Scalar field theory (d+1) dimensional generalization of the (0+1)d quantum mechanical path integral. $\varphi = \varphi(\vec{x},t) = \varphi(x)$

$$S[\varphi] = \int_{t_1}^{t_2} d^3x\, dt \left(\tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}(\partial_i\varphi)^2 + \tfrac{m^2}{2}\varphi^2 + \tfrac{\lambda}{4!}\varphi^4 \right) \text{ classical action}$$

The corresponding quantized fields $\hat{\varphi}(x)$ and conjugate momenta fulfill equal time commutation relations $[\hat{\varphi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = i\delta^{(3)}(x-y)$ $[\hat{\varphi},\hat{\varphi}] = [\hat{\pi},\hat{\pi}] = 0$ $\hat{\varphi}(\vec{x},t) = e^{i\hat{H}t}\varphi(\vec{x},0)e^{-i\hat{H}t}$ and the transition amplitude can be written as $\langle\varphi_2|U(t_2,t_1)|\varphi_1\rangle = \int_{\varphi_1}^{\varphi_2} D\varphi\, e^{iS[\varphi]}$ Eucl. $\Rightarrow \int D\varphi_E\, e^{-S(\varphi_E)}$ via analytic continuation of the time (d+1) Mink. $\rightarrow$ D Eucl. dimension

Discretization: $x_\mu = k_\mu a$ $k_\mu \in [0, N_\mu - 1]$ $a_\mu = a$ isotropic case

$$S_E[\varphi] \approx a^4 \sum_{x,\tau} \left[\frac{1}{2a^2}\left\{ (\varphi(\vec{x},\tau+a) - \varphi(\vec{x},\tau))^2 + \sum_i (\varphi(\vec{x}+a\hat{i},\tau) - \varphi(\vec{x},\tau))^2 + \frac{1}{2}m^2\varphi^2(x,\tau) + \frac{\lambda}{4!}\varphi^4(x,\tau) \right. \right]$$

using periodic boundary conditions: $\sum_x \nabla_F\phi = \sum_x \nabla_B\phi$
so that partial integration becomes $\sum_x \varphi(x)\nabla_F\varphi(x) = -\sum_x(\nabla_B\varphi)\varphi$
let us consider the dimensions of each building block:
$[S]=1$ since action is a number if $\hbar = c = 1$; kinetic term $\frac{[a]^D[\varphi]^2}{[a]^2} = 1$
$\Rightarrow [\varphi] = [a]^{\frac{2-D}{2}} \cdot [a]^{-1}\big|_{D=4} = [m]$ $\quad [\lambda] = 1\big|_{D=4}$

For numerical simulations advantageous to write action in dimensionless quantities: $a\varphi = (2\kappa)^{1/2}\phi$ $\quad a^2m^2 = \frac{1-2\lambda}{\kappa} - 8$ $\quad \lambda_0 = \frac{6\lambda}{\kappa^2}$

$$S[\phi] = \sum_x \left\{ -2\kappa \sum_{\mu=1}^{4} \phi(x)\phi(x+a\hat{\mu}) + \phi^2(x) + \lambda[\phi^2(x) - 1]^2 - \lambda \right\}$$

Analogously to the QM case we can construct a transfer matrix operator $\hat{T} = \exp[-a/2\,V(\hat{\varphi})]\exp[-a\tfrac{1}{2}\sum_x \hat{\pi}(x)]\exp[-a/2\,V(\hat{\varphi})]$ starting from the matrix elements $\langle\varphi(x,\tau+a)|\hat{T}|\varphi(x,\tau)\rangle$. Here it is explicitly possible to show that the resulting lattice Hamiltonian is hermitean.

Once again reflection positivity:

① Site reflection: Decompose the action into $S = S_0 + S_+ + S_-$, where S_0 contains only fields @ $\tau=0$, S_+ all fields with $\tau>0$ and $S_- = \Theta S_+$.
Note that such a decomposition only exists for scalar fields.

