $(\partial h)^2 \ll M^2 h^2$. Note that this inequality is not really mathematically correct, because for the kinetic term it refers to its size when evaluated for a given process. In that case, our Lagrangian becomes

$$\mathcal{L} \supset -\frac{M^2}{2}h^2 - \sqrt{\frac{\lambda}{2}}Mh^3 - \frac{\lambda}{8}h^4 + \frac{1}{2}(v+h)^2 \left|\frac{\partial\phi}{f}\right|^2 \tag{14}$$

Because the Higgs field h does not propagate, we can use its Euler–Lagrange equation $\partial\mathcal{L}/\partial h = 0$ to compute its (constant) field value. If we neglect its appearance in the kinetic term of ϕ (with a prefactor of order $vE^2/f \ll M$) we see that there is no linear term in h in the Lagrangian, which means that $\partial\mathcal{L}/\partial h$ is proportional to h, so one solution is $h(x) = 0$. Our weak–scale Lagrangian becomes simply the kinetic term for a massless scalar field ϕ. To obtain the correct normalization of this kinetic term for $h = 0$ we need to fix $f^2 = v^2$:

$$\mathcal{L} \supset \frac{1}{2}\frac{v^2}{f^2}(\partial\phi)^2 = \frac{1}{2}(\partial\phi)^2 \tag{15}$$

Going into the limit $M \to \infty$ has one profound consequence for our theory. Usually we attempt to construct renormalizable Lagrangians, *i.e.* Lagrangians which describe physics to arbitrarily high scales. Such a construction ensures for example that any transition amplitude is bounded from above at all energy scales, so that our theory is unitary at all energy scales. Now, in the large-M limit we have explicitely required $E/M \ll 1$, which means that we can still apply our theory to larger and larger energies, but not for a fixed value of M. We have to make sure that $E/M \ll 1$ always applies. This is the typical condition for an effective field theory — it only produces sensible predictions at energy scales below a given cut–off scale M. Or in other words, our theory is not anymore renormalizable or unitary.
Such a model breaking a gauge symmetry like the chiral symmetry is called a non-linear σ model, because of the non-linear dependence of Φ on the one remaining physical field ϕ. The σ field is our Higgs field, which can be decoupled, while the remaining massless field ϕ is usually referred to as the π field.

Let us now study the Yukawa terms in this limit and see if they still give rise to fermion masses. The original field Φ simply becomes $\sqrt{2}\Phi = f\exp(i\phi/f)$ with one fixed energy scale $f = v$. The complete Lagrangian modulo the potential term becomes

$$
\begin{aligned}
\mathcal{L} \supset &\, \overline{\psi}\, i \not{\partial}\psi + \frac{1}{2}(\partial\phi)^2 - \frac{gf}{\sqrt{2}} \left[\overline{\psi}_L \psi_R e^{+i\phi/f} + \overline{\psi}_R \psi_L e^{-i\phi/f} \right] \\
= &\, \overline{\psi}\, i \not{\partial}\psi + \frac{1}{2}(\partial\phi)^2 - \frac{gf}{\sqrt{2}} \left[\overline{\psi}_L \psi_R \left(1 + i\frac{\phi}{f}\right) + \overline{\psi}_R \psi_L \left(1 - i\frac{\phi}{f}\right) \right] + \mathcal{O}\left(\frac{1}{f^2}\right) \\
= &\, \overline{\psi}\, i \not{\partial}\psi + \frac{1}{2}(\partial\phi)^2 - \frac{gf}{\sqrt{2}}\, \overline{\psi}\psi - \frac{ig}{\sqrt{2}}\, \overline{\psi} \left(P_R^2 - P_L^2\right) \psi\, \phi + \mathcal{O}\left(\frac{1}{f^2}\right) \\
= &\, \boxed{\overline{\psi}\, i \not{\partial}\psi + \frac{1}{2}(\partial\phi)^2 - \frac{gf}{\sqrt{2}}\, \overline{\psi}\psi - \frac{ig}{\sqrt{2}}\, \overline{\psi}\gamma_5\psi\, \phi + \mathcal{O}\left(\frac{1}{f^2}\right)} \qquad \text{with} \qquad P_R - P_L = \gamma_5
\end{aligned}
\tag{16}
$$

In this form we can read off that ϕ is a massless pseudoscalar with a coupling strength $ig/\sqrt{2}$ which in terms of the fermion mass $m = fg/\sqrt{2}$ can be written as im/f. This relation between mass and pseudoscalar coupling is called Goldberger–Treiman relation. It can for example be verified in the case of the QCD pion's interaction in comparison to the nucleon masses.

This example of a non-linear sigma model illustrates how using a $SU(2)$ doublet scalar field we can give masses to fermions via Yukawa couplings. The chiral $SU(2)_L \times SU(2)_R$ symmetry is broken by the vacuum expectation value of the scalar field. Its radial excitations around the minimum we can decouple, while the massless scalar becomes a physical mode in our theory. On the other hand, we could of course use such a mode to give masses to gauge bosons, as seen before.

C. Goldstone's theorem

Those who know more about spontaneous symmetry breaking have noticed that using these two examples we have illustrated a few vital properties of Nambu–Goldstone bosons (NGB). Such massless physical states appear in many areas of physics and are described by Goldstone's theorem:

If a global symmetry group is spontaneously broken into a group of lower rank, its broken generators correspond to physical Goldstone modes. These fields transform non-linearly under the larger and linearly under the smaller group. They have to be massless, as the non-linear transformation only allows derivative terms in the Lagrangian.

If the spontaneous symmetry breaking induces gauge–boson masses, these massive degrees of freedom are 'eaten' Goldstone modes, and the mass is given by the vev breaking the larger symmetry. If the smaller symmetry is also broken, the NGBs become pseudo-NGB and acquire a mass of the size of this hard-breaking term.

For an alternative introduction into non–linear σ models and into Goldstone modes, you can have a look into the introduction of my little–Higgs lecture notes.

II. TECHNICOLOR

Technicolor is a way to break our electroweak symmetry and create masses for gauge bosons essentially using a non-linear sigma model, as we have seen it in the last section. In this example we have given the scalar field Φ a vacuum expectation value v through a potential, which is basically the Higgs mechanism. However, we know another way to break (chiral) symmetries through condensates — QCD. So let us review very few aspects of QCD which we will need later.

First, we should illustrate why an asymptotically free theory like QCD is a good model to explain electroweak symmetry breaking. For this we recall the main theoretical problem with the Higgs mechanism, *i.e.* spontaneous symmetry breaking with a fundamental scalar Higgs boson: If we think of our gauge theories as a stack of fundamental renormalizable field theories with some kind of cutoff scale (like for example the Planck scale) we can compute the quantum corrections to the Higgs mass with this cutoff. We find that the Higgs mass, and *only* the Higgs mass, corrections are quadratically divergent with the cutoff. This behavior is called the hierarchy problem between the electroweak scale v and for example the Planck mass. In other words, we introduce the Higgs boson to construct a renormalizable truly fundamental field theory perturbatively valid to all energies, and the Higgs mass itself spoils the high–energy behavior. The only easy way out is to tune the Higgs–mass counter term to cancel this cutoff dependence order by order, but this way we betray our original idea that small parameters in the Lagrangian cannot just occur, but need to be protected by some kind of symmetry. The alternative would be to postulate a UV completion of the Standard Model which cures this behavior and makes the complete theory consistent again. The most famous such completion is TeV–scale supersymmetry.

How can an interaction which becomes strong at small energies solve this problem — or why have we never heard of the hierarchy problem $\Lambda_{\rm QCD} \ll M_{\rm Planck}$? The inherent mass scale of QCD is $\Lambda_{\rm QCD} \sim 200$ MeV. It describes the scale at which the running QCD coupling constant $\alpha_s = g_s^2/(4\pi)$ becomes strong, *i.e.* perturbation theory in α_s breaks down, and quarks and gluons stop being QCD's physical degrees of freedom. At the leading one-loop level we can easily see where $\Lambda_{\rm QCD}$ comes from. Summing all gluon self–energy bubbles for example in the s–channel of the process $q\bar{q} \to q'\bar{q}'$ corresponds to the definition of an effective coupling

$$\alpha_s \to \alpha_s \, \left(1 - \frac{\alpha_s}{4\pi} \beta \log \frac{p^2}{\mu_R^2} \right) \to \alpha_s \, \left(1 + \frac{\alpha_s}{4\pi} \beta \log \frac{p^2}{\mu_R^2} \right)^{-1} \equiv \alpha_s^{\rm eff}(p^2) \tag{17}$$

where p^2 is the momentum flowing through the gluon propagator and μ_R is the (artifical) renormalization scale we are forced to introduce because we cannot write down a logarithm of a mass dimension. The form of the β function depends on the particle content of QCD, but not on the particle masses:

$$\beta = \frac{11}{3} N_c - \frac{2}{3} n_f > 0 \qquad \text{with} \qquad N_c = 3, \quad n_f = 5 \quad \text{(below the top threshold)} \tag{18}$$

This way, at large values of p^2 the denominator in parentheses becomes large and the effective running α_s becomes small, *i.e.* QCD is asymptotically free at large energies. We can relate the α_s values at two scales via

$$\frac{1}{\alpha_s(p^2)} = \frac{1}{\alpha_s(p_0^2)} \, \left(1 + \frac{\alpha_s(p_0^2)\beta}{4\pi} \, \log \frac{p^2}{p_0^2} \right) = \frac{1}{\alpha_s(p_0^2)} + \frac{\beta}{4\pi} \, \log \frac{p^2}{p_0^2} \stackrel{!}{=} \frac{\beta}{4\pi} \, \log \frac{p^2}{\Lambda_{\rm QCD}^2} \tag{19}$$

and parameterize its energy behavior using one dimensionful parameter $\Lambda_{\rm QCD}$. The functional form including $\Lambda_{\rm QCD}$ only reflects the general polynomial form of the one-loop running $\alpha_s^{-1}(p^2) = C_0 + C_1 \log p^2$. Practically, the value of $\Lambda_{\rm QCD}$ is extracted for example in a combined with with the parton densities. At leading order we can solve the above definition for $\Lambda_{\rm QCD}$:

$$\frac{1}{\alpha_s(p_0^2)} = \frac{\beta}{4\pi} \log \frac{p_0^2}{\Lambda_{\rm QCD}^2} \qquad \Leftrightarrow \qquad \log \frac{\Lambda_{\rm QCD}^2}{p_0^2} = -\frac{4\pi}{\beta} \, \frac{1}{\alpha_s(p_0^2)} \qquad \Leftrightarrow \qquad \frac{\Lambda_{\rm QCD}^2}{p_0^2} = \exp \left[-\frac{4\pi}{\beta} \, \frac{1}{\alpha_s(p_0^2)} \right] \tag{20}$$

This means that because QCD is not scale invariant, *i.e.* we have to introduce a renormalization scale in our perturbative expansion, the running of a dimensionless coupling constant can be translated into an inherent mass scale. This mass scale characterizes the theory, *e.g.* QCD, in the sense that $\alpha_s(p^2 = \Lambda_{\rm QCD}^2) \sim 1$ and for scales below $\Lambda_{\rm QCD}$ the theory will become strongly interacting. Note that first of all this scale could not appear if for some reason $\beta \simeq 0$ and that it secondly does not depend on any mass scale in the theory. This phenomenon of a logarithmically running coupling introducing a mass scale in the theory is called dimensional transmutation. It is the reason why there is no hierarchy problem between $\Lambda_{\rm QCD}$ and $M_{\rm Planck}$: if at a high scale we start from a strong coupling in the $10^{-2} \cdots 10^{-1}$ range the QCD scale will arrive at its known value without any need for fine tuning.

Just including the quark doublets and the covariant derivative describing the qqg interaction the QCD Lagrangian reads

$$\mathcal{L}_{\rm QCD} \supset \overline{\Psi}_L \, i \, \not{D} \Psi_L + \overline{\Psi}_R \, i \, \not{D} \Psi_R \tag{21}$$

We immediately see that it is symmetric under a chiral-type $SU(2)_L \times SU(2)_R$ symmetry. This symmetry forbids quark masses, *i.e.* it acts as a custodial symmetry for the tiny quark masses we measure for example for the valence quarks u, d. Because QCD is asymptotically free, at energies below roughly $\Lambda_{\rm QCD}$ the essentially massless quarks form condensates, *i.e.* two–quark operators will develop a vacuum expectation value $\langle \overline{\Psi} \Psi \rangle$. This operator spontaneously breaks the $SU(2)_L \times SU(2)_R$ symmetry into the (diagonal) $SU(2)$ of isospin. The valence quarks at low energies develop masses of the order of $m_{\rm nucleon}/3 \sim \Lambda_{\rm QCD}$, and the different composite color–singlet mesons and baryons become the relevant physical degrees of freedom. Their masses are of the order of the nucleon masses $m_{\rm nucleon} \sim 1$ GeV.
The only remaining massless particles are the NGBs from the breaking of $SU(2)_L \times SU(2)_R$, the pions. Their masses are not strictly zero, because the valence quarks do have a small mass of a few MeV. Their coupling strength (or decay

rate) is governed by f_π. It is defined by $\langle 0|j^5_\mu|\pi\rangle = i f_\pi p_\mu$, *i.e.* it parameterizes the breaking of the chiral symmetry via breaking the axial-vector-like $U(1)_A$. The axial current can be computed as $j^5_\mu = \delta\mathcal{L}/\delta(\partial_\mu \pi)$ and in the $SU(2)$ basis reads $j^5_\mu = \overline{\psi}\gamma_\mu \tau \psi/2$. From the measured decays of the light color–singlet QCD pion into two leptons we know that $f_\pi \sim 100$ MeV.

