Renormalization of the Fermi Surface

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Abstract

We review the role that renormalization plays in generating well–defined perturbation expansions for fermionic many–body models, particularly in the absence of rotation invariance.

1 The Model

I would like to discuss what is involved in producing well–defined perturbation expansions for fermionic many–body models, without first spending a long time explaining what a fermionic many–body model is. So, I would like you to pretend that you are interested in a class of models characterized by a function $\mathcal{A}(\psi, \bar\psi)$ (called the action) of two vectors, $\psi = (\psi_{k,\sigma})_{k\in M,\sigma\in\{\uparrow,\downarrow\}}$ and $\bar\psi = (\bar\psi_{k,\sigma})_{k\in M,\sigma\in\{\uparrow,\downarrow\}}$. Note that $\bar\psi$ is **NOT** the complex conjugate of ψ. It is just another vector that is totally independent of ψ. Usually, the components of vectors are labelled $1,\ 2,\ 3,\ \cdots$. However, the vectors ψ and $\bar\psi$ have their components labelled (k, σ) with σ only taking two possible values, $\uparrow$ and $\downarrow$ (think "spin up", "spin down") and k running over some as yet unspecified set M. I would like M to be (an open subset of) $\mathbb{R}^{d+1}$, with the zero component k_0 of k being interpreted as

*Research supported in part by the Natural Sciences and Engineering Research Council of Canada and by the Forschungsinstitut für Mathematik, ETH Zürich.

an energy and the final d components $\mathbf{k}$ being interpreted as (crystal) momenta. But I know that quite a few people are squeamish about this, so instead I will take M to be some finite subset of $\mathbb{R}^{d+1}$, with the understanding that I will eventually take an infinite volume limit in which M tends to (an open subset of) $\mathbb{R}^{d+1}$.

In our models, the quantities you measure are represented by other functions $f(\psi,\bar\psi)$ of the same two vectors and the value of the observable $f(\psi,\bar\psi)$ in the model with action $\mathcal{A}(\psi,\bar\psi)$ is given by the ratio of integrals

$$\left\langle f(\psi,\bar\psi)\right\rangle_{\mathcal{A}} = \frac{\int f(\psi,\bar\psi)\, e^{\mathcal{A}(\psi,\bar\psi)}\, \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}}{\int e^{\mathcal{A}(\psi,\bar\psi)}\, \prod_{k,\sigma} d\psi_{k,\sigma}\, d\bar\psi_{k,\sigma}} \tag{1}$$

The integrals are fermionic functional integrals. That is, linear functionals on a Grassmann algebra. But, if you are not already comfortable with fermionic functional integrals, pretend that they are ordinary integrals. It is common to choose the observable $f(\psi,\bar\psi)$ to be a monomial

$$f(\psi,\bar\psi) = \prod_{i=1}^{N} \psi_{p_i,\sigma_i}\bar\psi_{q_i,\tau_i}$$

The corresponding expected value $\left\langle f(\psi,\bar\psi)\right\rangle_{\mathcal{A}}$ is called the $2N$–point Euclidean Green's function. The p_i and q_i are called external momenta.

We are not interested in arbitrary actions $\mathcal{A}(\psi,\bar\psi)$. A typical action of interest is that corresponding to a gas of electrons, of strictly positive density, interacting through a two–body potential $u(\mathbf{x}-\mathbf{y})$. It is

$$\begin{aligned}\mathcal{A}_{\mu,\lambda} \;=\;& -\sum_{\sigma\in\{\uparrow,\downarrow\}}\int_M \tfrac{d^{d+1}k}{(2\pi)^{d+1}}\,\left(ik_0-(\tfrac{\mathbf{k}^2}{2m}-\mu)\right)\bar\psi_{k,\sigma}\psi_{k,\sigma} \\ & -\tfrac{\lambda}{2}\sum_{\sigma,\tau\in\{\uparrow,\downarrow\}}\int_M \prod_{i=1}^{4}\tfrac{d^{d+1}k_i}{(2\pi)^{d+1}}(2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4)\bar\psi_{k_1,\sigma}\psi_{k_3,\sigma}\hat u(\mathbf{k}_1-\mathbf{k}_3)\bar\psi_{k_2,\tau}\psi_{k_4,\tau}\end{aligned} \tag{2}$$

Here $\frac{\mathbf{k}^2}{2m}$ is the kinetic energy of an electron, μ is the chemical potential, which controls the density of the gas, and $\hat u$ is the Fourier transform of the two–body interaction. When M is a finite set, $\int_M \frac{d^{d+1}k}{(2\pi)^{d+1}}$ should be replaced by a Riemann sum and $\delta(k_1+k_2-k_3-k_4)$ should be replaced by a discrete approximation to the delta function.

