2. Simple Supersymmetric Lagrangeans

[Aitchison hep-ph/0505105]

Take step back and forget hierarchy problem for now;
assume a supersymmetric (as yet undefined) theory will not have it...

2.1 Supersymmetry as a general extended symmetry

[Haag, Lopuszanski, Sohnius 1975]

divide symmetry operators according to their Lorentz structure
$\left\{\begin{array}{l}\text{Lorentz scalar, e.g. charge, isospin} \\ \text{4-vector, e.g. space-time translations} \\ \text{antisymmetric tensor, e.g. angular momentum} \\ \rightsquigarrow \text{symmetric tensor } \not\downarrow \text{ not allowed due to Coleman-Mandula}\end{array}\right.$

$\Rightarrow$ one structure still allowed: spinor charge

i.e. $\boxed{Q\,|J\rangle = |J \pm \tfrac{1}{2}\rangle}$ with J spin of particle

⌜ used to be unthinkable, because spin fixes (Fermi- or Bose-Einstein) statistics and defines matter vs. interaction particles ⌟

$\Rightarrow$ goal for next 2 lectures:
Write down a supersymmetric version of scalar QED,
i.e. the simplest field theory with scalar ϕ_e and photon field A^μ

obvious observation

$$Q|\phi\rangle \sim |\tilde{\phi}\rangle$$

spin 0 $\quad$ spin $\frac{1}{2}$

degrees of freedom should not vanish

$\rightsquigarrow$ what is the fermionic partner of a complex scalar (2 d.o.f)?

$\rightsquigarrow$ Dirac spinor $(i\partial_\mu\gamma^\mu - m)\Psi = 0$

has 4 dimensions $\rightsquigarrow$ 4 d.o.f.

$\Rightarrow$ remember chirality projectors:

$$P_R = \frac{1+\gamma_5}{2} = \frac{1}{2}\left(1 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_L = \frac{1-\gamma_5}{2} = \frac{1}{2}\left(1 - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which means $\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$: $P_R \Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ $\qquad P_L \Psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$

fermion fields with 2 d.o.f on-shell or 4 d.o.f off-shell

with the Dirac equation

$$\sigma^\mu p_\mu \psi = m\chi$$
$$\bar{\sigma}^\mu p_\mu \chi = m\psi$$

now 2x2 Pauli matrices

note m mixes chiralities

$$\sigma^\mu = \left(1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\bar{\sigma}^\mu = \left(1, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

all hermitesch $(\sigma^\mu)^\dagger = \sigma^\mu$; $\sigma_i \sigma_i = 1$;

$\Rightarrow$ rewrite Lagrangian for massive fermion in Weyl spinors

$$\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \psi^\dagger i\sigma^\mu\partial_\mu\psi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi - m(\psi^\dagger\chi + \chi\psi)$$

for simplicity, let's start constructing a supersymmetric massless scalar QED...

2.2 Fermionic partner of complex scalar

$$\boxed{\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi}$$

pick either χ or ψ,
forget masses, remember Klein-Gordon equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \partial_\mu \partial^\mu \phi \equiv \Box \phi = 0$$

(o.k.)

Can this Lagrangian be supersymmetric under: $\begin{Bmatrix} \phi \\ \chi \end{Bmatrix} \xmapsto{\text{SUSY}} \begin{Bmatrix} \phi \\ \chi \end{Bmatrix} + \delta_\xi \begin{Bmatrix} \phi \\ \chi \end{Bmatrix}$

SUSY transformations

$$\boxed{\delta_\xi \phi = \xi^T (-i\sigma_2) \chi} = (\xi_1\ \xi_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = (\xi_1\ \xi_2) \begin{pmatrix} -\chi_2 \\ \chi_1 \end{pmatrix} = -\xi_1 \chi_2 + \xi_2 \chi_1$$

Lorentz invariant

$$=: \xi^a \chi_a = \xi \cdot \chi$$

$\rightsquigarrow$ ξ a spinor, Grassmann operator because SUSY flips spin
complex : $(\xi^T)^* = \xi^\dagger$ Hermitian conjugate

$$\delta_\xi \chi \stackrel{?}{=} (i\sigma^\mu \partial_\mu \phi) \xi$$

$[\delta_\xi \chi] = M^{3/2}$; $[\xi] = M^{-1/2}$; $[\phi] = M$

$\rightsquigarrow$ $\delta_\xi \chi \sim \phi \cdot \xi$ needs $\partial_\mu \phi$ with $[\partial_\mu] = M$

$\rightsquigarrow$ $\delta_\xi \chi \sim (\partial_\mu \phi) \xi$ needs $\sigma^\mu \partial_\mu$