There are two QCD parameters which we need to adjust when building the simplest technicolor model: the size of the new gauge group and the scale at which the asymptotically free theory becomes strongly interacting. In terms of the two parameters N_c and Λ_{QCD} there are scaling rules in QCD which are based on for example $\beta \propto N_c$ (and which strictly speaking do not hold arbitrarily well):

$$f_\pi \sim \sqrt{N_c}\,\Lambda_{\text{QCD}} \qquad\qquad \langle \overline{Q}Q\rangle \sim N_c\,\Lambda^3_{\text{QCD}} \qquad\qquad m_{\text{fermion}} \sim \Lambda_{\text{QCD}} \tag{22}$$

The Λ_{QCD} dependence simply follows from the mass dimension. The dimension of the vev is given by the mass dimension 3/2 of each fermion field.
The N_c dependence of f_π can be easily guessed: the pion decay rate is by definition proportional to f_π^2. The Feynman diagrams for this decay are (apart from the strongly interacting complications, parameterized by the appearance of f_π) the same as for the Drell–Yan process $q\bar{q} \to \gamma, Z$. The color structure of this process leads to an explicit factor of $\delta_{ab}\delta_{ab} = N_c$ and an averaging factor of $1/N_c$ for each of the quarks. Together, this gives a factor $1/N_c$ for a color singlet decaying to a non-colored photon, the pion decay rate is proportional to f_π^2/N_c. This means the pion decay constant scales like $f_\pi \sim \sqrt{N_c}$. The vev–operator represents two quarks exchanging a gluon at energy scales small enough for α_s to become large. The color factor (without any averaging over initial states) simply sums over all colors states for the color-singlet condensate, *i.e.* it is proportional to N_c. The fermion masses have nothing to do with color states and hence should not depend on the number of colors. For details you should ask a lattice gauge theorist, but we already get the idea how would should construct our high–scale version of QCD, dubbed technicolor.

A. Scaling up QCD

Let us work out the idea that a mechanism just like QCD condensates could be the underlying theory of the non-linear σ model described in the introduction. in contrast to QCD we now have a gauged custodial symmetry of the gauge–boson masses. The longitudinal modes of the massive W and Z bosons are then the NGBs (techni-pions) of the symmetry breaking induced by a condensate. The corresponding mass scale would have to be $\Lambda_{\text{T}} \sim f \sim v = 246$ GeV. Fermion masses we postpone to the next section — in the 70s, when technicolor was developed, all known fermions had masses of the order of GeV or much less, so they were to a good approximation massless compared to the gauge bosons.
To induce W and Z masses we write down the non–linear sigma model in its $SU(2)$ version, at this point without talking about the source of the vacuum expectation value f_{T} appearing in

$$\Phi = \frac{1}{\sqrt{2}}\, e^{i(\pi\cdot\tau)/f_{\text{T}}} \begin{pmatrix} f_{\text{T}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} f_{\text{T}} + i(\pi\cdot\tau) + \mathcal{O}(f_{\text{T}}^{-1}) \\ 0 \end{pmatrix} \tag{23}$$

As basis vectors we use the three Pauli matrices $\{\tau_j, \tau_k\} = 2\delta_{jk}$. We will in a second need their property

$$\left(\sum_j \tau_j\right)\left(\sum_k \tau_k\right) = \sum_{j<k} (\tau_j\tau_k + \tau_k\tau_j) + \sum_j \tau_j^2 = 3\,\mathbb{1} \qquad \Rightarrow \qquad (\tau\cdot\pi_1)\,(\tau\cdot\pi_2) = \sum_j \pi_{1,j}\pi_{2,j} = (\pi_1\cdot\pi_2) \tag{24}$$

The $SU(2)$-covariant derivative in the charge basis of the Pauli matrices

$$(\tau\cdot\pi) \equiv \sum_{(+,-,3)} \tau_j\pi_j = \frac{\tau^1 + i\tau^2}{\sqrt{2}}\,\frac{\pi^1 - i\pi^2}{\sqrt{2}} + \frac{\tau^1 - i\tau^2}{\sqrt{2}}\,\frac{\pi^1 + i\pi^2}{\sqrt{2}} + \tau^3\pi^3 = \sum_{(1,2,3)} \tau_j\pi_j \tag{25}$$

gives, when to simplify the formulas we for a moment forget about the $U(1)_Y$ contribution and only keep the non-zero upper entry:

$$\begin{aligned}
iD^\mu\Phi &= \left[i\partial^\mu - \frac{g_2}{2}\,(\tau\cdot W^\mu)\right]\,\frac{1}{\sqrt{2}}\left[f_{\rm T} + i(\tau\cdot\pi) + \mathcal{O}(f_{\rm T}^{-1})\right] \\
&= \frac{1}{\sqrt{2}}\,\left[-\partial^\mu(\tau\pi) - \frac{f_{\rm T}g_2}{2}(\tau\cdot W^\mu)\right] \\
(D_\mu\Phi)^\dagger D^\mu\Phi &= \frac{1}{2}\,\left[-\partial_\mu(\tau\pi) - \frac{f_{\rm T}g_2}{2}(\tau\cdot W_\mu)\right]\,\left[-\partial^\mu(\tau\pi) - \frac{f_{\rm T}g_2}{2}(\tau\cdot W^\mu)\right] \\
&\supset \frac{1}{2}(\partial\pi)^2 + \frac{f_{\rm T}g_2}{2}\,(W_\mu\cdot(\partial^\mu\pi))
\end{aligned} \tag{26}$$

If we also include the generator of the hypercharge $U(1)$ we find a mixing term between the techni-pions and the $SU(2)$ gauge bosons

$$\boxed{\mathcal{L} \supset \frac{g_2 f_{\rm T}}{2}\,W^+_\mu\partial^\mu\pi^- + \frac{g_2 f_{\rm T}}{2}\,W^-_\mu\partial^\mu\pi^+ + f_{\rm T}\,\left(\frac{g_2}{2}\,W^0_\mu + \frac{g_1}{2}\,B_\mu\right)\,\partial^\mu\pi^0} \tag{27}$$

This is precisely the mixing term from the massive–photon example which we need to absorb the NGBs into the massive vector bosons, with $f_{\rm T} = v$ from the known W and Z masses. We have strictly speaking not shown that the $f_{\rm T}$ appearing in the scalar field Φ is really the correctly normalized $f_{\rm T}$, defined as the decay constant of the techni-pions (and there is a lot of confusion about factors $\sqrt{2}$ in the literature which I will ignore in this sketchy argument). But if we assume this correct normalization then $f_{\rm T} \equiv v$ is the scaled-up version of f_π we see that technicolor is something like a scaled-up version of QCD by a factor $v/\Lambda_{\rm QCD} \sim 2000$.

This scaling factor we better compute in the more general case, where technicolor involves a gauge group $SU(N_T)$ instead of $SU(N_c)$ and N_D left-handed fermion doublets in the fundamental representation of $SU(N_T)$. To be able to write down Dirac masses for the fermions at the end of the day we also need $(2N_D)$ right-handed fermion singlets. If instead of one set of techni-pions we have N_D of them, we remember that the W, Z masses arise from the quadratic term associated with the techni-pion mixing above, proportional to g^2v^2. In the sum, the N_D techni-pions need to reproduce the measured mass squares, which means that the vacuum expectation value scales like $v \sim \sqrt{N_D}f_{\rm T}$. The known scaling rules then give:

$$f_{\rm T} \sim \sqrt{\frac{N_T}{N_c}}\,\frac{\Lambda_{\rm T}}{\Lambda_{\rm QCD}}\,f_\pi \qquad\qquad v = \sqrt{N_D}f_{\rm T} \sim \sqrt{\frac{N_D N_T}{N_c}}\,f_\pi \tag{28}$$

We can solve these scaling rules for the unknown technicolor parameters and obtain:

$$f_{\rm T} \sim \frac{v}{\sqrt{N_D}} \qquad\qquad \Lambda_{\rm T} \sim \Lambda_{\rm QCD}\,\frac{f_{\rm T}}{f_\pi}\,\sqrt{\frac{N_c}{N_T}} \sim v\,\frac{\Lambda_{\rm QCD}}{f_\pi}\,\sqrt{\frac{N_c}{N_D N_T}} \qquad \text{with} \quad v = 246~\text{GeV} \tag{29}$$

One simple example for such a technicolor model is the Susskind–Weinberg model. Its gauge group is $SU(N_T) \times SU(3)_c \times SU(2)_L \times U(1)_Y$. The matter fields forming the condensate which in turn breaks the electroweak symmetry we include one doublet ($N_D = 1$) of charged color-singlet techni-fermions $(u^T, d^T)_{L,R}$. In some ways this doublet and the two singlets look like a fourth generation of chiral fermions, but with different charges under all Standard–Model gauge groups: for example, their hypercharges Y need to be chosen such that gauge anomalies do not occur and we do not have to worry about non-perturbatively breaking any symmetries, namely $Y = 0$ for the left–handed doublet and $Y = 1/2, -1/2$ for u^T_R and d^T_R. The formula $Q = I_3 + Y/2$ then gives charges of $\pm 1/2$ to the heavy states u^T and d^T.
The additional $SU(N_T)$ gauge group gives us a running gauge coupling which becomes large at the scale $\Lambda_{\rm T}$. As a high-scale boundary condition we can for example choose $\alpha_s(M_{\rm GUT}) = \alpha_T(M_{\rm GUT})$. The beta function is modelled after the QCD case

$$\beta_{\rm QCD} = \frac{11}{3}N_c - \frac{2}{3}n_f \qquad\qquad \beta_{\rm T} = \frac{11}{3}N_T - \frac{4}{3}N_D \tag{30}$$

keeping in mind that N_D counts the doublets, while $n_f = 6$ counts the number of flavors at the GUT scale. This relation holds for a simple model, where quarks are only charged under $SU(3)_c$ and techniquarks are only charged under $SU(N_T)$. Of course, both of them can carry weak charges. Using the one-loop formula for $\Lambda_{\rm QCD}$ we can compute

$$\frac{\Lambda_{\rm T}^2}{\Lambda_{\rm QCD}^2} = \exp\left[-\frac{4\pi}{\beta_{\rm QCD}}\,\frac{1}{\alpha_s(m_{\rm GUT})}\right]\;\exp\left[+\frac{4\pi}{\beta_{\rm T}}\,\frac{1}{\alpha_T(m_{\rm GUT})}\right]$$
$$= \exp\left[\frac{4\pi}{\alpha_s(m_{\rm GUT})}\left(\frac{1}{\beta_{\rm T}}-\frac{1}{\beta_{\rm QCD}}\right)\right] = \exp\left[\frac{4\pi}{\alpha_s(m_{\rm GUT})}\,\frac{\beta_{\rm QCD}-\beta_{\rm T}}{\beta_{\rm T}\beta_{\rm QCD}}\right] \tag{31}$$

For $N_T = N_D = 4$ and $\alpha_s(M_{\rm GUT}) \sim 1/30$ we find $\Lambda_{\rm T} \sim 800\ \Lambda_{\rm QCD} \sim 165$ GeV. This gives a reasonable $v = 270$ GeV and generates the required hierarchy between v and $M_{\rm GUT}$ via dimensional transmutation.

At this stage, our fermion construction has two global chiral symmetries $SU(2)\times SU(2)$ and $U(1)\times U(1)$ protecting the techni-fermions from getting massive, which we will of course break together with the local weak $SU(2)_L \times U(1)_Y$ symmetry. Details about fermion masses we postpone to the next sections. Let us instead briefly look at the spectrum of our minimal model:
techniquarks — From the scaling rules we know that the techniquark masses will be of the order $\Lambda_{\rm T}$ as give above. Numerically, the factor $\Lambda_{\rm T}/\Lambda_{\rm QCD} \sim 800$ pushes the usual quark constituent mass to around 700 GeV for the minimal model with $N_T = 4$ and $N_D = 1$. Because of the $SU(N_T)$ gauge symmetry there should exist four–techniquark bound states (technibaryons) which are stable due to the asymptotic freedom of the $SU(N_T)$ symmetry. Those are not preferred by standard cosmology, so we should find ways to let them decay.
NGBs — Of course, from the breaking of the global chiral $SU(2)\times SU(2)$ and the $U(1)\times U(1)$ we will have four Goldstone modes. The three $SU(2)$ Goldstones are massless technipions, following our QCD analogy. Because we gauge the remaining Standard–Model subgroup $SU(2)_L$, they become the longitudinal polarizations of the W and Z boson, after all this is the entire idea behind this construction. The remaining $U(1)$ NGB also has an equivalent (η') in QCD, and its technicolor counter part acquires a mass though non–perturbative instanton breaking. Its mass can be estimates to ~ 2 TeV, so we are out of trouble.
more stuff — Just like in QCD we will have a whole zoo of additional technicolor vector mesons and heavy resonances, but all we need to know about them is that they are heavy (and therefore not a problem for example for cosmology) and that at this stage we should really move on and think about fermion masses...

B. Fermion masses: ETC

Before we move on, let us put ourselves into the shoes of the technicolor proponents in the 70s. They knew how QCD gives masses to protons, and the Higgs mechanism had nothing to do with it. Just copying this appealing idea of dimensional transmutation (without any hierarchy problem) once more they explained the measured W and Z masses. And just like in QCD, the masses of the four light quarks and the leptons are well below a GeV and could be anything, but not linked to weak–scale physics. And then people found the massive bottom quark and the even more massive top quark and it became clear that at least the top mass was very relevant to the weak scale. In this section we will very briefly discuss how this challenge to technicolor basically removed it from the list of models people take seriously — until extra dimensions came and brought it back...