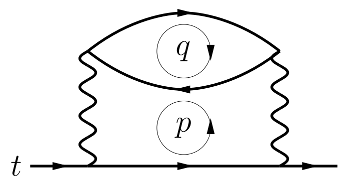
More generally, when the electron gas is subject to a periodic potential due to a crystal lattice and when the electrons are interacting with the motion of the crystal lattice through the mediation of harmonic phonons, the action is of the form

$$\mathcal{A}_\lambda \;= -\sum_\sigma \int_M \tfrac{d^{d+1}k}{(2\pi)^{d+1}}\,\left(ik_0-E(\mathbf{k})\right)\bar\psi_{k,\sigma}\psi_{k,\sigma} - \lambda\mathcal{V} \tag{3}$$

where $E(\mathbf{k})$ is the dispersion relation minus the chemical potential μ. There really should also be a sum over a band index n, but it will not play a role here, so I have suppressed it. The form of the interaction is not very important either, so I will delay writing it out.

2 The Problem

In the infinite volume limit, $\left\langle f(\psi,\bar\psi)\right\rangle_{\mathcal{A}_\lambda}$ is too complicated to evaluate explicitly, except when the coupling constant λ is zero. But you can fairly easily evaluate, in terms of ordinary integrals of the type taught in first and second year calculus, $\left\langle f(\psi,\bar\psi)\right\rangle_{\mathcal{A}_\lambda}$ and all of its derivatives with respect to λ when $\lambda = 0$. There is a standard mnemonic device that uses Feynman diagrams as a compact notation for these ordinary integrals. Here is one of the simplest Feynman diagrams and the integral it represents.



$$= -\lambda^2 \int \tfrac{d^{d+1}p}{(2\pi)^{d+1}} \tfrac{d^{d+1}q}{(2\pi)^{d+1}}\ \hat u(\mathbf{p})\hat u(-\mathbf{p})\ \tfrac{1}{iq_0-E(\mathbf{q})}\ \tfrac{1}{i(p_0+q_0)-E(\mathbf{p}+\mathbf{q})}\ \tfrac{1}{i(p_0+t_0)-E(\mathbf{p}+\mathbf{t})}$$

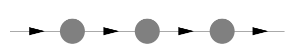
Observe that

- The integrals are pretty complicated.
- The integration variables are momenta.
- Each line of the diagram has an associated momentum, which is a linear combination of integration variables and external momenta (the momenta of the $\psi_{k,\sigma}$'s and $\bar\psi_{k,\sigma}$'s of the observable f) determined by conservation of momentum.
- The integrand contains, for each line of the diagram, a factor (called a propagator) like $\frac{1}{ik_0-E(\mathbf{k})}$ where k is the momentum associated with the line and the denominator $ik_0 - E(\mathbf{k})$ appears in the part of the action $\mathcal{A}_\lambda$ that is quadratic in $\psi_{k,\sigma},\ \bar\psi_{k,\sigma}$.

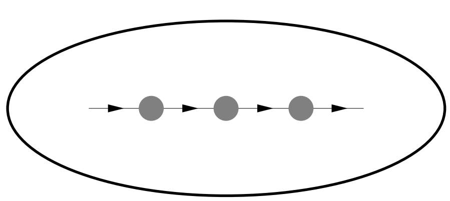
The problem is that the denominator $ik_0 - E(\mathbf{k})$ can vanish, making the integrand singular. In fact these singularities are often sufficiently strong to destroy integrability. Contributions to

$$\tfrac{\partial^n}{\partial\lambda^n} \left\langle f(\psi,\bar\psi)\right\rangle_{\mathcal{A}_\lambda} \Big|_{\lambda=0}$$

corresponding to Feynman diagrams that contain subdiagrams of the form



where each blob represents an arbitrary two–legged subdiagram, are not well–defined. The reason is that, in the integral that such a Feynman diagram represents, conservation of momentum forces the momentum flowing in each of the lines of the string to be the same. Hence all of the factors $\frac{1}{ik_0-E(\mathbf{k})}$ in the integrand that correspond to lines of the string are the same and the integral looks like



$$= \int dk \quad \cdots \quad \frac{1}{[ik_0 - E(\mathbf{k})]^p} \quad \cdots$$

The left hand side represents a generic Feynman diagram that contains a string of p lines. The right hand side gives the value of that diagram, though only the factors in the integrand associated to the lines of the string are given explicitly. Because $E(\mathbf{k})$ vanishes on a $d-1$ dimensional surface, called the Fermi surface, $\frac{1}{[ik_0-E(\mathbf{k})]^p}$ is not locally integrable for any $p \ge 2$. To see this make a change of variables from $(k_0, \mathbf{k})$ to

$$x = k_0 \qquad y = E(\mathbf{k}) \qquad \vec{\phi} = d-1 \text{ angular variables}$$

and then to

$$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\tfrac{y}{x} \qquad \vec{\phi} = d-1 \text{ angular variables}$$