$\rightsquigarrow$ not yet Lorentz invariant; $\xi \mapsto i\sigma_2 \xi^*$

$$\Rightarrow \boxed{\delta_\xi \chi = -A \cdot i\sigma^\mu (i\sigma_2 \xi^*)(\partial_\mu \phi)}$$

unknown normalization, c-number

n.b.: sign of A different from Aitchison

before we tackle Lagrangian we need Hermitian conjugate

$$\delta_\xi \phi^\dagger = \chi^\dagger (i\sigma_2) \xi^* \qquad \text{in analogy to } (A \cdot B)^\dagger = B^\dagger A^\dagger$$

$$\delta_\xi \chi^\dagger = -A \, (\partial_\mu \phi^\dagger)(\xi^T i\sigma_2) i\sigma^\mu$$

by brute force

$$\delta_\xi \mathcal{L} = \partial_\mu (\delta_\xi \phi^\dagger) \partial^\mu \phi + \partial_\mu \phi^\dagger \partial^\mu (\delta_\xi \phi) + (\delta_\xi \chi^\dagger) i\bar\sigma^\mu \partial_\mu \chi + \chi^\dagger i \bar\sigma^\mu \partial_\mu (\delta_\xi \chi)$$

$$= \partial_\mu (\chi^\dagger i\sigma_2 \xi^*) \partial^\mu \phi - \partial_\mu \phi^\dagger \partial^\mu (\underset{R}{\xi^T} i\sigma_2 \chi)$$

$$- A (\partial_\mu \phi^\dagger)(\xi^T i\sigma_2 i\sigma^\mu)(i\bar\sigma^\nu \partial_\nu \chi) - A \chi^\dagger i\bar\sigma^\nu \partial_\nu i\sigma^\mu i\sigma_2 \xi^* (\partial_\mu \phi) \overset{?}{\doteq} 0$$

$$\xi^* \text{ only} \quad \partial_\mu \chi^\dagger i\sigma_2 \xi^* \partial^\mu \phi + iA\, \chi^\dagger \underbrace{\bar\sigma^\nu \partial_\nu \sigma^\mu}_{} \sigma_2 \xi^* (\partial_\mu \phi)$$

$$\left[\ \bar\sigma^\nu \partial_\nu \sigma^\mu \partial_\mu = (\partial_0 + \vec\sigma \cdot \vec\nabla)(\partial_0 - \vec\sigma \vec\nabla) = \partial_0^2 - \vec\nabla^2 = \partial_\mu \partial^\mu\ \right]$$

$$= i \partial_\mu \chi^\dagger \sigma_2 \xi^* (\partial^\mu \phi) + iA\, \chi^\dagger \sigma_2 \xi^* (\partial_\mu \partial^\mu \phi)$$

$$= i \partial_\mu (\chi^\dagger \sigma_2 \xi^* (\partial^\mu \phi))$$

$\boxed{A=1}$

total derivative (same for ξ^T)

$$\boxed{\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i\bar\sigma^\mu \partial_\mu \chi \quad \text{is supersymmetric}}$$

2.3 SUSY algebra

ϕ and χ form a SUSY multiplet ($\Leftrightarrow$ transform into each other under δ_ξ)

$\rightsquigarrow$ transform into themselves under $\delta_\xi \delta_\eta$ or $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)$?

$\rightsquigarrow$ can we deduce some general operator algebra from $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)$?

brute force again:

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi = -\delta_\eta (\xi^T i\sigma_2 \chi) - (\xi \leftrightarrow \eta) = -i \xi^T \sigma_2 \delta_\eta \chi - (\xi \leftrightarrow \eta)$$

$$= -i \xi^T \sigma_2 (-i\sigma^\mu i\sigma_2 \eta^*) \partial_\mu \phi - (\xi \leftrightarrow \eta)$$

$$= -i \xi^T \sigma_2 \sigma^\mu \sigma_2 \eta^* \partial_\mu \phi - (\xi \leftrightarrow \eta)$$

$$= -i \xi^T (\bar\sigma^\mu)^T \eta^* \partial_\mu \phi - (\xi \leftrightarrow \eta) \qquad \text{using } \sigma_2 \sigma^\mu \sigma_2 = (\bar\sigma^\mu)^T$$

$$= -i (\xi^T \bar\sigma^{\mu T} \eta^*)^T \partial_\mu \phi - (\xi \leftrightarrow \eta) \qquad \text{just a c-number } C^T \equiv C$$

$$= +i \eta^\dagger \bar\sigma^\mu \xi^* \partial_\mu \phi - (\xi \leftrightarrow \eta) \qquad (-) \text{ from Grassmann } \xi$$

$$\Rightarrow \quad \boxed{(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi = i(\eta^\dagger \bar\sigma^\mu \xi^* - \xi^\dagger \bar\sigma^\mu \eta)\, \partial_\mu \phi}$$

units: $[\phi] = M$; $[\eta^\dagger] = M^{-1/2} = [\xi]$; $[\partial_\mu \phi] = [\partial_\mu][\phi] = M \cdot M = M^2$

now the same for χ:
$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\chi \overset{?}{=} i(\eta^\dagger \bar\sigma^\mu \xi - \xi^\dagger \bar\sigma^\mu \eta)\partial_\mu \chi$$

that would mean we have a operator algebra relation
$$[\delta_\eta, \delta_\xi] \overset{?}{=} i(\eta^\dagger \bar\sigma^\mu \xi - \xi^\dagger \bar\sigma^\mu \eta)\partial_\mu$$

and it would map $\chi \leftrightarrow \phi$ and we could use this to write a proper algebra definition of the SUSY generators...