Extended technicolor is a version of the original idea of technicolor which attempts to solve two problems: create fermion masses for three generations of quarks and leptons and let the heavy techniquarks decay, to avoid stable technibaryons. From the introduction we in principle know how to obtain a fermion mass from Yukawa couplings, but to write down the Yukawa coupling to the sigma field or to the TC condensate we need to write down some Standard–Model and technifermion operators. This is what ETC offers a framework for.

First, we need to introduce some kind of multiplets of matter fermions. Just as before, the techniquarks, like all matter particles have $SU(2)_L$ and $U(1)_Y$ or even $SU(2)_R$ quantum numbers. However, there is no reason for them all to have a $SU(3)_c$ charge, because we would prefer not to change $\beta_{\rm QCD}$ too much. Similarly, the Standard–Model particles do not have a $SU(N_T)$ charge. This means we should write matter multiplets with explicitely assigned color and technicolor charges. This means:

$$\left(\,Q^T_{a=1..N_T},\;Q^{(1)}_{j=1,\dots,N_c},\;Q^{(2)}_{j=1,\dots,N_c},\;Q^{(3)}_{j=1,\dots,N_c},\;L^{(1)},\;L^{(2)},\;L^{(3)}\,\right) \tag{32}$$

These multiplets replace the usual $SU(2)_L$ and $SU(2)_R$ singlets and doublets in the Standard Model. The upper indices denote the generation, the lower indices count the N_T and N_c fundamental representations. In the minimal

model $N_T = 4$ this multiplet has $4+3+3+3+1+1+1 = 16$ entries. In other words, we have embedded $SU(N_T)$ and $SU(N_c)$ in a local gauge group $SU(16)$. If without further discussion we also extend the Standard–Model group by a $SU(2)_R$ gauge group, the complete ETC symmetry group is $SU(16) \times SU(2)_L \times SU(2)_R$, where we omit the additional $U(1)_{B-L}$ throughout the discussion.
A technicolor condensate will now break $SU(2)_L \times SU(2)_R$, while leaving $SU(3)_c$ untouched. If we think of the generators of the ETC gauge group as (16×16) matrices we can put a (4×4) block of $SU(N_T)$ in the upper left corner and then three (3×3) copies of $SU(N_c)$ on the diagonal. The last three rows/columns can be the unit matrix. Once we break $SU(16)_{\rm ETC}$ to $SU(N_T)$ and the Standard–Model gauge groups, the NGBs corresponding to the broken generators obtain masses of the order of $\Lambda_{\rm ETC}$. This breaking should on the way produce the correct fermion masses. The remaining $SU(N_T) \times SU(2)_L \times U(1)_Y$ will then break the electroweak symmetry through a $SU(N_T)$ condensate and create the measured W and Z masses as described in the last section.

In this construction we will have ETC gauge bosons which for example in the quark sector couple $(\overline{Q^T}\gamma_\mu T_{\rm ETC} Q^T)$, $(\overline{Q^T}\gamma_\mu T_{\rm ETC} Q)$ and $(\overline{Q}\gamma_\mu T_{\rm ETC} Q)$ currents. Here, $T_{\rm ETC}$ stands for the $SU(16)_{\rm ETC}$ generators. The multiplets Q^T and Q replace the $SU(2)_{L,R}$ singlet and doublets, which means the $T_{\rm ETC}$ include for example the chiral projectors. Below the the ETC breaking scale $\Lambda_{\rm ETC}$ these currents become four–fermion interactions, just like a Fermi interaction in the electroweak theory:

$$\frac{(\overline{Q^T}\gamma_\mu T^a_{\rm ETC} Q^T)\ (\overline{Q^T}\gamma^\mu T^b_{\rm ETC} Q^T)}{\Lambda^2_{\rm ETC}} \qquad \frac{(\overline{Q^T}\gamma_\mu T^a_{\rm ETC} Q)\ (\overline{Q}\gamma^\mu T^b_{\rm ETC} Q^T)}{\Lambda^2_{\rm ETC}} \qquad \frac{(\overline{Q}\gamma_\mu T^a_{\rm ETC} Q)\ (\overline{Q}\gamma^\mu T^b_{\rm ETC} Q)}{\Lambda^2_{\rm ETC}} \tag{33}$$

The mass scale in this effective theory can be linked to the mass of the ETC gauge bosons and their gauge coupling and should be of the order $1/\Lambda_{\rm ETC} \sim g_{\rm ETC}/M_{\rm ETC}$. Let us see what these kind of interactions predict at energy scales below $\Lambda_{\rm ETC}$, which means somewhere around the weak scale, where we have data. Because currents are much harder to interpret, we first Fierz–rearrange these operators and then pick out three relevant classes of scalar operators.

Maybe at this stage I should very briefly repeat without proof what a Fierz transformation is. We start from scalar operators based on spinors in a Lagrangian. The complete set is defined schematically written as:

$$\mathcal{L} \supset \left(\overline{\psi} A_j \psi\right)\ \left(\overline{\psi} A^j \psi\right) \qquad \text{with} \qquad A_j = \mathbb{1},\ \gamma_5,\ \gamma_\mu,\ \gamma_5\gamma_\mu,\ \sigma_{\mu\nu} \tag{34}$$

The multi-index j implies summing over all open indices in the diagonal combination $A_j A^j$. These five types of (4×4) matrices form a basis of all real (4×4) matrices which can occur in the Lagrangian. Note that in the equation above we have not specified anything about the spinors. If they carry charges, the $\overline{\psi}$ and the ψ have to cancel in the entire term, but of course not inside each current, *i.e.* there is more than one scalar operator of this type with a given set of spinors.
If we now specify the spinors and exchange them in one of the terms in the Lagrangian, we should be able to write the new (1,4,3,2) scalar term (or any new scalar term, for that matter) as a linear combination of the scalar basis elements (1,2,3,4):

$$\left(\overline{\psi}_1 A_i \psi_4\right)\ \left(\overline{\psi}_3 A_i \psi_2\right) = \sum_j C_{ij}\ \left(\overline{\psi}_1 A_j \psi_2\right)\ \left(\overline{\psi}_3 A_j \psi_4\right) \tag{35}$$

Note that in this notation we have ignored the normal-ordering of the spinors in the Lagrangian. It is easy to show $C \cdot C = \mathbb{1}$. All we need to know is the value of the coefficients C_{ij}, which I will list for completeness reasons, but without using them at all in the technicolor context:

	$\mathbb{1}$	γ_5	γ_μ	$\gamma_5\gamma_\mu$	$\sigma_{\mu\nu}$
$\mathbb{1}$	$-1/4$	$-1/4$	$-1/4$	$1/4$	$-1/8$
γ_5	$-1/4$	$-1/4$	$1/4$	$-1/4$	$-1/8$
γ_μ	-1	1	$1/2$	$1/2$	0
$\gamma_5\gamma_\mu$	1	-1	$1/2$	$1/2$	0
$\sigma_{\mu\nu}$	-3	-3	$1/2$	0	$1/2$

(36)

Applying this transformation to the three quark–techniquark four–fermion operators listed above we certainly obtain scalar $(A = \mathbb{1})$ operators by Fierz–transforming the three current $(A = \gamma_\mu)$ operators listed above. Because we are model builders, these are the only operators we will discuss in this context, and which will give us all the information we need:

$$\boxed{\frac{(\overline{Q^T}\,T^a_{\rm ETC}\,Q^T)\;(\overline{Q^T}\,T^b_{\rm ETC}\,Q^T)}{\Lambda^2_{\rm ETC}}} \qquad \boxed{\frac{(\overline{Q^T_L}\,T^a_{\rm ETC}\,Q^T_R)\;(\overline{Q_R}\,T^b_{\rm ETC}\,Q_L)}{\Lambda^2_{\rm ETC}}} \qquad \boxed{\frac{(\overline{Q_L}\,T^a_{\rm ETC}\,Q_R)\;(\overline{Q_R}\,T^b_{\rm ETC}\,Q_L)}{\Lambda^2_{\rm ETC}}} \tag{37}$$

Note that we have now picked certain chiralities of the Standard Model fields and the technifermions. Let us go through these operators once after the other in the following section.

C. Killing technicolor

From the title of this part it is fairly obvious that not all of the operators listed above will be our friends. On the other hand, we need them to give masses to the Standard–Model fermions, which means we have to live with their additional constraints:

(1) Once technicolor becomes strongly interacting and forms condensates of the kind $\left\langle \overline{Q^T} Q^T \right\rangle \propto \Lambda_{\rm T}^3$ the first operator will lead to masses for all TC generators which do not commute with the (broken) ETC generators. Without going into the details we know from the scalar operators that these masses have to be proportional to $1/\Lambda_{\rm ETC}$. The TC condensate will be proportional to N_T, which means that by dimensional analysis these masses will be $m \sim N_T \Lambda_{\rm T}^2/\Lambda_{\rm ETC}$. This mechanism will be very useful once we go beyond the minimal $N_T = 4, N_D = 1$ structure of technicolor, which predicts massless pseudoscalar NGBs which do not get eaten by the weak gauge bosons, so-called techni-axions. ETC has a mechanism to give these particles a mass of the order $\Lambda_{\rm T}$. So the first scalar operator is our friend.

(2) Condensating the techniquarks in the second operator will according to the QCD scaling rules give us fermion mass terms of the kind

$$\mathcal{L} \supset \frac{N_T \Lambda_{\rm T}^3}{\Lambda^2_{\rm ETC}}\, \overline{Q_L} q_R \equiv m_Q\, \overline{Q_L} q_R \qquad \Leftrightarrow \qquad \Lambda_{\rm ETC} \sim \sqrt{\frac{N_T \Lambda_{\rm T}^3}{m_Q}} \sim \begin{cases} 2\ {\rm TeV} & m_Q = 1\ {\rm GeV} \\ 200\ {\rm GeV} & m_Q = 100\ {\rm GeV} \end{cases} \tag{38}$$

for $N_T = 4$ and $\Lambda_{\rm T} = 100$ GeV. Remember that Dirac mass terms involve a left–right mixing, which means that they form an $SU(2)$ doublet, which in turn means that gauge invariance forces us to couple them to a techniquark doublet as well. From the numbers above we see that this operator appears to be our friend for light quarks, but it becomes problematic for the top quark, where $\Lambda_{\rm ETC}$ needs to be probably too low for current constraints.
Moreover, the operator responsible for the top mass can be fierzed into a fermion–technifermion current which can occur for either chirality

$$\left(\overline{Q^T_L} Q^T_R\right)\ (\overline{Q_R} Q_L) \quad \to \quad \left(\overline{Q^T_L}\gamma_\mu Q_L\right)\ (\overline{Q_R}\gamma^\mu Q^T_R) \quad \text{and} \quad \left(\overline{Q^T_L}\gamma_\mu Q_L\right)\ (\overline{Q_L}\gamma^\mu Q^T_L) \tag{39}$$

where we omitted the prefactor $g^2_{\rm ETC}/M^2_{\rm ETC}$. Of course, until now we have identified the right–handed Standard Model field with the right-handed top singlet. But because of the $SU(2)_R$ symmetry which as we will see later it necessary to avoid electroweak precision data as a custodial symmetry, we can rotate this $t_{L,R}$ into a $b_{L,R}$. So the operator we are looking at it of the kind

$$\frac{g^2_{\rm ETC}}{M^2_{\rm ETC}} \left(\overline{Q^T_L}\gamma_\mu b_L\right)\ (\overline{Q_L}\gamma^\mu Q^T_L) \tag{40}$$

where the techniquarks carry the index T. This operator induces a coupling of a charged ETC gauge boson to $T_L b_L$. Such a diagram contributes to the decay $Z \to b\bar{b}$, where the two outgoing b quarks exchange a heavy charged ETC gauge boson and this propagator is pinched after integrating out the ETC gauge bosons. It contributes to the effective bbZ coupling

$$g_L = \frac{e}{s_w c_w} \left(-\frac{1}{2} + \frac{s_w^2}{3} \right) \to g_L - \frac{\xi^2}{4}\, \frac{\Lambda_{\rm T}^2}{\Lambda^2_{\rm ETC}}\, \frac{e}{s_w c_w} = g_L - \frac{\xi^2}{4}\, \frac{m_t}{N_T \Lambda_{\rm T}}\, \frac{e}{s_w c_w} \tag{41}$$

The angle ξ describes a possible mixing between the W and the ETC gauge boson. Unless we find a good argument why the different gauge boson cannot mix at all, this contribution will be considerably too big for the LEP measurement

of $R_b = \Gamma_Z(b\bar{b})/\Gamma_Z(\text{hadrons})$. Note that this constraint from B decay will affect any theory which induces a top mass through a partner of the top quark and allows for a general set of (fierzed) operators corresponding to this mass term, not just extended technicolor.
The way out of these problem with $1/M_{\text{ETC}}$ operators we can read off the formula: we need to increase Λ_{ETC} while at the same time still getting the correct m_t. This can be achieved by so-called walking technicolor, which we will not discuss here, though.