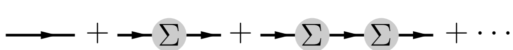
Denoting by $J(x, y, \vec{\phi})$ the Jacobian of the first change of variables

$$\begin{aligned}
\int d^{d+1}k \; \frac{h(k)}{[ik_0 - E(\mathbf{k})]^p} &= \int dx\, dy\, d^{d-1}\vec{\phi}\; J(x,y,\vec{\phi})\; \frac{h}{[ix-y]^p} \\
&= \int dr\, d\theta\, d^{d-1}\vec{\phi}\; r\; \frac{Jh}{[ire^{i\theta}]^p} \\
&= \int dr\; \frac{1}{r^{p-1}} \int d\theta\, d^{d-1}\vec{\phi}\; (-i)^p e^{-ip\theta} Jh \\
&= \infty \qquad \text{for generic } h \text{ if } p \ge 2
\end{aligned}$$

We hasten to emphasize that this divergence does **not** signal that $\left\langle f(\psi,\bar{\psi})\right\rangle_{\mathcal{A}_\lambda}$ is ill–defined. It signals that $\left\langle f(\psi,\bar{\psi})\right\rangle_{\mathcal{A}_\lambda}$ is not C^∞ in λ. Furthermore, we shall shortly see that if we are a bit more careful about the dependence on the parameter λ that we introduce in our models, we can recover smoothness in λ.

3 The Root of the Problem

The lack of integrability that we identified in the last section is associated with strings of two–legged subdiagrams. In fact, it is true, though not obvious, that all of the integrals giving the value of any diagram, that does not contain a nontrivial two–legged subdiagram, converge. On a formal level, we can prevent strings from ever appearing by blocking the sum of all diagrams. First, compute the sum of all strings and call it $\frac{1}{ik_0-E(\mathbf{k})-\Sigma(k)}$:



$$\begin{aligned}
&= \tfrac{1}{ik_0-E(\mathbf{k})} + \tfrac{1}{ik_0-E(\mathbf{k})}\Sigma(k)\tfrac{1}{ik_0-E(\mathbf{k})} + \tfrac{1}{ik_0-E(\mathbf{k})}\Sigma(k)\tfrac{1}{ik_0-E(\mathbf{k})}\Sigma(k)\tfrac{1}{ik_0-E(\mathbf{k})} + \cdots \\
&= \tfrac{1}{ik_0-E(\mathbf{k})-\Sigma(k)}
\end{aligned}$$

where the proper self–energy $\Sigma(k)$ is the sum of all two–legged diagrams that cannot be disconnected by cutting a single line. Such diagrams are called two–legged 1PI diagrams. Define a skeleton diagram to be a diagram that does not contain any nontrivial two–legged subdiagrams. Then, formally (or when the set M of allowed momenta is finite so that all Feynman diagrams have trivially well–defined values),

$$\sum_{\text{all diagrams } G} \text{value of } G\text{, using propagator } \frac{1}{ik_0 - E(\mathbf{k})}$$

$$= \sum_{\substack{\text{skeleton} \\ \text{diagrams } G}} \text{value of } G\text{, using propagator } \frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(k)}$$

This formula is almost obvious, provided that you don't think about it enough to realise that Σ has only been defined .

$$\Sigma = \sum_{\substack{\text{all two–legged} \\ \text{1PI skeleton} \\ \text{diagrams } G}} \text{value of } G\text{, using propagator } \frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(\mathbf{k})} \tag{4}$$

