Exercise sheet 3

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Exercise 1

In this exercise we will solve Laplace's equation ($\nabla^2\Phi = 0$) as a first application of a boundary value problem. In order to make things simpler let's focus on the 2−dim problem, so $\Phi = \Phi(x, y)$. We want to solve the boundary value problem in a region of space S bounded by ∂S. When the potential is known on ∂S, then the solution is unique and the potential, and consequently, the electric field is determined everywhere in S. In this case, S should be taken to be bounded by a square with sides of length l. Our system would be the 2−dim projection of two parallel conducting plates inside a cubic box. You should consider the square boundary as grounded and the conductors are at constant potential V_1 and V_2 respectively.

- Find the finite difference equation that approximates Laplace's equation by Taylor expanding $\Phi(x \pm \delta x, y)$ and $\Phi(x, y \pm \delta y)$ (assuming that $\delta x = \delta y = a$). You should arrive at the following equation
$$\Phi^*(x,y) = \frac{1}{4}\left(\Phi(x+\delta x, y) + \Phi(x-\delta, y) + \Phi(x, y+\delta y) + \Phi(x, y-\delta y)\right).$$
- Map the coordinates of the lattice sites to integers to integers (i, j) such as
$$x_i = (i-1)a,$$
$$y_j = (j-1)a,$$
where $i, j = 1, \ldots, L$.
Your new equation should read
$$\Phi^*(i,j) = \frac{1}{4}\left(\Phi(i+1, j) + \Phi(i-1, j) + \Phi(i, j+1) + \Phi(i, j-1)\right).$$
- The steps that should be taken are the following
  - Fix the Lattice size.
  - Mark the sites corresponding to conductors.
  - Choose carefully an initial trial solution for the vacuum. You can try $\Phi(x, y) = 0$.
  - The data structure should involve a two-dimensional array of doubles for the potential of length L. You should also include a logical array that checks if a given site is a conductor (returning "true") or if is a site belonging to the vacuum (returning "false").
  - Sweep the lattice and enforce the discretized Laplace equation.
  - Keep sweeping successively until two successive sweeps result in a tiny change of $\Phi(x, y)$.
  - As discussed in the class the errors in the solution obtained by the Jacobi and Gauss-Seidel methods decrease only slowly and in a monotonic manner. We want to try the method of successive overrelaxation where
$$\Phi^*(i,j) = (1-s)\Phi(i,j) + \frac{s}{4}\left(\Phi(i+1, j) + \Phi(i-1, j) + \Phi(i, j+1) + \Phi(i, j-1)\right),$$
where the optimal value for s depends on the problem but $s \in (1.2, 1.4)$ gives good results.
  - Save your results in a file and print them with gnuplot (using pm3d and hidden3d).

Hint: Remember to test your code at each stage of its development.

Exercise 2

In this problem you should generalize the previous code to take into account Poisson's equation. You should add $\rho(i, j)$ in your partial differential equation and repeat the previous steps (the charge density should be given by the total charge Q -given as input- divided by the area).