(3) The third operator on the list does not include any techniquarks, but all combinations of four–fermion couplings of light quarks. In the Standard Model such operators are very strongly limited, in particular when they involve different flavors of quarks. Typical operators of this form which are strongly constrained are

$$\frac{1}{\Lambda^2_{\text{ETC}}}\,(\bar{s}\gamma^\mu d)\;(\bar{s}\gamma_\mu d) \qquad\qquad \frac{1}{\Lambda^2_{\text{ETC}}}\,(\bar{\mu}\gamma^\mu e)\;(\bar{e}\gamma_\mu \mu) \tag{42}$$

They are examples for flavor–changing neutral currents, *i.e.* couplings of a neutral gauge boson to two different fermion generations. Note that if we only allow two different generations in any of the operators, Fierz transformations will distribute them into all other operators. The currently strongest constraints come from kaon physics, for example the mass splitting between the K^0 and the $\overline{K}^0$. Its limit $\Delta M_K \lesssim 3.5\cdot 10^{-12}$ MeV implies $M_{\text{ETC}}/(g_{\text{ETC}}\theta_{sd}) \gtrsim 600$ TeV in terms of the Cabibbo angle θ_{sd}. We can translate such a lower bounds on Λ_{ETC} into an upper bound on fermion masses we can construct in our minimal model. $\Lambda_{\text{ETC}} > 10^3$ TeV simply translates in a maximum fermion mass which we can explain in this model: $m \lesssim 4$ MeV for $\Lambda_{\text{T}} \lesssim 1$ TeV. This is obviously not good news.

The last problem ETC runs into has to do with electroweak precision data, namely the two parameters S and T. While I will probably not be able to cover this in the lecture, let met briefly sketch a really nice introduction into electroweak precision observables from Csaba Csaki's lecture which I believe he found in an article by Cliff Burgess. If we allow for deviations from the Standard–Model gauge sector, but limit ourselves to only dimension–four operators in the Lagrangian we can write down the additional terms

$$\mathcal{L} \supset -\frac{\Pi'_{\gamma\gamma}}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{\Pi'_{WW}}{2}\hat{W}_{\mu\nu}\hat{W}^{\mu\nu} - \frac{\Pi'_{ZZ}}{4}\hat{Z}_{\mu\nu}\hat{Z}^{\mu\nu} - \frac{\Pi'_{\gamma Z}}{4}\hat{F}_{\mu\nu}\hat{Z}^{\mu\nu} - \Pi_{WW}\,\hat{m}^2_W\hat{W}^+_\mu\hat{W}^{-\mu} - \frac{\Pi_{ZZ}}{2}\,\hat{m}^2_Z\hat{Z}^+_\mu\hat{Z}^{-\mu} \tag{43}$$

The field strengths $\hat{F}_{\mu\nu}, \hat{W}_{\mu\nu}, \hat{Z}_{\mu\nu}$ correspond to the photon and the W and Z gauge bosons, *i.e.* the fields $\hat{A}_\mu, \hat{W}_\mu, \hat{Z}_\mu$. The hats on the field are necessary, because these kinetic terms and therefore the fields do not (yet) have the canonical normalization. If we assume that the parameters $\Pi'_{\gamma\gamma}, \Pi'_{WW}, \Pi'_{ZZ}$ and $\Pi'_{\gamma Z}$ are small, we can express the hatted gauge–boson fields in terms of the properly normalized fields as

$$\hat{A}_\mu = \left(1 - \frac{\Pi'_{\gamma\gamma}}{2}\right) A_\mu + \Pi'_{\gamma Z} Z_\mu \qquad \hat{W}_\mu = \left(1 - \frac{\Pi'_{WW}}{2}\right) W_\mu \qquad \hat{Z}_\mu = \left(1 - \frac{\Pi'_{ZZ}}{2}\right) Z_\mu \tag{44}$$

which means for example for the terms proportional to $\Pi'_{\gamma Z}$:

$$\begin{aligned}
-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\Big|_{\gamma Z} &= -\frac{1}{4}\left(\partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu\right)\,\left(\partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu\right)\Big|_{\gamma Z} \\
&= -\frac{1}{4}\left(\partial_\mu(A + \Pi'_{\gamma Z}Z)_\nu - \partial_\nu(A + \Pi'_{\gamma Z}Z)_\mu\right)\,\left(\partial_\mu(A + \Pi'_{\gamma Z}Z)_\nu - \partial_\nu(A + \Pi'_{\gamma Z}Z)_\mu\right)\Big|_{\gamma Z} \\
&= -\frac{\Pi'_{\gamma Z}}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\,\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right) - \frac{\Pi'_{\gamma Z}}{4}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)\,\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \mathcal{O}(\Pi'^2) \\
&= -\frac{\Pi'_{\gamma Z}}{2}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)\,\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \mathcal{O}(\Pi'^2) \\
&= -\frac{\Pi'_{\gamma Z}}{2}Z_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\Pi' x^2) = -\frac{\Pi'_{\gamma Z}}{2}\hat{Z}_{\mu\nu}\hat{F}^{\mu\nu} + \mathcal{O}(\Pi'^2)
\end{aligned} \tag{45}$$

So the two contributions to $Z - \gamma$ mixing indeed cancel each other. This brings the kinetic terms in the Lagrangian given above into the canonical form

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - (1 + \Pi_{WW} - \Pi'_{WW})\,\hat{m}^2_W W^+_\mu W^{-\mu} - \frac{1}{2}(1 + \Pi_{ZZ} + \Pi'_{ZZ})\,\hat{m}^2_Z Z^+_\mu Z^{-\mu} \tag{46}$$

The Z mass is given in terms of the additional small parameters $m_Z^2 = (1 + \Pi_{ZZ} + \Pi'_{ZZ})\,\hat{m}_Z^2$. Just as in the usual Lagrangian we can link the two gauge–boson masses through the (hatted) weak mixing angle $\hat{m}_W = \hat{c}_w \hat{m}_Z$, and in terms of this mixing angle we can compute the muon decay constant. The relation we obtain is:

$$\hat{s}_w^2 = s_w^2 \left[1 + \frac{c_w^2}{c_w^2 - s_w^2} \; \left(\Pi'_{\gamma\gamma} - \Pi'_{ZZ} - \Pi_{WW} + \Pi_{ZZ} \right) \right] \tag{47}$$

With all these corrections the W–mass term in the Lagrangian reads

$$\begin{aligned}
\mathcal{L} \supset & -(1 + \Pi_{WW} - \Pi'_{WW})\; \hat{m}_W^2 \; W^+_\mu W^{-\mu} = -(1 + \Pi_{WW} - \Pi'_{WW})\; \hat{c}_w^2 \; \hat{m}_Z^2 \; W^+_\mu W^{-\mu} \\
&= -(1 + \Pi_{WW} - \Pi'_{WW}) \; \left[1 - \frac{s_w^2}{c_w^2 - s_w^2} \; \left(\Pi'_{\gamma\gamma} - \Pi'_{ZZ} - \Pi_{WW} + \Pi_{ZZ} \right) \right] \; c_w^2 \; (1 - \Pi_{ZZ} + \Pi'_{ZZ})\; m_Z^2 \; W^+_\mu W^{-\mu} \\
&= \left[1 - \Pi'_{WW} + \Pi'_{ZZ} + \Pi_{WW} - \Pi_{ZZ} - \frac{s_w^2}{c_w^2 - s_w^2} \left(\Pi'_{\gamma\gamma} - \Pi'_{ZZ} - \Pi_{WW} + \Pi_{ZZ} \right) \right] \; m_Z^2 \; W^+_\mu W^{-\mu} \\
&= \left[1 - \frac{\alpha S}{2(c_w^2 - s_w^2)} + \frac{c_w^2 \alpha T}{c_w^2 - s_w^2} + \frac{\alpha U}{4 s_w^2} \right] \; m_Z^2 \; W^+_\mu W^{-\mu}
\end{aligned} \tag{48}$$

In the last step we have defined three typical combinations of the different correction factors as

$$\begin{aligned}
\alpha S &= 4 s_w^2 c_w^2 \; \left(-\Pi'_{\gamma\gamma} + \Pi'_{ZZ} - \Pi'_{\gamma Z} \frac{c_w^2 - s_w^2}{c_w s_w} \right) \\
\alpha T &= \Pi_{WW} - \Pi_{ZZ} \\
\alpha U &= 4 s_w^4 \; \left(\Pi'_{\gamma\gamma} - \frac{\Pi'_{WW}}{s_w^2} + \Pi'_{ZZ} \frac{c_w^2}{s_w^2} - 2 \Pi'_{\gamma Z} \frac{2 c_w}{s_w} \right)
\end{aligned} \tag{49}$$

These three so-called Peskin–Takeuchi can be understood fairly easily: the S parameter corresponds to a shift of the Z mass. This is not quite as obvious because it seems to also involve anomalous terms involving the photon's kinetic term, but we have to remember that the weak mixing angle is defined such that the photon is massless (*i.e.* corresponds to the unbroken $U(1)_Q$), while all mass terms are absorbed in the Z boson. The T parameter obviously compares contributions to the W and Z masses. Since the custodial $SU(2)$ symmetry precisely protects this mass ration, usually referred to as $\rho = 1$, the T parameter measures the amount of custodial symmetry violation. To get an idea how additional fermions contribute to S and T I just quote the contributions from the heavy fermion doublet:

$$\Delta S = \frac{N_c}{6\pi} \; \left(1 - 2Y \log \frac{m_t^2}{m_b^2} \right) \qquad \qquad \Delta T = \frac{N_c}{4\pi s_w^2 c_w^2 m_Z^2} \; \left(m_t^2 + m_b^2 - \frac{2 m_t^2 m_b^2}{m_t^2 - m_b^2} \log \frac{m_t^2}{m_b^2} \right)$$

$$\begin{aligned}
\Delta\rho &= \frac{N_c G_F}{8\sqrt{2}\pi^2} \; \left(m_t^2 + m_b^2 - \frac{2 m_t^2 m_b^2}{m_t^2 - m_b^2} \log \frac{m_t^2}{m_b^2} \right) \\
&= \frac{N_c}{8\sqrt{2}\pi^2} \; \frac{\sqrt{2} e^2}{8 s_w^2 c_w^2 m_Z^2} \; \left(2 m_b^2 + \delta - \frac{2(m_b^2 + \delta) m_b^2}{\delta} \log \left(1 + \frac{\delta}{m_b^2} \right) \right) \qquad m_t^2 = m_b^2 \\
&= \frac{N_c}{4\pi s_w^2 c_w^2 m_Z^2} \; \frac{e^2}{16\pi} \; \left(1 + \mathcal{O}(\delta^2) \right)
\end{aligned} \tag{50}$$

Where $Y = 1/6$ for Standard–Model quarks and $Y = -1/2$ for Standard–Model leptons. The ρ parameter is defined in terms of the W and Z masses and is one at tree level

$$\rho = \frac{m_W^2}{c_w^2 m_Z^2} = 1 \tag{51}$$

One of the main differences between ρ and T is the reference point, where $\rho = 1$ refers to its tree-level value and $T = 0$ is often chosen for some kind of light Higgs mass and including the Standard–Model top-bottom corrections. For a slightly longer discussion of such contributions to the ρ parameter or ΔT, just have a look into my little–Higgs notes.

Let us now get to the constraints on technicolor models from the very strongly constrained S, T plane. The central point in this plane $S = T = 0$ is somewhat conventional, because the Standard Model predicts for example two sources for finite T: the Higgs boson itself as well as the mass splitting between up-type and down-type quarks (like

the bottom and top quarks). Moreover, the electroweak precision constraints typically form a diagonal ellipse in the $S-T$ plane. But unless we can rely on a clear correlations, we can assume that models which to not predict $-0.15 < \Delta S < 0.25$ and $-0.1 < \Delta T < 0.3$ on the diagonal are ruled out with 95% C.L. For $S = 0$ or $T = 0$ the range of the respective other parameter is typically out to ± 0.1.
From the formulas we know that all we need to compute for S and T are the photon and W, Z self energies. Self energies from a field theoretical point of view can be considered part of the renormalization of a field, because whatever we do we need to reproduce the canonically normalized kinetic terms. If we introduce new particles with $SU(2)_L \times U(1)_Y$ quantum numbers, all of these particles will contribute to these self–energy loops. From the appearance of N_c in the formulas above we see that all these contributions simply add, unless the up-type and down-type contributions cancel. This is for example the case for a chiral fourth generation, just as a side remark.
In technicolor models, the singlet techniquarks will contribute to the S parameter each with a factor $N_T/(6\pi) \sim N_T/20 \sim 0.2$, assuming the minimal model with $N_D = 1$. This number can barely be tolerated if it is accompanied with $\Delta T \sim 0.2$, due to the diagonal ellipse structure of the current constraints. Constructing an appropriate model with an up-type and down-type is a challenge to technicolor model building in the minimal models. More complex models easily get to $\Delta S \sim \mathcal{O}(1)$, which is firmly ruled out, no matter what kind of ΔT we manage to obtain. These electroweak constraints are typically considered the last blow to technicolor models, even though we should mention that good model builders will find ways to construct models around almost any constraint, even the deadly list of technicolor constraints listed above. Only once we see (or do not see) a fundamental light Higgs at the LHC will we know...