Set aside, for the time being, the question of the solubility of (4) in the infinite volume limit. Assuming that the solution exists and that $\Sigma(k)$ is at all reasonable, the propagator $\frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(k)}$ is locally integrable. But if, in the process of expanding

$$\left\langle f(\psi, \bar{\psi}) \right\rangle_{\mathcal{A}_\lambda} = \sum_{n=0}^{\infty} \tfrac{\lambda^n}{n!} \tfrac{\partial^n}{\partial \lambda^n} \left\langle f(\psi, \bar{\psi}) \right\rangle_{\mathcal{A}_\lambda} \Big|_{\lambda=0}$$

we expand

$$\frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(k)} = \sum_{n=0}^{\infty} \frac{1}{ik_0 - E(\mathbf{k})} \left(\frac{\Sigma(k)}{ik_0 - E(\mathbf{k})} \right)^n \tag{5}$$

no term of the right hand side, except $n = 0$, is locally integrable. The problem arises because we are attempting to expand the interacting propagator $\frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(k)}$, which has a singularity when $k_0 = 0$ and $\mathbf{k}$ is on the interacting Fermi surface

$$F_\lambda = \{ \ \mathbf{k} \mid E(\mathbf{k}) + \Sigma((0, \mathbf{k})) = 0 \ \}$$

in powers of the free propagator $\frac{1}{ik_0 - E(\mathbf{k})}$ which has a singularity when $k_0 = 0$ and $\mathbf{k}$ is on the free Fermi surface

$$F_0 = \{ \ \mathbf{k} \mid E(\mathbf{k}) = 0 \ \}$$

In practice, implementation of the above resummation algorithm is not completely trivial, because it is not easy to verify the solubility of (4). In fact, it is far from obvious that the right hand side of (4) is even once differentiable with respect to Σ, because differentiating $\frac{1}{ik_0 - e(\mathbf{k}) - \Sigma(k)}$ once with respect to Σ produces a string of length two. And you will certainly not be able to solve (4) for $\Sigma(k, \lambda) = \sum_{r=1}^{\infty} \lambda^r \Sigma_r(k)$ as a formal power series in λ, because the right hand side is certainly not C^∞ in Σ. However, there is a procedure that implements, at least the important part of, the above resummation algorithm and that can live mostly in the land of formal power series. The procedure is renormalization.

4 Renormalization

Suppose that we are interested in a model with some prescribed $E(\mathbf{k})$ and λ. Pretend, temporarily, that you know the proper self–energy $\Sigma(k, E, \lambda)$ for this model. Write $E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k})$ where $e(\mathbf{k})$ has the property that its zero set $\{ \mathbf{k} \mid e(\mathbf{k}) = 0 \}$ coincides with the interacting Fermi surface

$$F_\lambda = \{ \mathbf{k} \mid E(\mathbf{k}) + \Sigma((0, \mathbf{k}), E, \lambda) = 0 \}$$

The condition $\{ \mathbf{k} \mid e(\mathbf{k}) = 0 \} = F_\lambda$ does not uniquely determine the decomposition $E = e + \delta e$. It only forces

$$\delta e(\mathbf{k}) + \Sigma((0, \mathbf{k}), E, \lambda) = 0 \text{ for all } \mathbf{k} \in F_\lambda \tag{6}$$

Suppose that we have found a decomposition $E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k})$ satisfying (6). If we expand

$$\begin{aligned} \frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(k)} &= \frac{1}{ik_0 - e(\mathbf{k}) - \delta e(\mathbf{k}) - \Sigma(k)} \\ &= \sum_{n=0}^{\infty} \frac{1}{ik_0 - e(\mathbf{k})} \left(\frac{\delta e(\mathbf{k}) + \Sigma(k)}{ik_0 - e(\mathbf{k})} \right)^n \end{aligned} \tag{7}$$

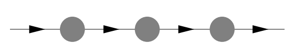
the numerator $\delta e(\mathbf{k}) + \Sigma(k)$ vanishes on F_λ, the zero set of the denominator, and, under reasonable regularity conditions, the ratio $\frac{\delta e(\mathbf{k}) + \Sigma(k)}{ik_0 - e(\mathbf{k})}$ is bounded. Then each term in the expansion is locally integrable. Now rewrite

$$\begin{aligned} \mathcal{A} &= -\sum_\sigma \int \tfrac{d^{d+1}k}{(2\pi)^{d+1}} \, (ik_0 - E(\mathbf{k})) \bar{\psi}_{k,\sigma} \psi_{k,\sigma} - \lambda \mathcal{V} \\ &= -\sum_\sigma \int \tfrac{d^{d+1}k}{(2\pi)^{d+1}} \, (ik_0 - e(\mathbf{k})) \bar{\psi}_{k,\sigma} \psi_{k,\sigma} - \lambda \mathcal{V}' \end{aligned} \tag{8}$$

where

$$\lambda \mathcal{V}' = \lambda \mathcal{V} + \int \tfrac{d^{d+1}k}{(2\pi)^{d+1}} \, \delta e(\mathbf{k}) \bar{\psi}_{k,\sigma} \psi_{k,\sigma}$$

and view $\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \, \delta e(\mathbf{k}) \bar{\psi}_{k,\sigma} \psi_{k,\sigma}$ as part of the interaction, rather than as part of the free action, which determines the propagator. This changes the value of