We need to compute $\langle(\Theta F)F\rangle = \prod_x \int d\varphi(x)\, e^{-S_0}\, \Theta(Fe^{-S_+})\, Fe^{-S_+}$. When writing $\tilde{F} = \prod_{t>0} \int d\varphi(x)\, F e^{-S_+}$ one can show that

$$\langle(\Theta_s F)F\rangle = \int \prod_{t=0} d\varphi(x)\, e^{-S_0} |\tilde{F}|^2 \geq 0 \Rightarrow \hat{T}^2 \geq 0$$

Link reflection positivity needs a different decomposition:

$$S = S_c + S_+' + S_-' \quad \text{with } S_c = -2\kappa \sum_x \phi(\vec{x},a)\phi(\vec{x},0)$$

Then one expand e^{-S_c} to arrive at the expression

$$\langle(\Theta_e F)F\rangle = \sum_{n=0}^{\infty} \kappa^n \sum_i c_{ni} \left| \prod_{t>0} \int d\phi(x)\, S_c^i F e^{-S_+'} \right|^2 \geq 0 \text{ if } \kappa > 0$$

$\Rightarrow$ for a "hopping parameter" $\kappa > 0$ $\hat{T}$ is positive.

<u>Free scalar field</u>: Action can be written as matrix $S = \frac{1}{2}\sum_{x,y} S_{xy}\varphi_x\varphi_y$

$$S_{xy} = -\sum_z \left[\sum_\mu (\bar{\delta}_{z+a\hat{\mu},x} - \bar{\delta}_{z,x})(\bar{\delta}_{z+a\hat{\mu},y} - \bar{\delta}_{z,y}) + m^2 \bar{\delta}_{z,x}\bar{\delta}_{z,y} \right] \quad \bar{\delta}_{n,m} = 0 \;\; n \neq m \bmod N, \;\; = 1 \;\; n = m \bmod N$$

Similar to continuum theory define a generating functional $Z[J] = \int D\varphi \exp[S + \sum_x J_x\varphi_x]$ and introduce the free propagator G_{xy} as inverse of S: $S_{xy}G_{yz} = -\bar{\delta}_{x,z}$. With the field redefinition $\varphi_x \to \varphi_x + \sum_y G_{xy}J_y$ we have $Z[J] = Z[0]\exp[\frac{1}{2}G_{xy}J_xJ_y]$ with

$$Z[0] = \int D\varphi \exp(-\tfrac{1}{2}G^{-1}_{xy}\varphi_x\varphi_y) = \frac{1}{\sqrt{\det G}} = \exp(\tfrac{1}{2}\log\det G)$$

Let us solve for G in Fourier space (setting $a=1$)

$$S_{p,-q} = \sum_{x,y} e^{-ipx+iqy} S_{x,y} = -\sum_z \left[\sum_\mu (e^{-ip\hat{\mu}}-1)(e^{iq\hat{\mu}}-1) + m^2\right] e^{ipz+iqz}$$

$$= S_p \bar{\delta}_{p,q} \quad \text{with } -S_p = m^2 + \sum_\mu (2 - 2\cos(p_\mu)) \qquad p_\mu = n_\mu \frac{2\pi}{Na}$$

Since S is diagonal in Fourier space, can directly invert it:

$$G_{p,-q} = G_p \bar{\delta}_{p,q} \qquad G_p = \frac{1}{m^2 + \sum_\mu (2-2\cos(p_\mu))} = \frac{1}{m^2 + a^{-2}\sum_\mu (2 - 2\cos(p_\mu a))}$$

to reintroduce explicit a dependence: $p \to ap$ $\;m \to am$ $\;G_p \to a^2 G(p)$

In the free theory continuum limit gives correct expression

$$G(p) \overset{a\to 0}{=} \frac{1}{m^2 + p^2 + O(a^2)}.$$

Correlation functions $\langle\varphi_x\rangle = \frac{\partial \log Z}{\partial J_x}\Big|_{J=0} = 0$

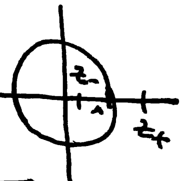
$$\langle\varphi_x\varphi_y\rangle = \frac{\partial^2 \log Z}{\partial J_x \partial J_y}\Big|_{J=0} = G_{x,y}$$

Let us compute the coordinate space expression @ $T=0$

$$G(x-y) = G_{xy} = \frac{1}{N^3}\int_{-\pi}^{\pi}\frac{dp_4}{(2\pi)}\sum_p e^{ip(x-y)}\,G(p) = \frac{1}{V}\sum_p e^{i\vec{p}(\vec{x}-\vec{y})}\underbrace{\int_{-\pi}^{\pi}\frac{dp_4}{2\pi}\,\frac{e^{ip_4\tau}}{2b-2\cos p_4}}_{I}$$

$b = \lambda + \frac{1}{2}\left(m^2 + \sum_{j=1}^{3} 4\sin^2 p_j/2\right)$. To solve I introduce $z = e^{ip_4}$

$$I = -\int_{\mathcal{C}}\frac{dz}{2\pi i}\,\frac{z^\tau}{z^2-2bz+1}$$ with integration contour on unit circle