III. SYMMETRY BREAKING BY BOUNDARY CONDITIONS

A much more recent idea of electroweak symmetry breaking which will, however, have to deal with the same kind of experimental constraints, is electroweak symmetry breaking from a fifth dimension. In other words, we extend our usual picture of space-time by an additional spacial coordinate, *i.e.* $\mu = 0, 1, 2, 3$ becomes $M = 0, 1, 2, 3, 5$. Giving the additional fifth dimension the index '5' instead of '4' is meant to avoid confusion. Of course, we have to construct our model such that for example gravitational measurements cannot detect the fact that there is this additional dimension. This will be one of the requirements on the extra dimension, which at this stage we will not discuss. For a very simple introduction into extra–dimensional theories and their benefits in solving the hierarchy problem you could have a look into my lecture notes. In the following three lectures we will limit ourselves to a new mechanism of breaking electroweak symmetry without introducing a Higgs field. In a way, this concept is more revolutionary than technicolor, because as we have seen in the very beginning, we can always think of a non-linear sigma model as the special case of a decoupled fundamental Higgs boson. Using extra–dimensional boundary condition really does not resemble the usual Higgs mechanism anymore.

Before we break electroweak symmetry, we need to get a general feel for a field theory which involves a higher–dimensional space (called bulk) and four–dimensional boundaries. Therefore, let us look at the action of a simple scalar field in five dimensions. Naively, we just write down a Lagrangian which we integrate over five dimensions of space-time:

$$S_{\rm bulk} = \int d^4x \int dy \, \mathcal{L}_5 = \int d^4x \, \mathcal{L}_4 \tag{52}$$

Already from this formula we know that our counting of powers of mass will be different - if the action still has mass dimension zero, then the Lagrangian $\mathcal{L}_5$ now has to have mass dimension five instead of four.
Gravitational constraints suggest that the extra dimension cannot be arbitrarily large, because it would modify Newton's gravity at very large distances (or very low energies), and such modifications are ruled out by everything we know about how our solar system or our galaxy works. Moreover, to get any mileage out of boundary conditions we need to give our extra dimension such boundaries, which means a finite size. A finite–size additional dimension we can obtain from an infinite dimensions two ways: either we think of it as a repeated interval, or we think of it as running around a circle, where the ends are simply identified. The latter leads us to the concept of an orbifold compactification which defines a brane. However, in comparison to the most general boundaries, such an orbifold compactification limits the set of possible boundary conditions, so we will instead stick to a general boundary setup. In both cases we can write the size of the fifth dimension as $y = 0...\pi R$.
The simplest field we can write down is a scalar field with a kinetic term and a potential, so our action reads:

$$S_{\rm bulk} = \int d^4x \int_0^{\pi R} dy \, \left(\frac{1}{2} (\partial^M \phi)^2 - V(\phi) \right) = \int d^4x \int_0^{\pi R} dy \, \left(\frac{1}{2} (\partial^\mu \phi)^2 - \frac{1}{2} (\partial^5 \phi)^2 - V(\phi) \right) \tag{53}$$

Because the additional dimensions is a space dimension the metric tensor g_{MN} is $(+,-,-,-,-)$. The trouble with this Lagrangian is that the kinetic term means that this scalar field has a mass dimension 3/2, but on the other hand it is not clear what we could do instead.

A. Fields on the boundary

Trying to derive the equations of motion from this action will bring in the boundaries. The variation of the action is

$$
\begin{aligned}
0 \stackrel{!}{=} \delta S_{\text{bulk}} &= \int d^4x \int_0^{\pi R} dy \, \left((\partial^\mu \phi)(\partial_\mu \delta\phi) - \frac{\partial}{\partial\phi} V(\phi)\, \delta\phi - (\partial^5\phi)(\partial_5 \delta\phi) \right) \\
&= \int d^4x \left[\int_0^{\pi R} dy \, \left(-\partial^\mu \partial_\mu \phi - \frac{\partial}{\partial\phi} V(\phi) + \partial^5 \partial_5 \phi \right) \delta\phi - (\partial_5\phi)\delta\phi \Big|_0^{\pi R} \right] \\
&= \int d^4x \left[\int_0^{\pi R} dy \, \left(-\partial^M \partial_M \phi - \frac{\partial}{\partial\phi} V(\phi) \right) \delta\phi - (\partial_5\phi)\delta\phi \Big|_0^{\pi R} \right]
\end{aligned}
\tag{54}
$$

We have simply integrated by parts in all five dimensions. In contrast to the four usual dimension where our Hilbert space is defined such that all fields vanish at the infinite boundary we cannot require such a thing for the fifth dimension. Instead, we need to keep the surface term in the variation of the action, which will generically give us boundary terms from the originally five–dimensional Lagrangian. The first condition we read off this variation is the five–dimensional bulk equation of motion $\partial_M \partial^M \phi = -\partial V / \partial\phi$.

In addition, the boundary term if the variation of the action has to vanish, which gives us the choice of two boundary conditions:

$$
\boxed{\partial_5 \phi \Big|^{0,\pi R} = 0} \quad \text{(Neumann)} \qquad \text{or} \qquad \boxed{\phi \Big|^{0,\pi R} = 0} \quad \text{(Dirichlet)}
\tag{55}
$$

There is in principle be a third possibility, namely that the contributions from both boundaries cancel, but this would force is to treat the two boundaries equal, which as we will see later is not what we want.

From this short argument we see that it would be useful to study the behavior of additional Lagrangian terms only on the boundary, to modify such boundary conditions. For example, what happens, if we add a boundary mass term?

$$
S = S_{\text{bulk}} - \int d^4x \, \frac{1}{2} M \phi^2 \Big|^0 - \int d^4x \, \frac{1}{2} M \phi^2 \Big|^{\pi R}
\tag{56}
$$

The masses on the two boundaries can of course be different. Looking at the formula above we have gotten ourselves into trouble, because the usual four–dimensional mass terms would be M^2. However, this M^2 would need to have mass dimension one to arrive at the usual dimension–four Lagrangian in four dimensions. The variational principle gives us

$$
\begin{aligned}
\delta S &= \delta S_{\text{bulk}} - \int d^4x \, M\phi \, \delta\phi \Big|^0 - \int d^4x \, M\phi \, \delta\phi \Big|^{\pi R} \\
&= \int d^4x \left[\int_0^{\pi R} dy \, (\cdots) \, \delta\phi - \partial_5\phi \, \delta\phi \Big|_0^{\pi R} - M\phi \, \delta\phi \Big|^0 - M\phi \, \delta\phi \Big|^{\pi R} \right] \\
\Leftrightarrow \qquad & \partial_5 \phi - M\phi \Big|^0 = 0 \qquad \text{and} \qquad \partial_5\phi + M\phi \Big|^{\pi R} = 0
\end{aligned}
\tag{57}
$$

This form is interesting, because it interpolates between the two possible boundary conditions in the absence of the mass term: for $M = 0$ we recover the Neumann BC, while for $M \to \infty$ we are left with the Dirichlet BC. Note again that these conditions really do not look like equations of motion on the boundary because of mass dimension of the scalar field. In fact, they look much more like a Dirac equation, which makes no sense for scalars, but then they are not equations of motion either.

Moving on, let us try a boundary kinetic term on one of the boundaries:

$$
S = S_{\text{bulk}} + \int d^4x \, \frac{1}{2M} (\partial_\mu \phi)(\partial^\mu \phi) \Big|^{\pi R}
\tag{58}
$$

Note that on the four–dimensional boundary we are using the four–dimensional derivative of course. The variational principle now gives us — as usually integrating by parts and keeping the factor two from the symmetric squared kinetic term:

$$\begin{aligned} \delta S &= \delta S_{\rm bulk} + \int d^4x \; \frac{1}{M} (\partial_\mu \phi)(\partial^\mu \delta\phi) \Big|^{\pi R} \\ &= \int d^4x \left[\int_0^{\pi R} dy \; (\cdots) \; \delta\phi - \partial_5 \phi \; \delta\phi \Big|_0^{\pi R} - \frac{1}{M} (\partial_\mu \partial^\mu \phi) \; \delta\phi \Big|^{\pi R} \right] \quad \Leftrightarrow \quad \partial_5 \phi = - \frac{1}{M} \partial_\mu \partial^\mu \phi \Big|^{\pi R} = 0 \end{aligned} \tag{59}$$

Remembering the bulk equation of motion $\partial_M \partial^M \phi = 0$ we can re-write this boundary conditions as $\partial_5 \phi = -(\partial_5)^2 \phi / M$. On the other boundary, the relative sign would simply change. This form has an interesting consequence: if we want the second-derivative operator $\phi'' \equiv (\partial_5)^2 \phi$ to be hermitian $(f, g'') = (f'', g)$ we have to redefine the scalar product on the space of five–dimensional wave functions including a boundary term. Csaba nicely derives this in his lecture.

As the final step we will move away from the scalar toy model and introduce a five–dimensional photon field into our theory:

$$S = \int d^5x \; \left(-\frac{1}{4} F_{MN} F^{MN} \right) = \int d^5x \; \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\mu 5} F^{\mu 5} \right) \tag{60}$$

There would be the additional F_{55} term, but it vanishes due to the antisymmetric nature of $F_{MN} = \partial_M A_N - \partial_N A_M$. The additional term including the fifth component of the gauge field becomes

$$\begin{aligned} S &= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; F_{\mu 5} F^{\mu 5} \\ &= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; (\partial_\mu A_5 - \partial_5 A_\mu) \; \left(\partial^\mu A^5 - \partial^5 A^\mu \right) \\ &= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; \left[+\partial_\mu A_5 \, \partial^\mu A^5 + \partial_5 A_\mu \, \partial^5 A^\mu - 2 \partial_\mu A_5 \, \partial^5 A^\mu \right] \\ &= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; \left[-A_5 \, \partial_\mu \partial^\mu A^5 + \partial_5 A_\mu \, \partial^5 A^\mu + 2 A_5 \, \partial_\mu \partial^5 A^\mu \right] \\ &= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; \left[-A_5 \, \partial_\mu \partial^\mu A^5 - A_\mu \, \partial_5 \partial^5 A^\mu - 2 \partial^5 A_5 \, \partial_\mu A^\mu \right] - \frac{1}{2} \int d^4x \; \left[A_\mu \, \partial^5 A^\mu + 2 A_5 (\partial_\mu A^\mu) \right]_0^{\pi R} \end{aligned} \tag{61}$$

again after integrating by parts first in the four–dimensional space (with vanishing boundary terms) and then in the fifth dimension . The first term in the last line is obviously a kinetic terms for the scalar field A_5. The second term will after a Kaluza–Klein decomposition (*i.e.* a discrete Fourier transform in the periodic fifth dimension) become a mass term for our photon in five dimensions. We can schematically write the five–dimensional wave functions by separating variables

$$A_\mu(x,y) = \bar{A}_\mu(x) f(y) \sim \sum_n \hat{A}_\mu^{(n)}(x) \; e^{iny/R} \quad \Rightarrow \quad \partial_5^2 A_\mu(x,y) = \sum_n \partial_5^2 \hat{A}_\mu^{(n)}(x) \; e^{iny/R} = - \sum_n \frac{m^2}{R^2} \hat{A}_\mu^{(n)}(x) \; e^{iny/R} \tag{62}$$

Which means that if we write our five–dimensional photon field as an effective theory in four dimensions we obtain towers of massive photons whose mass is given by the inverse size of our fifth dimension. Note, however, that we have to clearly distinguish between two kinds of photon masses. The KK excitations will be massive, but this does not mean that we break the symmetry of our Lagrangian. In particular, there will be a zero mode $n = 0$ with vanishing mass. Which means we still have to find a mechanism for electroweak symmetry breaking. The role of the KK excitations will become obvious later, when we discuss unitarity in these models.

From the formula above it also becomes clear what role boundary conditions play: Dirichlet boundary conditions ($A = 0$) mean sine-type behavior, while Neumann boundary conditions ($\partial_5 A = 0$) mean cosine at the respective boundaries. This implies that if we want to write down a zero mode, *i.e.* a constant wave function in the y dimensions which corresponds to $\exp(ny/R)$ for $n = 0$, we need a Neumann-Neumann setup on the two boundaries.

For the third term in eq.(61) we have to briefly remember something about gauge theories which I also had to read again for example in the book by Peskin and Schroeder. It obviously mixes the scalar field and the photon. The

same thing happens if we write down the usual Higgs mechanism: the NGB will mix with the transverse degrees of freedom of the gauge boson which will then eat it as its longitudinal component. Such a term we do not want in the Lagrangian — the definition of the gauge boson should instead absorb this term into the massive gauge boson. This we can achieve in a general $(R-\xi)$ gauge: gauge fixing means including a gauge–fixing term with including the Lagrangian multiplier $1/\xi$. You can find a discussion of this gauge for the Standard–Model Higgs mechanism in Peskin & Schroeder section 21.1. For example, for an abelian massive photon we introduce a gauge–fixing term $(\partial_\mu A^\mu - \xi e v \phi)^2/(2\sqrt{\xi})$ to cancel the photon–NGB mixing and fix the photon gauge at the same time. The third term from the gauge fixing gives us a mass for the NGB $m_\phi^2 = \xi (ev)^2 = \xi m_A^2$ (in terms of the photon mass). Since this mass is gauge dependent the NGB it not a well-defined physical degree of freedom, and it can be decoupled by choosing $\xi \to \infty$, which is called unitary gauge. In that gauge the NGB survives only as the longitudinal component of the massive photon, but does not appear in the Lagrangian anymore.
Precisely the same way we now introduce a gauge–fixing term in the five dimensional space (bulk):

$$
\begin{aligned}
S_{\text{GF,bulk}} &= \frac{1}{2\xi} \int d^4x \int_0^{\pi R} dy \; \left(\partial_\mu A^\mu - \xi \, \partial_5 A^5 \right)^2 \\
&= \int d^4x \int_0^{\pi R} dy \; \left[\frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \partial_\mu A^\mu \, \partial_5 A^5 + \frac{\xi^2}{2} (\partial_5 A^5)^2 \right] \\
&= \int d^4x \int_0^{\pi R} dy \; \left[\frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \partial_\mu A^\mu \, \partial_5 A^5 - \frac{\xi^2}{2} A_5 \partial_5 \partial^5 A^5 \right] + \frac{\xi^2}{2} \int d^4x \; A_5 \, \partial_5 A^5 \Big|_0^{\pi R}
\end{aligned}
\tag{63}
$$

The usual gauge fixing term $(\partial_\mu A^\mu)^2$ appears for the transverse degrees of freedom of the massless photon. The second term cancels the mixing term between A_μ and A_5. What is interesting is the last term in $S_{\text{GF,bulk}}$: there is no need to fix the gauge for the scalar field A^5, and if we compute the equation of motion for A_5 using the variational principle for the contributions to δS proportional to δA_5 it includes a term $\xi^2 \partial_5 A^5 \, \partial^5 (\delta A_5)$. After integrating by parts this leads to $\xi^2 (\partial_5)^2 A^5$ appearing in the equation of motion for A_5, which is nothing but a massive KK tower. The KK masses will become infinitely large in unitary gauge $\xi \to \infty$, so that the entire A_5 tower as a physical mode decouples from the theory. Instead, its degrees of freedom now give KK masses to the excitation of the four–dimensional gauge field A_μ. Note that a possible zero mode in the A_5 tower would be linked to a finite mass for the lowest (*i.e.* Standard Model) gauge boson. Dependent on the boundary conditions such a zero might or might not appear. We will discuss the role of such a A_5 zero term when we discuss ways to break electroweak symmetry.