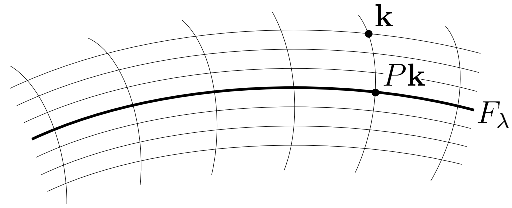


from $\frac{1}{ik_0 - E(\mathbf{k})} \left(\frac{\Sigma(k)}{ik_0 - E(\mathbf{k})} \right)^n$, which is not locally integrable, to $\frac{1}{ik_0 - e(\mathbf{k})} \left(\frac{\delta e(\mathbf{k}) + \Sigma(k)}{ik_0 - e(\mathbf{k})} \right)^n$, which is.

In practice, $\Sigma(k, E, \lambda)$ is not known ahead of time, so we reorder the procedure. First, fix a function $e(\mathbf{k})$ and forget, again temporarily, about $E(\mathbf{k})$. Next, we will define a function $\delta e(\mathbf{k}) = \delta e(\mathbf{k}, e, \lambda)$ which obeys

$$\delta e(\mathbf{k}) + \Sigma((0, \mathbf{k}), e + \delta e, \lambda) = 0 \text{ for all } \mathbf{k} \text{ with } e(\mathbf{k}) = 0$$

This can be done by defining any reasonable projection P onto the interacting Fermi surface $F = \{ \mathbf{k} \mid e(\mathbf{k}) = 0 \}$, as in the following figure,



and setting

$$\begin{aligned}\delta e(\mathbf{k}) &= -\sum \text{all 2–legged 1PI diagrams}\big|_{k_0=0,\ P\mathbf{k}} \\ &= -\Sigma((0,P\mathbf{k}),e+\delta e,\lambda) \end{aligned} \tag{9}$$

Recall that "1PI" means that the diagram cannot be disconnected by cutting one line. Of course (9) is an implicit equation for δe. However, unlike (4), the solubility of (9) in perturbation theory is trivial because Σ is $O(\lambda)$ and because our construction has been designed to provide the needed regularity. So, for a given $e(\mathbf{k})$, it is a relatively simple matter to construct a suitable $\delta e(\mathbf{k},e,\lambda)$ and then define a corresponding $E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k},e,\lambda)$. But, to end up with the prescribed $E(\mathbf{k})$ of the model that we fixed at the beginning of this section, we must still solve

$$E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k},e,\lambda) \tag{10}$$

for

$$e(\mathbf{k}) = e(\mathbf{k},E,\lambda)$$

Because $\delta e(\mathbf{k},e,\lambda)$ does not depend smoothly on its second argument, the solubility of (10) is a somewhat delicate problem. We discuss it in the next section.

5 Results

We have above developed a vague picture in which the expected values $\langle f\rangle$ of observables, when viewed as functions of λ with E held fixed are not smooth. But all derivatives with respect to λ, with e rather than E held fixed, are well-defined. We now quote some more precise statements that firm up this picture. We consistently make hypotheses that are stronger (sometimes substantially stronger) than necessary in order to simplify the statements as much as possible. The cutoff expected values are

$$\left\langle f(\psi,\bar\psi)\right\rangle_\kappa = \frac{\int f(\psi,\bar\psi)e^{\mathcal{A}_{I,\kappa}(\psi,\bar\psi)}\,d\mu_{C,\kappa}(\psi_{k,\sigma},\bar\psi_{k,\sigma})}{\int e^{\mathcal{A}_{I,\kappa}(\psi,\bar\psi)}\,d\mu_{C,\kappa}(\psi_{k,\sigma},\bar\psi_{k,\sigma})}$$

where

- $d\mu_{C,\kappa}$ is the Grassmann Gaussian measure with mean zero and covariance

  $$\delta_{\sigma,\sigma'}\frac{\rho(|k|/\aleph)}{ik_0-e(\mathbf{k})}$$

  This measure combines the measure $\prod_{k,\sigma} d\psi_{k,\sigma}\,d\bar\psi_{k,\sigma}$ of (1) with the "free" part of the action. Precise hypotheses on $e(\mathbf{k})$ will be given later. The role of the function $\rho(|k|/\aleph)$ will be discussed shortly.