but we are still on foot:
$$\delta_\eta \delta_\xi \chi = -\delta_\eta i\sigma^\mu i\sigma_2 \xi^* \partial_\mu \phi = +\sigma^\mu \sigma_2 \xi^* \partial_\mu(\delta_\eta \phi)$$ δ_η only acts on SUSY fields
$$= \sigma^\mu \sigma_2 \xi^* \partial_\mu(-\eta^T i\sigma_2 \chi) = -i\sigma^\mu \sigma_2 \xi^* \eta^T \sigma_2 \partial_\mu \chi$$
$= \dots\dots$ [Aitchison eqs. (241) to (247)]
$$= -i\eta(\xi^\dagger \bar\sigma^\mu \partial_\mu \chi) + i\eta^T \sigma_2 \sigma^\mu \sigma_2 \xi^* \partial_\mu \chi$$
$$= -i\eta(\underbrace{\xi^\dagger \bar\sigma^\mu \partial_\mu \chi}_{\text{c-number}}) - i\underbrace{\xi^\dagger \bar\sigma^\mu \eta^*}_{\text{c-number}} \partial_\mu \chi$$

$\Rightarrow$ only second term: $[\delta_\eta, \delta_\xi]\chi = i(\eta^\dagger \bar\sigma^\mu \xi^* - \xi^\dagger \bar\sigma^\mu \eta^*)\partial_\mu \chi$ is what we want.

only first term $[\delta_\eta, \delta_\xi]\chi = -i\eta(\xi^\dagger \bar\sigma^\mu \partial_\mu \chi) + i\xi(\eta^\dagger \bar\sigma^\mu \partial_\mu \chi)$ is not appreciated

Have a closer look at d.o.f.: no Dirac equation etc. $\Leftrightarrow$ off-shell

ϕ: complex scalar, 2 d.o.f $\;\lightning\;$ χ: complex Weyl spinor, 4 d.o.f

$\rightsquigarrow$ not matching, but another scalar field would do it: F

$\rightsquigarrow$ choose $\delta_\xi F$ and F's contribution to $\delta_\xi \phi$, $\delta_\xi \chi$ such that

(1) our operator algebra holds, i.e. "closes" for SUSY multiplet $\{\phi, \chi, F\}$

(2) $\mathcal{L}(\phi, \chi, F)$ is still SUSY-invariant.

avoid couplings etc. : $\mathcal{L}_F \sim F^\dagger F$ (dagger because field operator)

$\Rightarrow \quad [F] = M^2$

equation of motion

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F^\dagger)} \right) - \frac{\partial \mathcal{L}}{\partial F} = -F^\dagger \stackrel{!}{=} 0$$

auxiliary field. can be set to any number, does not propagate.

$$\boxed{\delta_\xi F = -i\, \xi^\dagger \bar\sigma^\mu \partial_\mu \chi}$$

from dimensions start from $\delta_\xi F \sim \xi \partial_\mu \chi$

$\rightsquigarrow$ match index $\delta_\xi F \sim \xi \bar\sigma^\mu \partial_\mu \chi$

$\rightsquigarrow$ Lorentz invariance $\delta_\xi F \sim \xi^\dagger \bar\sigma^\mu \partial_\mu \chi$

$$\Rightarrow \quad \delta_\xi F^\dagger = +i (\partial_\mu \chi^\dagger) \bar\sigma^\mu \xi$$

to attempt a cure for the $[\delta_\eta, \delta_\xi]\chi \neq i(\eta^\dagger \bar\sigma^\mu \xi^* - \xi^\dagger \bar\sigma^\mu \eta^*)\partial_\mu \chi$

we assume

$$\boxed{\delta_\xi \chi = \sigma^\mu \sigma_2 \xi^* \partial_\mu \phi + \xi F}$$

$\delta_\xi \chi \big|_F \sim \xi F$ already does it!

and keep $\boxed{\delta_\xi \phi = -i\, \xi^T \sigma_2 \chi}$ because $[\delta_\eta, \delta_\xi]\phi \stackrel{!}{=} i(\cdots)\partial_\mu \phi$