We know that we are not living in five but in four dimensions. Which means that we should have a careful look at the action on the boundaries in eq.(61). After fixing the gauge in the bulk, there is also a dangerous boundary mixing term of the type $A_5(\partial_\mu A^\mu)$. Again, we have to introduce a gauge fixing term, now on the boundary

$$
\begin{aligned}
S_{\text{GF,bound}} &= \frac{1}{2\hat{\xi}} \int d^4x \; \left(\partial_\mu A^\mu \pm \hat{\xi} A_5 \right)^2 \Big|^{0,\pi R} \\
&= \int d^4x \; \left[\frac{1}{2\hat{\xi}} (\partial_\mu A^\mu)^2 \Big|^{0,\pi R} + \frac{\hat{\xi}}{2} A_5^2 \Big|^{0,\pi R} - (\partial_\mu A^\mu) A_5 \Big|^{0} + (\partial_\mu A^\mu) A_5 \Big|^{\pi R} \right] \\
&= \int d^4x \; \left[\frac{1}{2\hat{\xi}} (\partial_\mu A^\mu)^2 \Big|^{0,\pi R} + \frac{\hat{\xi}}{2} A_5^2 \Big|^{0,\pi R} + (\partial_\mu A^\mu) A_5 \Big|_0^{\pi R} \right]
\end{aligned}
\tag{64}
$$

Note the difference between the upper and lower notation of the boundary terms. The last term precisely cancels the boundary mixing term. We can now combine S_{bound} from the original Lagrangian and from the two gauge fixing terms:

$$
\begin{aligned}
S_{\text{bound}} &= -\frac{1}{2} \int d^4x \; \left[A_\mu \, \partial^5 A^\mu + 2 A_5 (\partial_\mu A^\mu) \right]_0^{\pi R} + \frac{\xi^2}{2} \int d^4x \; A_5 \, \partial_5 A^5 \Big|_0^{\pi R} \\
&\quad + \int d^4x \; \left[\frac{1}{2\hat{\xi}} (\partial_\mu A^\mu)^2 \Big|^{0,\pi R} + \frac{\hat{\xi}}{2} A_5^2 \Big|^{0,\pi R} + (\partial_\mu A^\mu) A_5 \Big|_0^{\pi R} \right] \\
&= -\frac{1}{2} \int d^4x \; A_\mu \, \partial^5 A^\mu \Big|_0^{\pi R} + \frac{\xi^2}{2} \int d^4x \; A_5 \, \partial_5 A^5 \Big|_0^{\pi R} + \int d^4x \; \left[\frac{1}{2\hat{\xi}} (\partial_\mu A^\mu)^2 \Big|^{0,\pi R} + \frac{\hat{\xi}}{2} A_5^2 \Big|^{0,\pi R} \right]
\end{aligned}
\tag{65}
$$

For this action we can compute the variation, which needs to be zero. The two gauge parameters ξ in the bulk and $\hat{\xi}$ on the boundary do not have to be identical. To simplify the results we can use the unitary gauge on the boundary

$\hat{\xi} \to \infty$ and find for the terms proportional to the variation of A_5

$$0 \stackrel{!}{=} \delta S_{\text{bound}}\Big|_{A^5} = \int d^4x \, \left[\frac{\xi}{2} \partial_5 A^5 \, \delta A_5 \Big|_0^{\pi R} + \frac{\xi}{2} A^5 \, \partial_5(\delta A_5) \Big|_0^{\pi R} + \hat{\xi} A^5 \, \delta A_5 \Big|^{0,\pi R} \right]$$
$$\sim \hat{\xi} \int d^4x \; A^5 \, \delta A_5 \Big|^{0,\pi R} \qquad \Leftrightarrow \qquad \boxed{A^5 \Big|^{0,\pi R} = 0} \tag{66}$$

while the condition on $\partial_5 A^5$ we would have gotten from the gauge fixing in the bulk does not contribute anymore. The second term proportional to $\partial_5 \delta A_5$ looks funny at first, but it is taken care of by the boundary condition $A^5 = 0$. Secondly, the variational contributions proportional to the regular photon field A^μ are:

$$0 \stackrel{!}{=} \delta S_{\text{bound}}\Big|_{A^\mu} = \int d^4x \, \left[-\frac{1}{2} \delta A_\mu \, \partial_5 A^\mu \Big|_0^{\pi R} - \frac{1}{2} A_\mu \, \partial_5 \delta A^\mu \Big|_0^{\pi R} + \frac{1}{\hat{\xi}} (\partial_\nu A^\nu) \, (\partial_\mu \delta A^\mu) \Big|^{0,\pi R} \right]$$
$$= \int d^4x \, \left[-\frac{1}{2} \delta A_\mu \, \partial_5 A^\mu \Big|_0^{\pi R} - \frac{1}{2} A_\mu \, \partial_5 \delta A^\mu \Big|_0^{\pi R} - \frac{1}{\hat{\xi}} (\partial_\mu \partial_\nu A^\nu) \, \delta A^\mu \Big|^{0,\pi R} \right]$$
$$\sim \int d^4x \, \left[-\frac{1}{2} \partial_5 A^\mu \, \delta A_\mu \Big|_0^{\pi R} - \frac{1}{2} A_\mu \, \partial_5 \delta A^\mu \Big|_0^{\pi R} \right] \qquad \Leftrightarrow \qquad \boxed{\partial_5 A^\mu \Big|^{0,\pi R} = 0} \tag{67}$$

Because we fix $\partial_5 A^\mu$ on the boundaries, it does not contribute in the second term of δS_{bound}, like any other constant would not contribute. According to our very brief look at zero modes this set of boundary conditions means that after Fourier–transforming the fifth dimension there will be a zero mode for the photon A_μ, while due to the Dirichlet boundary conditions the scalar mode A_5 will not have a zero mode. It will only occur with finite KK masses, which are eaten by the massive KK gauge bosons. In other words, we expect a massless Standard–Model photon with a massive KK tower, but no additional A_5 fields.
Looking back at S_{bound} we see that the two sets of boundary conditions and in addition the boundary unitary gauge $\hat{\xi} \to \infty$ implies $S_{\text{bound}} = 0$. All we have to consider for our five–dimensional QED is the bulk action in eq.(61).

$$S = \int d^5x \, \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(\partial_\mu A_5 \, \partial^\mu A^5 + \partial_5 A_\mu \, \partial^5 A^\mu - 2 \partial_\mu A_5 \, \partial^5 A^\mu \right) \right] \tag{68}$$

B. Breaking the gauge symmetry on the boundaries

Since we now know how the basics of a five–dimensional version of QED, let us see what happens if we break the gauge symmetry — in this case the $U(1)$ — on the boundaries. From the introduction we know how to do this; let us add a non-linear sigma model on the two boundaries.

$$S = \int d^4x \, \mathcal{L}_4 \qquad \mathcal{L}_4 \supset \int_0^{\pi R} dy \, \left[|D_\mu \Phi|^2 - \frac{\lambda}{2} \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 \right] \qquad \Phi \sim \frac{v}{\sqrt{2}} \, e^{i\pi/v} \tag{69}$$

Again, in these notes am using the technicolor version of the Higgs potential with a prefactor $\lambda/2$, instead simply λ as I use it in my Higgs notes or Csaba uses it as well... In the last step we have already decoupled the physical Higgs field and chosen $\lambda \to \infty$, with finite v. The two Higgs fields on the two boundaries should of course be labelled differently, and the parameters λ and v will not be the same for both of them. To keep things short I will only spell out the action for $y = \pi R$. This gives us the bulk contributions we computed before, remembering that in unitary gauge and with given boundary conditions $\mathcal{L}_{\text{bound}} = 0$:

$$\mathcal{L}_4 = \mathcal{L}_{4,\text{bulk}} + \mathcal{L}_{4,\sigma}$$
$$= \int_0^{\pi R} dy \, \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A_5)^2 - \frac{1}{2} (\partial_5 A_\mu)^2 + \partial_\mu A_5 \partial^5 A^\mu \right] + \frac{1}{2} \left(\partial_\mu \pi - v A_\mu \right)^2 \Big|^{\pi R} \tag{70}$$

The sigma-field contribution we simply copy from eq.(1) with $ef \to v$. Note that writing down the boundary terms we can see that if A_μ now has mass dimension 3/2, we need to assign mass dimension $[v] = M^{1/2}$ and $[\pi] = M^1$. In contrast to our earlier discussion we now use a general $(R - \xi)$ gauge, which means we need to introduce gauge–fixing terms to cancel the $(A_5 - A_\mu)$ and $(\pi - A_\mu)$ mixing terms

$$\begin{aligned}
\mathcal{L}_{4,\mathrm{GF}} &= -\frac{1}{2\xi}\int_0^{\pi R} dy\ \left(\partial_\mu A^\mu - \xi\partial_5 A^5\right)^2 - \frac{1}{2\hat\xi}\left(\partial_\mu A^\mu + \hat\xi(v\pi + A_5)\right)^2\Big|^{\pi R} \\
&= -\frac{1}{2\xi}\int_0^{\pi R} dy\ \left[(\partial_\mu A^\mu)^2 - \xi\,\partial_\mu A^\mu\ \partial_5 A^5 + \xi^2(\partial_5 A^5)\right] - \frac{1}{2\hat\xi}\left(\partial_\mu A^\mu + \hat\xi(v\pi + A_5)\right)^2\Big|^{\pi R} \\
&= -\frac{1}{2\xi}\int_0^{\pi R} dy\ \left[(\partial_\mu A^\mu)^2 - \xi\,\partial_\mu A^\mu\ \partial_5 A^5 - \xi^2 A_5\partial_5^2 A^5)\right] - \frac{\xi}{2}A_5\partial_5 A^5\Big|_0^{\pi R} - \frac{1}{2\hat\xi}\left(\partial_\mu A^\mu + \hat\xi(v\pi + A_5)\right)^2\Big|^{\pi R} \qquad (71)
\end{aligned}$$

Again, we have copied the bulk contribution from eq.(63) and added the appropriate term needed for the NGB contributions. After adding these gauge–fixing terms the bulk action involving only the gauge field A_μ is

$$\begin{aligned}
\mathcal{L}_{A^\mu} &= \int_0^{\pi R} dy\ \left[-\frac{1}{4}\left((\partial_\mu A_\nu)^2 + (\partial_\nu A_\mu)^2 - 2(\partial_\mu A_\nu)(\partial^\nu A^\mu)\right) - \frac{1}{2}(\partial_5 A_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right] \\
&= \frac{1}{2}\int_0^{\pi R} dy\ \left[-(\partial_\mu A_\nu)^2 + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - (\partial_5 A_\mu)^2 - \frac{1}{\xi}(\partial_\mu A^\mu)^2\right] \\
&= \frac{1}{2}\int_0^{\pi R} dy\ \left[A_\nu\partial^\mu\partial_\mu A^\nu - A_\nu\partial_\mu\partial^\nu A^\mu + A_\mu\partial_5\partial^5 A^\mu + \frac{1}{\xi}A_\nu\partial^\nu\partial_\mu A^\mu\right] \quad - \frac{1}{2}A_\mu\partial_5 A^\mu\Big|_0^{\pi R} \\
&= \frac{1}{2}\int_0^{\pi R} dy\ A_\nu\left[g^{\mu\nu}(\partial^\rho\partial_\rho - \partial^\mu\partial^\nu + g^{\mu\nu}\partial_5\partial^5 + \frac{1}{\xi}\partial^\nu\partial^\mu\right]A_\mu \quad - \frac{1}{2}A_\mu\partial_5 A^\mu\Big|_0^{\pi R} \\
&= \frac{1}{2}\int_0^{\pi R} dy\ A_\nu\left[g^{\mu\nu}(\partial^\rho\partial_\rho + \partial_5\partial^5) - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\mu \quad - \frac{1}{2}A_\mu\partial_5 A^\mu\Big|_0^{\pi R} \qquad (72)
\end{aligned}$$

What we see in the last line it simply the gauge–boson propagator in $(R - \xi)$ gauge, now including the KK term. **tp: for some reason this ∂_5^2 has a weird sign...?** The corresponding bulk equation of motion for the scalar component in the absence of any additional mass terms arises from gauge fixing: $\mathcal{L}_5 \supset -\xi/2\ (\partial_5 A^5)^2$.