The relevant pole for the residue theorem is z_-. $z_\pm = b \pm \sqrt{b^2-\lambda}$

Conventionally one writes $z_- = e^{-\omega}$ $\cosh\omega = b$ $\omega = \log(b+\sqrt{b^2-1})$

$$\Rightarrow I = \frac{e^{-\omega\tau}}{2\sinh\omega} \Rightarrow G(\vec{x},\tau) = \frac{1}{V}\sum_p \frac{e^{i\vec{p}\vec{x}-\omega\tau}}{2\sinh\omega} \overset{a\to 0}{=} \frac{1}{V}\sum_p \frac{e^{i\vec{p}\vec{x}-\omega\tau}}{m^2+p^2+\mathcal{O}(a^2)}$$

Again we have that $G(x,\tau)$ is a collection of exponentially damped terms and smallest ω survives for late τ. $G(\tau\to\infty) \approx e^{-m\tau}$

Defining the correlation length via the decay of G we have $\xi = 1/m$.

In practice $\phi(\vec{p},t) = \frac{1}{N^{3/2}}\sum_x e^{-i\vec{p}\vec{x}}\phi(\vec{x},t) \rightarrow \langle\phi(\vec{p},t)\phi(-\vec{p},t')\rangle = G(\vec{p},t-t')$

Lattice formulation provided explicit regularization of the QFT, all manipulations above were carried out on well defined expressions. But the No-Free-Lunch theorem still applies: We need to recover the continuum limit in a meaningful way in an interacting theory.

<u>Towards the continuum limit in interacting scalar theory</u>

Question: If we take $a\to 0$, does the limit exist and do we recover an interacting field theory there?

To make sense of this question let us introduce the concept of "renormalized quantities", which are supposed to be physical and posess a well defined continuum limit and "bare quantities" that can take any value (or even diverge) as $a\to 0$.

Renormalization group picture. Bare parameters are those in the classical Lagrangian, they define the poles in the "tree-level" propagator etc. Quantum fluctuations introduce deviations from these values and the pole of the propagator should approach the physical value for $a\to 0$

In our case: bare quantities $\phi = \phi_0 \quad m^2 = m_0^2 \quad \lambda = \lambda_0$

Define the renormalized quantities from interacting correlation functions.
$$\sum_x a^4 \langle \phi_0(x) \phi_0(0) \rangle e^{-ipx} \equiv \frac{Z_R}{m_R^2 + p^2 + \mathcal{O}(p^4)}$$

using $\chi_2 = \sum_x a^4 \langle \phi_0(x) \phi_0(0) \rangle$ and $\mu_2 = 2 \sum_x a^4 x^2 \langle \phi_0(x) \phi_0(0) \rangle$

we have $m_R^2 = \frac{\chi_2}{\mu_2} \quad Z_R = \frac{\chi_2^2}{\mu_2}$ (field renormalization)

For the coupling one uses four-point functions
$$\chi_4 = a^4 \sum_x a^4 \sum_y a^4 \sum_z \langle \phi_0(x) \phi_0(y) \phi_0(z) \phi(0) \rangle \quad \lambda_R = -\frac{1}{6} \frac{\Lambda}{\mu_2^2} \left(\chi_4 - 3V \chi_2^2 \right)$$

Take the following point of view: If we keep the renormalized quantities fixed at their physical values (e.g. Higgs mass) m_R, λ_R does there exist $a^2 m_0^2(a)$ and $\lambda_0(a)$ such that

a) $a m_R \to 0$ ($\equiv$ continuum limit) b) $\lambda_R \neq 0$ (non-trivial theory)

Note that a) entails that the correlation length of the theory has to diverge. If the theory becomes trivial then $\lambda_R \to 0$ in the continuum limit and one recovers only the free system.

<u>Consider the following:</u>

① $a m_R \to 0 \quad \leftrightarrow \quad \frac{\xi}{a} \sim \frac{1}{m_R a} \to \infty$

To reach the continuum limit, the lattice theory needs to exhibit a continuous second order phase transition. Requires the study of its phase diagram in terms of κ, λ.

② Universality

The properties of 2nd order phase transitions are solely determined by the field content, the symmetries and the dimensionality of the system. The diverging ξ becomes the only relevant scale and most microscopic details do not matter anymore. Many different systems belong to the same universality class (e.g. Ising and $\phi \in \mathbb{R}^N$).