<u>Lagrangian</u>:

$$\delta_\xi F^\dagger F = (\delta_\xi F^\dagger) F + F^\dagger \delta_\xi F$$

$$= i (\partial_\mu \chi^\dagger) \bar\sigma^\mu \xi F - i F^\dagger \xi^\dagger \bar\sigma^\mu \partial_\mu \chi$$

add to total derivative $i\partial_\mu (\chi^\dagger \bar\sigma^\mu \xi F)$

$$\delta_\xi (\chi^\dagger i \bar\sigma^\mu \partial_\mu \chi)\big|_F = (\delta_\xi \chi^\dagger) i \bar\sigma^\mu \partial_\mu \chi + i \chi^\dagger \bar\sigma^\mu \partial_\mu (\delta_\xi \chi)\big|_F$$

$$= i F^\dagger \xi^\dagger \bar\sigma^\mu \partial_\mu \chi + i \chi^\dagger \bar\sigma^\mu \xi \partial_\mu F$$

cancel

$\Rightarrow$ Lagrangian changes by total derivative, action is preserved by SUSY

(that was (2))

check our operator algebra

$$
\begin{aligned}
[\delta_\eta, \delta_\xi] F &= \delta_\eta \delta_\xi F - (\eta \leftrightarrow \xi) \\
&= \delta_\eta\left(-i\xi^\dagger \bar\sigma^\mu \partial_\mu \chi\right) - (\eta\leftrightarrow\xi) = -i\xi^\dagger\bar\sigma^\mu\partial_\mu(\delta_\eta\chi) - (\eta\leftrightarrow\xi) \\
&= -i\xi^\dagger\bar\sigma^\mu\partial_\mu\left(\sigma^\nu\sigma_2\eta^*\partial_\nu\phi + \eta F\right) - (\eta\leftrightarrow\xi) \\
&= -i\,\underbrace{\xi^\dagger\bar\sigma^\mu\sigma^\nu\sigma_2\eta^*}\partial_\mu\partial_\nu\phi - i\xi^\dagger\bar\sigma^\mu\eta\,\partial_\mu F - (\eta\leftrightarrow\xi) \\
&= -i\xi^\dagger\bar\sigma^\mu\eta\,\partial_\mu F + i\eta^\dagger\bar\sigma^\mu\xi\,\partial_\mu F
\end{aligned}
$$

first term vanishes with $(\eta\leftrightarrow\xi)$

$$
\begin{aligned}
[\delta_\eta,\delta_\xi]\chi\Big|_F &= \delta_\eta\,\xi F - (\xi\leftrightarrow\eta) \\
&= \xi\left(-i\eta^\dagger\bar\sigma^\mu\partial_\mu\chi\right) - (\xi\leftrightarrow\eta) \\
&= -i\,\xi\left(\eta^\dagger\bar\sigma^\mu\partial_\mu\chi\right) + i\eta\left(\xi^\dagger\bar\sigma^\mu\partial_\mu\chi\right)
\end{aligned}
$$

cancels precisely the unwanted term from p.11

$$\Rightarrow \quad \boxed{[\delta_\eta,\delta_\xi] = i\left(\eta^\dagger\bar\sigma^\mu\xi^* - \xi^\dagger\bar\sigma^\mu\eta^*\right)\partial_\mu}$$

holds for all components of Susy multiplet $\{\phi,\chi,F\}$

2.4 Wess-Zumino Model

$$\boxed{\mathcal{L}_{WZ} = \partial_\mu\phi^\dagger\partial^\mu\phi + \chi^\dagger i\bar\sigma^\mu\partial_\mu\chi + F^\dagger F}$$

with massless Weyl fermion and massless charged scalar

F can be made use of once we introduce interactions

$$
\begin{aligned}
\mathcal{L}_{WZ}^{(int)} &= \mathcal{L}_{WZ}^{(free)} + W_i(\phi,\phi^\dagger)F_i - \frac{1}{2}W_{ij}(\phi,\phi^\dagger)\chi_i\chi_j + h.c. \\
&= \mathcal{L}_{WZ}^{(free)} + \left(m_{ij}\phi_j + \frac{1}{2}Y_{ijk}\phi_j\phi_k\right)F_i + \dots
\end{aligned}
$$

gives equation of motion from $\mathcal{L} \sim W_iF_i + W_i^\dagger F_i^\dagger + F_i^\dagger F_i$

$$\frac{\partial\mathcal{L}}{\partial F_i} = 0 \quad\Leftrightarrow\quad F_i^\dagger = -W_i \;,\quad F_i = -W_i^\dagger$$

$$\Rightarrow \quad \mathcal{L}_{WZ}^{(int)} = \mathcal{L}_{WW}^{(free)} + |W_i|^2 - \frac{1}{2}\left(W_{ij}\chi_i\chi_j + h.c.\right)$$

So F is of use...