To compute the boundary conditions for A_μ, we can for example collect all boundary contributions at $y = \pi R$ after removing the $(A_5 - A_\mu)$ mixing:

$$\begin{aligned}
\mathcal{L}_{A^\mu} &= -(\partial_\mu\pi)(vA_\mu) + \frac{1}{2}(vA_\mu)^2 \quad - \frac{1}{2\hat\xi}(\partial_\mu A^\mu)^2 - \frac{1}{\hat\xi}(\partial_\mu A^\mu)\,\hat\xi\,(v\pi) \quad - \frac{1}{2}A_\mu\partial_5 A^\mu \\
&= -(\partial_\mu\pi)(vA_\mu) + \frac{1}{2}(vA_\mu)^2 - \frac{1}{2\hat\xi}(\partial_\mu A^\mu)^2 + A^\mu\partial_\mu(v\pi) - \frac{1}{2}A_\mu\partial_5 A^\mu \\
&= \frac{1}{2}v^2 A_\mu A^\mu - \frac{1}{2\hat\xi}(\partial_\mu A^\mu)^2 - \frac{1}{2}A_\mu\partial_5 A^\mu \\
&\sim \frac{1}{2}A_\mu\left(v^2 - \partial_5\right)A^\mu \qquad (73)
\end{aligned}$$

In unitary gauge, this determines the boundary condition at πR and correspondingly at $y = 0$ to be

$$\boxed{(\partial_5 \mp v^2)A_\mu\Big|^{0,\pi R} = 0} \qquad (74)$$

Remember that now $[v] = M^{1/2}$. From the general scalar boundary–mass case we expect that adding a boundary–mass for the photon indeed means that the new boundary conditions will become an interpolation of Dirichlet and Neumann conditions. What is new in this formula is that the mass scale is given by v, the vacuum expectation value breaking electroweak symmetry on the boundaries. In other words, in the unbroken phase $v = 0$ the photon field has to obey Neumann boundary conditions $\partial_5 A_\mu = 0$, while in the broken phase $v \neq 0$ it will follow Dirichlet boundary conditions $A_\mu = 0$. We know that this means that only in the unbroken phase it will have a zero mode. We can turn this argument around: a physical photon field with a Dirichlet boundary condition $A_\mu = 0$ and hence without a zero mode is indeed a sign for a broken symmetry on the respective boundary.

If a Dirichlet boundary condition for the physical gauge–boson field is indeed a sign for a broken symmetry, some combination of A_5 and the NGB π has to provide the degrees of freedom to make the photon (including its zero mode) massive. The boundary terms for A_5 and π after removing all mixing terms and including a boundary mass m with $[m] = M^1$ for π are

$$\begin{aligned}\mathcal{L}_{A^5,\pi} &= \frac{1}{2}(\partial_\mu \pi)^2 - \frac{\xi}{2} A_5 \partial_5 A^5 - \frac{\hat{\xi}}{2}(v\pi + A_5)^2 - \frac{m^2}{2}\pi^2 \\ &\sim -\frac{\xi}{2} A_5 \partial_5 A^5 - \frac{\hat{\xi}}{2}\left[\left(v^2 + \frac{m^2}{\hat{\xi}}\right)\pi^2 + 2v\pi A_5 + A_5^2\right]\end{aligned} \tag{75}$$

We then find for the π's boundary conditions in combination with A_5

$$0 \stackrel{!}{=} \frac{\partial}{\partial \pi}\Big[\cdots\Big] = 2\left(v^2 + \frac{m^2}{\hat{\xi}}\right)\pi + 2vA_5 \qquad \Leftrightarrow \qquad \left(v^2 + \frac{m^2}{\hat{\xi}}\right)\pi + vA_5 \Big|^{0,\pi R} = 0 \tag{76}$$

The same way we can compute the boundary conditions for A_5 in terms of both scalar fields:

$$\begin{aligned}0 &\stackrel{!}{=} -\frac{\xi}{2}\partial^5 A_5 - \frac{\hat{\xi}}{2}\left[2v\pi + 2A_5\right] \\ &= -\frac{\xi}{2}\partial^5 A_5 - \hat{\xi} A_5 + \hat{\xi} v\,\frac{vA_5}{v^2 + m^2/\hat{\xi}} \\ &= -\left[\frac{\xi}{2}\,\partial^5 + \hat{\xi}\frac{m^2/\hat{\xi}}{v^2 + m^2/\hat{\xi}}\right] A_5\end{aligned} \tag{77}$$

From there we can read off the boundary condition for the scalar component A_5

$$\boxed{\left(\partial_5 \mp \frac{\hat{\xi}}{\xi}\,\frac{m^2/\hat{\xi}}{v^2 + m^2/\hat{\xi}}\right) A_5 \Big|^{0,\pi R} = 0} \tag{78}$$

For unitary gauge on the boundaries $\hat{\xi} \to \infty$ we know from the last example without boundary scalars that indeed we should find Dirichlet boundary conditions $A_5 = 0$. In that limit the NGB mass terms become suppressed, because these degrees of freedom are not physical and should be eaten by the gauge bosons.

If we want to study the behavior of the NGB in the bulk we can go into unitary gauge in the bulk $\xi \to \infty$. We see that breaking the symmetry on the boundaries shifts the A_5 boundary conditions from originally Dirichlet ($A_5 = 0$) in eq.(67) to Neumann ($\partial_5 A_5 = 0$). This means that A_5 now can develop a zero mode, which provides the necessary degree of freedom for the photon which in the presence of v cannot include a zero mode any longer!
In general we see a pattern for the boundary conditions of the gauge boson and of the scalar A_5 when we break the symmetry on the boundaries. In the unbroken symmetry the gauge boson will have a zero mode, which corresponds to Neumann BC, while we have seen that the scalar mode's Dirichlet BC do not allow for a zero mode. After symmetry breaking, the Dirichlet BC for the gauge boson forbids their zero mode, but the scalar A_5 can include a zero mode, provided the symmetry is broken on both boundaries. The necessary degree of freedom for this zero mode comes from the boundary scalar π. This implies that the boundary conditions for the scalar component have to be the opposite of the vector's conditions, simply exchanging Neumann and Dirichlet BCs. It also means, that in the absence of massless scalars we should concentrate on Neumann–Neumann and Dirichlet–Neumann boundary conditions on our two boundaries $y = 0, \pi R$.

This mechanism now allows us to write down a very simple toy model for breaking a gauge symmetry by boundary conditions. We start with three gauge bosons, corresponding to a $SU(2)$ gauge group in the bulk. We will try to make two of them (W, Z) heavy while leaving the third (γ) massless. The massless photon is simple, because we know that we need Neumann–Neumann boundary conditions:

$$\partial_5 A_\mu^3 \Big|^{0,\pi R} = 0 \qquad \Rightarrow \qquad \hat{A}_\mu^3 \sim \cos\frac{ny}{R} \qquad \Rightarrow \qquad \boxed{m_{A^3}^{(n)} = \frac{n}{R} = 0, \frac{1}{R}, \frac{2}{R}\cdots} \tag{79}$$

For the other two gauge bosons it is sufficient to require a Dirichlet boundary condition at at least one of the boundaries. The choice of the second boundary condition will then affect the mass of the first KK excitation and the mass ratios of the higher excitations:

$$\partial_5 A_\mu^{1,2}\Big|^{0} = 0 \qquad A_\mu^{1,2}\Big|^{\pi R} = 0 \qquad \Rightarrow \qquad \hat{A}_\mu^{1,2} \sim \cos\frac{(2n+1)y}{2R} \qquad \Rightarrow \qquad \boxed{m_{A^{1,2}}^{(n)} = \frac{n + 1/2}{R} = \frac{1}{2R}, \frac{3}{2R}\cdots} \tag{80}$$

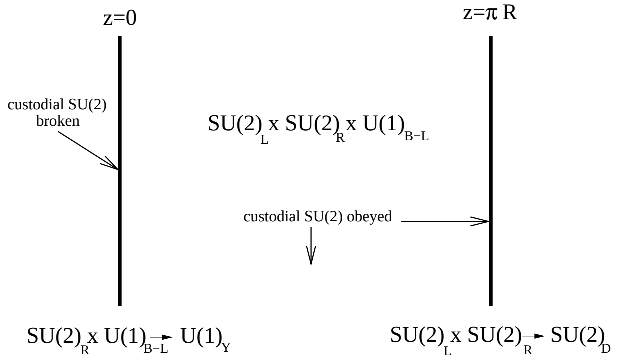

FIG. 1: Symmetry–breaking pattern of the Higgsless toy model, Figure stolen from Csaba's notes.

As discussed above, the boundary conditions for the scalar components are exactly the opposite of those for the vectors derived here. We have deliberately not chosen any pure Dirichlet–Dirichlet boundary setup for the gauge fields, because the corresponding scalar would then have purely Neumann boundary conditions, which would imply an unwanted massless scalar zero mode in the model.
This means, we indeed built a model with a massless photon and a W and Z with the same mass terms. Because of the factors of two between the ZZ and the W^+W^- mass terms in the Lagrangian we predict $m_Z/m_W = 2$ and for their first KK modes $m_{Z'}/m_Z = 2$ and $m_{W'}/m_W = 3$.
Nothing of that is anywhere close to reality, but we also have many aspect of the model to play with, so let us see what we can do better. At this point we can for the first time see why knowing technicolor and its problems helps us building models which break electroweak symmetry through boundary conditions: if we want to survive the electroweak precision constraints we need to protect the relevant observables using symmetries in our model.

C. A toy model with custodial symmetry

From the section on electroweak precision data we know that the S and T parameters in the gauge sector are very small. We also remember that the parameter T measures the different contributions to the W and Z masses from quantum corrections to their propagators. In the Standard Model there are two sources of this global $SU(2)$ symmetry breaking: in the Feynman diagrams contributing to m_Z we either find pure bottom or pure top loops, while m_W corrections include mixed bottom–top contributions. Modulo prefactors we can either say $\Delta T \sim 0$ or $\rho \sim 1$ defined as $\rho = m_W^2/(c_w^2 m_Z^2)$. For $m_b \neq m_t$ we find the contributions shown in eq.(50). In addition, electroweak symmetry breaking giving the Higgs doublet a vev in one doublet component also breaks the $SU(2)$ symmetry protecting $T = 0$. We can think of the complete symmetry of the Lagrangian with a protected value of $T = 0$ as $SU(2)_L \times SU(2)_R$. At this stage, none of them needs to be gauged, even though we know that $SU(2)_L$ at some point will be gauged. If both global $SU(2)_{LR}$ are unbroken, the left-right mixing Dirac masses of quark doublets will be degenerate $m_b = m_t$. If following the example of the chiral $U(1)_L \times U(1)_R$ symmetry we are now willing to re-align the two $SU(2)$ symmetries such that Dirac masses only break one of the combinations, there will be a remaining (diagonal) $SU(2)_D$ to protect T. To construct a realistic model of electroweak symmetry breaking we need to combine the electroweak symmetry and the custodial $SU(2)_D$ symmetry.

Let us first collect the maximal symmetry structure of the Standard Model. We start from the $SU(2)_L$ symmetry of the unbroken Lagrangian and expand it to $SU(2)_L \times SU(2)_R$ which protects the ρ parameter. In contrast to $SU(2)_L$ we do not need to gauge the global $SU(2)_R$, since we know there are no $SU(2)_R$ gauge bosons. But there is an additional gauged $U(1)_Y$ which we need for the abelian electromagnetic symmetry, under which left-handed and right-handed fermions are charged. So our unbroken electroweak symmetry can be viewed as a subset of the left-right symmetry $SU(2)_L \times SU(2)_R \supset SU(2)_L \times U(1)_Y$, where $SU(2)_R$ now needs to be gauged.
In the presence of fermions we finally need to add another global symmetry which gives us the fermions' hypercharges. They need to be protected by a global symmetry to avoid anomalies, *i.e.* quantum effects violating the $(B-L)$ number conservation. Again, this $U(1)_{B-L}$ does not need to be gauged, unless we embed $U(1)_Y \subset SU(2)_R \times U(1)_{B-L}$. In our model we will start from this complete unbroken $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ gauge symmetry in the bulk. The five–dimensional gauge bosons we denote as $A_M^{(L)}, A_M^{(R)}, B_M$. On the two boundaries we will break this maximal symmetry group into the electroweak $SU(2)_L \times U(1)_Y$ and into the custodial $SU(2)_D$ subgroups.