- The parameter κ in $d\mu_{C,\kappa}$ specifies the infrared cutoff and will ultimately be sent to zero. The infrared cutoff can be imposed in many different ways without affecting the results that follow. For concreteness, put the model in a periodic spatial box of side $\frac{1}{\kappa}$, so that the spatial components $\mathbf{k}$ of momenta run over $2\pi\kappa\mathbb{Z}^d$, and set the temperature to κ so that k_0 runs over $\pi\kappa(2\mathbb{Z}+1)$. Thus the set of allowed momenta is $M_\kappa = \{\ k \in \pi\kappa(2\mathbb{Z}+1) \times 2\pi\kappa\mathbb{Z}^d \mid \rho(|k|/\aleph) \neq 0\ \}$.

- The interaction part $\mathcal{A}_{I,\kappa}$ of the action is given by

$$\kappa^{d+1} \sum_{\substack{k\in M_\kappa \\ \sigma\in\{\uparrow,\downarrow\}}} \delta e(\mathbf{k},\lambda)\bar\psi_{k,\sigma}\psi_{k,\sigma}$$
$$\qquad + \tfrac{\lambda}{2}\kappa^{3(d+1)} \sum_{\substack{k_i\in M_\kappa \\ \sigma,\tau\in\{\uparrow,\downarrow\}}} \delta_{k_1+k_2,k_3+k_4}\bar\psi_{k_1,\sigma}\psi_{k_3,\sigma}\,\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle\,\bar\psi_{k_2,\tau}\psi_{k_4,\tau}$$

  where the sum of all counterterms $\delta e(\mathbf{k},\lambda)$ will be chosen later.

- The function ρ is a C_0^∞ function that is one in a neighbourhood of zero and $\aleph$ is fixed but arbitrary. Thus $\rho(|k|/\aleph)$ restricts consideration to momenta in a large compact set that contains the momenta $\{\ k \in \mathbb{R}^{d+1} \mid k_0 = 0,\ e(\mathbf{k}) = 0\ \}$ that are important for the problem under consideration. This restriction, which did not appear in (8), can be easily removed under suitable hypotheses on the behaviour of $e(\mathbf{k})$ and $\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle$ near infinity, but I don't want to draw attention away from the important regime by stating these hypotheses.

The following Theorem states that the perturbation expansion coefficients for the Euclidean Green's functions and the proper self–energy exist under very weak hypotheses on $e(\mathbf{k})$ and $\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle$. A natural norm for measuring the size of the $2N$–point functions is

$$|\!|G_N|\!| = \sum_{\sigma_j,\tau_j\in\{\uparrow,\downarrow\}} \int \prod_{j=1}^N dp_j dq_p\ |G_N(p_1,\sigma_1,\cdots,q_N,\tau_N)|$$

When G_N is only defined on a discrete set of momenta, extend it to be piecewise constant on $\mathbb{R}^{2N(d+1)}$ so that integrals are replaced by their Riemann sum approximants.

Theorem 1 ([FKST, Theorem I.2]) *Assume*

H1) $e(\mathbf{k})$ *is* C^1

H2) $\nabla e(\mathbf{k}) \neq 0$ *for all* k *with* $e(\mathbf{k}) = 0$

H3) $\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle$ *is real,* C^1 *and invariant under time reversal. That is,*

$$\langle k_1,k_2|\mathrm{V}|k_3,k_4\rangle = \langle Tk_1,Tk_2|\mathrm{V}|Tk_3,Tk_4\rangle$$

where $T(k_0,\mathbf{k}) = (-k_0,\mathbf{k})$.

There is a formal power series $\delta e(\mathbf{k},\lambda) = \sum_{r=1}^{\infty} \lambda^r \delta e_r(\mathbf{k})$, independent of κ, such that the following holds. Expand the $2N$–point Euclidean Green's functions and the proper self–energy

$$G_{N,\kappa}(\,\cdot\,,\lambda) = \sum_{r=1}^{\infty} \lambda^r G_{N,\kappa,r}(\,\cdot\,)$$
$$\Sigma_{\kappa}(\,\cdot\,,\lambda) = \sum_{r=1}^{\infty} \lambda^r \Sigma_{\kappa,r}(\,\cdot\,)$$

as formal power series in the coupling constant λ. Here each $\cdot$ refers to the appropriate set of spins and momenta. Then the limits $G_{N,0,r} = \lim_{\kappa\to 0} G_{N,\kappa,r}$ and $\Sigma_{0,r} = \lim_{\kappa\to 0} \Sigma_{\kappa,r}$ exist and, for every $0 \le \epsilon < 1$,

$$\sup_{\kappa} \big| G_{N,\kappa,r}(\,\cdot\,) \big| < \infty$$
$$\sup_{\kappa} \kappa^{-\epsilon} \big| G_{N,\kappa,r}(\,\cdot\,) - G_{N,0,r}(\,\cdot\,) \big| < \infty$$
$$\sup_{p,\kappa} |\Sigma_{\kappa,r}(p,\sigma)| < \infty$$
$$\sup_{p,\kappa} \kappa^{-\epsilon} |\Sigma_{\kappa,r}(p,\sigma) - \Sigma_{0,r}(p,\sigma)| < \infty$$