We know how to break symmetries on the boundaries from the last section. For the massless B gauge boson we require Neumann BCs while for the massive $SU(2)_L$ gauge bosons we assume a mixed set:

$$\partial_5 B_\mu \Big|^{0,\pi R} = 0 \qquad \partial_5 A_\mu^{(L)} \Big|^{0} = 0 \qquad A_\mu^{(L)} \Big|^{\pi R} = 0 \tag{81}$$

This is the same model as before, which means it will wrongly give us $m_Z/m_W = 2$, so we have to modify this setup. What we would hope to achieve is implementing the custodial $SU(2)_D$ on the boundary which describes our TeV-scale physics. For $y = \pi R$ we therefore replace $A^{(L,R)}$ by $(cA^{(R)} + sA^{(L)})$ and $(-sA^{(R)} + cA^{(L)})$ where the '+' combination corresponds to the unbroken $SU(2)_D$. The mixing angle we write in terms of $c \equiv g_{5,R}$ and $s \equiv g_{5,L}$. For the boundary conditions at $y = \pi R$ this implies:

$$\partial_5 B_\mu \Big|^{\pi R} = 0 \qquad \partial_5 \left(g_{5,L} A^{(L)} + g_{5,R} A^{(R)} \right) \Big|^{\pi R} = 0 \qquad \left(g_{5,L} A^{(L)} - g_{5,R} A^{(R)} \right) \Big|^{\pi R} = 0 \tag{82}$$

The still unbroken electroweak symmetry $SU(2)_L \times SU(2)_R \times U(1)_{B-L} \to SU(2)_L \times U(1)_Y$ we realize on the other boundary. Note that we will not discuss in detail how the large symmetry in the bulk will be broken on the two boundaries, but we do know how to write a non-linear sigma model on the boundaries. Also note that the electroweak symmetry $SU(2)_L \times U(1)_Y$ is not broken anywhere directly to $U(1)_Y$, this will happen by the boundary conditions automatically.
Our setup of the two $U(1)$ symmetries implies a mixing between B_μ and $A_\mu^{(R)}$. Again, we define the unbroken $U(1)_Y$ gauge boson as one of the linear combinations $(cA^{(R,3)} + sB)$ and break the other linear combination $(-sA^{(R,3)} + cB)$. The mixing angles are now $c \equiv g'_5$ and $s \equiv g_{5,R}$. This gives us for $y = 0$:

$$\partial_5 A_\mu^{(L)} \Big|^{0} = 0 \qquad \partial_5 \left(g_{5,R} B + g'_5 A^{(R,3)} \right) \Big|^{0} = 0 \qquad \left(g'_5 B - g_{5,R} A^{(R,3)} \right) \Big|^{0} = 0 \qquad A_\mu^{(R,12)} \Big|^{0} = 0 \tag{83}$$

The one remaining question is what to do with the two remaining fields $A^{(R)}$ at $y = 0$. We do not want a zero mode for the corresponding gauge fields, so we give them a Dirichlet BC there. This setup produces precisely the symmetries in Csaba's Fig. 1. Notice that we do not have to specify the boundary conditions for the scalar fifth components, because they are as usually fixed by exchanging Neumann and Dirichlet conditions.

We first see that each five–dimensional field combines different types of boundary conditions and that by construction the zero-mode photon will be built out of components of B_μ and $A_\mu^{(R,3)}$ mixed at $y = 0$ and $A_\mu^{(L,3)}$ and $A_\mu^{(R,3)}$ mixed at $y = \pi R$. This linear combination is the only field with purely Neumann boundary conditions. The physical Z boson will come from the '$-$' combination of $A_\mu^{(L,3)}$ and $A_\mu^{(R,3)}$ at $y = \pi R$, which has mixed boundary conditions. Its mass we can in principle compute:

$$m_Z^{(n)} = m_0 + \frac{n}{R} = m_0, \left(m_0 + \frac{1}{R} \right) \cdots \qquad m_0 = \frac{1}{\pi R} \arctan \sqrt{1 + \frac{g'^2}{g^2}} \tag{84}$$

The mass scale lifting the first Z mode off the zero mode is given in terms of the gauge couplings, which acted as the mixing angles in the rotation to a massless photon. This is really the same thing we know as the weak mixing angle in the Standard Model. Similarly, we can compute the W boson masses. To make the analysis of the KK states easier we can identify $g_{5,L} \equiv g_{5,R} = g$. This allows us to combine $A^{(L)}$ and $A^{(R)}$ into $A^{(\pm)}$, which should describe the $W^\pm$ gauge bosons.

$$m_W^{(n)} = \frac{2n+1}{4R} = \frac{1}{4R}, \frac{3}{4R} \cdots \qquad \Rightarrow \qquad \frac{m_W^{(0)}}{m_Z^{(0)}} = \frac{\pi^2}{16} \left[\arctan \sqrt{1 + \frac{g'^2}{g^2}} \right]^{-2} \sim 0.85 \qquad \Rightarrow \qquad \boxed{\rho = \frac{m_W^{(0)}}{c_w^2 m_Z^{(0)}} \sim 1.10} \tag{85}$$

This is really not bad a result. At this stage we will have to believe that we can adjust the result by for example bending our flat space and incorporating our setup in a Randall–Sundrum model. Such a model is built as a five–dimensional theory with two branes, usually referred to as a Planck brane at $y = 0$ and a TeV brane at $y = \pi R$. The difference will be that we cannot simply write sine and cosine Fourier series for the wave functions in the warped fifth dimension, but that we have to solve a differential equation which will give us Bessel functions (except for zero modes like the photon). In the usual RS language we can then play around with the location $y = b$ of the TeV brane and the warp factor k, to adjust the gauge boson masses. In Csaba's lecture he replaces the warp factor in the metric $\exp(-A(z)) = 1/(1+kz)^2$ by $(R/z)^2$ with $R \sim 1/M_{\text{Planck}}$. The TeV-scale in the RS models arises as $M_{\text{Planck}} \exp(-kb)$ which can be written as $R' \sim 1/\text{TeV}$. In that case the KK mass scale is given in terms of $1/R'$, but including logarithms of the type $\log R/R'$ from the Bessel functions, so we have parameters to play with. Instead of discussing in detail how such a Randall–Sundrum embedding works we will move on and see how the KK towers of massive electroweak gauge bosons behave in the usual unitarity argument for a light fundamental Higgs boson.

D. Unitarity and KK excitations

One of the ways to introduce a Higgs boson is the complete unitarization of a theory with massive gauge bosons, *e.g.* from a non-linear sigma model. The classical example is the scattering process of longitudinal $W_L W_L \to W_L W_L$, where we can express the W polarization vector in terms of the energy and momentum as

$$\epsilon_\mu = \left(\frac{|\vec{p}|}{M}, \frac{E}{M} \frac{\vec{p}}{|\vec{p}|} \right) \propto E \qquad p_\mu^{(\text{in})} = \left(E, 0, 0, \pm\sqrt{E^2 - M^2} \right) \propto E$$
$$p_\mu^{(\text{out})} = \left(E, \pm\sqrt{E^2 - M^2} \sin\theta, 0, \pm\sqrt{E^2 - M^2} \cos\theta, \right) \propto E \tag{86}$$

We have indicated the energy behavior of the longitudinal components. If we now compute the scattering amplitude at high energies we find that for example the contact interaction contributes proportional to the maximum power $\mathcal{A} \propto E^4$. However, with the s,t,u-channel gauge-boson exchange diagrams this E^4 term cancels due to gauge invariance. What we are left with is $\mathcal{A} \propto E^2$, which still means that the transition amplitude diverges at high energies and will at some point violate perturbative unitarity. The old argument for the existence of a Higgs boson with a mass smaller than the scale at which unitarity is violated (the TeV scale) is that such a Higgs boson with all the proper couplings will unitarize the $W_L W_L \to W_L W_L$ scattering process. In my notes on Higgs searches you can see for example how to compute this behavior using the equivalence theorem between gauge bosons and Goldstone bosons. The obvious question is: how will our theory without any fundamental Higgs boson cure this fundamental problem with massive gauge bosons?

Csaba explicitly writes the form for the leading E^4 term in the amplitude of four Standard–Model gauge boson with index n and the exchanged KK tower k:

$$\mathcal{A}^{(4)} = i \left(g_{nnnn}^2 - \sum_k g_{nnk}^2 \right) \left[f^{abe} f^{cde} \left(3 + 6\cos\theta - \cos^2\theta \right) + 2 f^{ace} f^{bde} \left(3 - \cos^2\theta \right) \right] \tag{87}$$

No masses appear in this form, only coupling constants. This dangerous contribution vanishes only if the couplings fulfill the appropriate sum rule. The coupling between different KK modes is given by the overlap of their wave functions in the fifth dimension

$$g_{mnk} = g_5 \int dy\; f_m(y) f_n(y) f_k(y) \qquad\qquad g_{mnkl} = g_5^2 \int dy\; f_m(y) f_n(y) f_k(y) f_l(y) \tag{88}$$

The Fourier transforms of the wave functions have a completeness relation

$$\sum_k f_k(y) f_k(z) = \delta(y - z) \tag{89}$$

which we can use to show the <u>couplings sum rule</u> starting from the left-hand side

$$\begin{aligned}
\sum_k g_{nnk}^2 &= g_5^2 \sum_k \left(\int_0^{\pi R} dy\; f_n^2(y) f_k(y) \right) \left(\int_0^{\pi R} dz\; f_n^2(z) f_k(z) \right) \\
&= g_5^2 \int_0^{\pi R} dy \int_0^{\pi R} dz\; f_n^2(y) f_n^2(z) \left(\sum_k f_k(y) f_k(z) \right) \\
&= g_5^2 \int_0^{\pi R} dy \int_0^{\pi R} dz\; f_n^2(y) f_n^2(z) \delta(y - z) \\
&= g_5^2 \int_0^{\pi R} dy\; f_n^4(y) \qquad \Rightarrow \qquad \boxed{\sum_k g_{nnk}^2 = g_{nnnn}}
\end{aligned} \tag{90}$$

Assuming that this sum rule — which really does not have anything to do with a Higgs boson, only with gauge invariance between 3-point and 4-point couplings — we can write a compact form of the second diverging term in the amplitude:

$$\mathcal{A}^{(2)} = \frac{i}{m_n^2} \left(4 g_{nnnn} m_n^2 - 3 \sum_k g_{nnk}^2 m_k^2 \right) \left[-f^{abe} f^{cde} \sin^2 \frac{\theta}{2} + f^{ace} f^{bde} \right] \stackrel{!}{=} 0 \qquad (91)$$

Again, there is a mass–couplings sum rule given by the first parentheses. It involves KK masses as well as the gauge couplings, which is different from the Higgs mechanism. In other words, the KK tower with all couplings fixed properly plays the role of the Higgs boson in the Standard Model. The problem is that while the Higgs mass can be chosen such that its effects come in beyond the scale of unitarity violation, the KK tower involves an infinite sum over states with arbitrarily high masses. This implies a cutoff scale of our effective theory, but then we always knew there would be such a cutoff, namely the fundamental Planck scale, above which we cannot use the KK effective theory to compute scattering effects.

If we had more time we would at this point need to talk about fermion masses in this model. The problem starts long before writing down Yukawa terms in five dimensions, namely with the extension of chiral fermions into more than four dimensions. In four dimensions spinors are another representation of the Lorentz group. We can express the 4×4 matrices γ_μ in terms of the 2×2 Pauli matrices σ_j and $-\mathbb{1}$ and define the transformation

$$x^\mu \to [x] = x^0 - x^j \sigma^j = \begin{bmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^2 - ix^2 & x^0 + x^3 \end{bmatrix} \qquad (92)$$

which is nothing but a Lorentz transformation. When we write fermions in five dimensions we need to extend the corresponding Dirac gamma–matrix basis $\gamma_\mu \to \gamma_M$. There is even a candidate for the fifth gamma matrix, namely γ_5. The problem is that this γ_5 appears in the chiral projectors $(\mathbb{1} \pm \gamma_5)/2$, which means that it mixes chiralities. This means that Lorentz transformations to not respect chirality. If we write the Dirac equation in five dimensions, the derivative ∂_5 will just like the mass term mix left-handed and right-handed Weyl fermions. Once we Fourier-transform the fifth dimensions into a KK tower this is not surprising — after all ∂_5 is nothing but a mass term. But to learn more about writing down Yukawa couplings and making them into fermion masses you will need to read Csaba's review or some of the original papers for example by Tim Tait and friends...

One last word concerning these fermions. From extended technicolor we remember that giving the top quark a mass using a dimension–six operator leads to problems with the effective $Zb\bar{b}$ coupling. This happens because of the $SU(2)_L$ symmetry in combination with a chiral or custodial $SU(2)_R$ symmetry. In extra–dimensional models we will define a mass for all fermions via their position in the fifth dimension and a wave–function overlap with something playing the role of a sigma field. By construction, we incorporate the $SU(2)_L$ and the $SU(2)_D$ symmetries, which means we will run into precisely the same problem as extended technicolor did. Unless our really bright model–building colleagues manage to solve this problem at some stage.

Literature: In particular on the first part of the course there are many lectures available by all the experts on the field. As usual, I find the TASI lecture notes the most useful, but not the only good source

- A very extensive introduction into technicolor and its successors can be found in hep-ph/0203079. Note that this writeup is almost 200 pages long, but at least the first half of them are really instructive. Most of my notes on technicolor are based on this review.
- A shorter and also very modern introduction into technicolor is Sekhar Chivucula's hep-ph/0011264. If you have already understood something and would like to refresh your memory on the ideas behind technicolor, it is great.
- The short introduction on electroweak symmetry breaking from boundary conditions is based on Csaba Csaki's, Jay Hubisz's and Patrick Meade's TASI lecture hep-ph/0510275. I have no idea how Csaba managed to teach all this material in four lectures, but I always had the suspicion that he is an extraordinarily good teacher.
- And finally, for an introduction to electroweak precision data there is the usually nicely written TASI lecture, in that case by James Wells: hep-ph/0512342. James even teaches how to compute loops leading to S and T contributions, so go and have a look.