Stronger hypotheses are required for the map from $e(\mathbf{k})$ to $e(\mathbf{k}) + \delta e(\mathbf{k}, e, \lambda)$ to be injective. It does not make sense to ask for a formal power series inverse for that map, because $\delta e(\mathbf{k}, e(\,\cdot\,), \lambda)$ is NOT C^∞ in $e(\,\cdot\,)$. Instead, we truncate the formal power series expansion of δe at an arbitrary power R and treat the result as a true function of λ, rather as a formal power series in λ.

Theorem 2 ([FST1, Theorem I.4]) *Assume*

H1') *$e(\mathbf{k})$ is C^2*

H2) *$\nabla e(\mathbf{k}) \neq 0$ for all k with $e(\mathbf{k}) = 0$*

H3') *$\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle$ is real, C^2 and invariant under time reversal.*

H4) *The Fermi surface has no flat pieces (for a precise definition see [FST1, Assumption A3])*

Then for all $R \in \mathbb{N}$, $\delta e^{(R)}(\mathbf{k}, \lambda, e) = \sum_{r=1}^{R} \lambda^r \delta e_r(\mathbf{k})$ is C^1 in $\mathbf{k}$. Furthermore the Fréchet derivative $D_e \delta e^{(R)}$ of $\delta e^{(R)}$ with respect to e exists and obeys, for all $h \in C^1$,

$$\sup_{\mathbf{k}} |\langle D_e \delta e^{(R)}, h\rangle(\mathbf{k})| \le \mathrm{const}|\lambda| \sup_{\mathbf{k}} |h(\mathbf{k})|$$

Thus the derivative of $\delta e^{(R)}(\mathbf{k}, e(\,\cdot\,), \lambda)$ with respect to $e(\,\cdot\,)$ is a bounded linear operator from C^0 to C^0, whose norm, for sufficiently small λ, is strictly smaller than one. The set of e satisfying $(H1',\ H2,\ H4)$ is open, so if e_1 and e_2 are close enough together, the above Theorem applies to all e on the line from e_1 to e_2. Then $e_1 + \delta e^{(R)}(e_1) = e_2 + \delta e^{(R)}(e_2)$ implies that $(\mathbb{1} + \mathbb{L})(e_2 - e_1) = 0$, where $\mathbb{L}h = \int_0^1 dt\ \big\langle D_e \delta e^{(R)}((1-t)e_1 + t e_2), h \big\rangle$. Since $D_e \delta e^{(R)}$ has norm less than one for small λ, we have

Theorem 3 ([FST1, Theorem I.7]) *Let $\mathcal{D}$ be a convex set of dispersion relations e obeying ($H1'$, $H2$, $H3'$, $H4$) with uniform bounds. For each $R \in \mathbb{N}$, there is $\lambda_R > 0$ such that for all $\lambda \in (-\lambda_R, \lambda_R)$, the map $e \mapsto e + \delta e^{(R)}$ is injective on $\mathcal{D}$.*

To get invertibility we have to strengthen the hypotheses still more. Let $|\cdot|_j$ denote the usual norm on C^j.

Theorem 4 *Let $G > 1$ and $\mathcal{E} \subset C^2$ be an open set of dispersion relations e fulfilling*

H1") $|e|_2 < G$ *and* $e(\mathbf{p}) = e(-\mathbf{p})$ *for all* $\mathbf{p}$

H2) $|\nabla e(\mathbf{p})| > \frac{1}{G}$ *for all* $\mathbf{p}$ *within a distance* $\frac{1}{G}$ *of* $\{ \mathbf{k} \mid e(\mathbf{k}) = 0 \}$

H3') $\langle k_1, k_2 | \mathrm{V} | k_3, k_4 \rangle$ *is real,* C^2 *and invariant under time reversal.*

H4") *The Fermi surface* $\{ \mathbf{k} \mid e(\mathbf{k}) = 0 \}$ *has curvature at least* $\frac{1}{G}$ *(for a precise definition, see [FST2, Hypotheses (H3)–(H5)])*

Then

a) *[FST3, Theorem 1.1]* $\delta e^{(R)} \in C^2$ *for all* $e \in \mathcal{E}$.

b) *[FST4] Let $\mathcal{E}' \subset \mathcal{E}$ be the set of all dispersion relations e whose distance from the boundary of $\mathcal{E}$ is at least $1/G$. For each $R \in \mathbb{N}$, there is $\lambda_R > 0$ and a map $\mathcal{R} : (-\lambda_R, \lambda_R) \times \mathcal{E}' \to \mathcal{E}$ such that for all $(\lambda, E) \in (-\lambda_R, \lambda_R) \times \mathcal{E}'$, $e = \mathcal{R}(\lambda, E)$ solves $E = e + \delta e^{(R)}$.*

Set $K(e)(\mathbf{k}) = \delta e(\mathbf{k}, e, \lambda)$. To prove part b) of Theorem 4 using the usual implicit function theorem, one would need to prove an estimate like

$$|K(e_1) - K(e_2)|_2 \le \alpha |e_1 - e_2|_2$$

for some $\alpha < 1$. To prove such an estimate one needs to be able to take three derivatives of K, one with respect to e and two with respect to the external momentum $\mathbf{k}$. We can only control $3 - \epsilon$ derivatives. Fortunately, the following implicit function theorem only requires control of $2 + \epsilon$ derivatives.

Proposition 5 ([FST4]) *Le $N \subset \mathbb{R}^d$ be open, $E \in C^2(N)$ and the strictly positive constants α, β, ϵ, A, C obey*

$$\alpha < 1, \ \beta < \alpha^\epsilon$$

Define

$$\mathcal{B} = \{ \ e \in C^2(N) \mid |e - E|_2 \le A \ \}$$

Assume that the map $K : \mathcal{B} \to C^2(N)$ obeys

a) $|K(e)|_2 \le A$

b) $|K(e_1) - K(e_2)|_1 \le \alpha |e_1 - e_2|_1$

c) $|K(e_1) - K(e_2)|_2 \leq C|e_1 - e_2|_1^\epsilon + \beta|e_1 - e_2|_2$

for all $e,\ e_1,\ e_2 \in \mathcal{B}$*. Then there is a unique* $e \in \mathcal{B}$ *such that* $E = e + K(e)$*.*

Proof: Define $\phi(e) = E - K(e)$. By hypothesis a) ϕ is defined on $\mathcal{B}$ and has range contained in $\mathcal{B}$. Set $e_0 = E$ and define the sequence

$$e_n = \phi(e_{n-1})$$

in $\mathcal{B}$. Denote $\delta_n = e_n - e_{n-1}$. By hypotheses a) and b), for every $n \geq 1$

$$\begin{aligned} |\delta_n|_1 &= |e_n - e_{n-1}|_1 = |\phi(e_{n-1}) - \phi(e_{n-2})|_1 \\ &\leq \alpha|\delta_{n-1}|_1 \leq \alpha^{n-1}|\delta_1|_1 = \alpha^{n-1}|\phi(e_0) - e_0|_1 \\ &= \alpha^{n-1}|E - K(E) - E|_1 \leq\ A\alpha^{n-1} \end{aligned}$$

Hence, by hypothesis c), for every $n \geq 2$

$$\begin{aligned} |\delta_n|_2 &= |e_n - e_{n-1}|_2 = |\phi(e_{n-1}) - \phi(e_{n-2})|_2 \\ &\leq C|\delta_{n-1}|_1^\epsilon + \beta|\delta_{n-1}|_2 \\ &\leq C\left(A\alpha^{-2}\right)^\epsilon \alpha^{\epsilon n} + \beta|\delta_{n-1}|_2 \end{aligned}$$

We make the inductive hypothesis that there is a constant D such that $|\delta_n|_2 \leq D\alpha^{\epsilon n}$. This is satisfied for $n = 1$ if D is chosen so that

$$A \leq D\alpha^\epsilon$$

and is preserved under induction if

$$C\left(A\alpha^{-2}\right)^\epsilon \alpha^{\epsilon n} + \beta D\alpha^{\epsilon(n-1)} \leq D\alpha^{\epsilon n}$$

Hence we need

$$D \geq A\alpha^{-\epsilon}$$

and

$$D \geq C\left(A\alpha^{-2}\right)^\epsilon + \beta D\alpha^{-\epsilon}$$

Such a D exists since $\beta\alpha^{-\epsilon} < 1$.

Consequently, since $\alpha < 1$, the series $\sum \delta_n$ converges in C^2. The sum, $e = \lim_{n\to\infty} e_n$ is a fixed point of ϕ and hence obeys

$$e = E - K(e)$$

Uniqueness of the solution follows immediately from hypothesis a) since $\alpha < 1$. □

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