Max-Planck-Institut für Festkörperforschung

Fermionic superfluids I

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Max-Planck-Institute for Solid State Research, Stuttgart

Discussions: Metzner Group, FOR 723, Heidelberg, Jena

LadenbuRG, November 20st, 2008

Predicament

- *From Merriam-Webster Online:*

Main Entry:
pre·dic·a·ment
Pronunciation:
\pri-□di-kə-mənt, *1 is usually* □pre-di-kə-\
Function:
noun
Etymology:
Middle English, from Late Latin *praedicamentum,* from *praedicare*
Date:
14th century
2**:** condition , state ; *especially* **: a difficult, perplexing, or trying situation**

Key topics

RG approach to fermionic superfluids (60 min.)

- Fermionic approach with initial symmetry-breaking field (10 min.)
- Hubbard-Stratonovich approach with bosonic field (50 min.)

Overview & Outlook (15 min.)

- Superfluid Kosterlitz-Thouless phase
- Goldstone renormalization of fermion self-energy
- BCS-Bose crossover

Normal-to-superfluid QCP (45 min.)

- Preliminary results for non-Fermi liquid behavior at QCP
- Coupled RG for fermions and order-parameter fluctuations at quantum criticality

Agenda Fermionic superfluids I (today)

1. History of Fermionic Superfluids / Flows to strong coupling
2. Attractive Hubbard Model
3. Fermionic functional RG approach
4. Hubbard-Strato-novich approach
5. Truncation and cutoff choice
6. Analytical and numerical results
7. Summary Fermionic Super-fluids I

Agenda Fermionic superfluids II (tomorrow)

- Recap Fermionic Superfluids I
- Outlook for (truly) functional RG studies
- Two experi-ments
- Non-Fermi liquid behavior at QCP‘s
- Coupled Fermi-Bose RG at normal-to-superfluid QCP
- Summary Fermionic Super-fluids I & II

Agenda Fermionic superfluids I

- **History of Fermionic Superfluids/ Flows to strong coupling**
- **Attractive Hubbard Model**
- **Fermionic functional RG approach**
- **Hubbard-Strato-novich approach**
- **Truncation and cutoff choice**
- **Analytical and numerical results**
- **Summary Fermionic Super-fluids I**

(partial) History of Fermionic Superfluids

Theory of Superconductivity*

J. Bardeen, L. N. Cooper,† and J. R. Schrieffer‡
Department of Physics, University of Illinois, Urbana, Illinois
(Received July 8, 1957)

Volume 71, Number 19 PHYSICAL REVIEW LETTERS 8 November 1993

Crossover from BCS to Bose Superconductivity: Transition Temperature and Time-Dependent Ginzburg-Landau Theory

C. A. R. Sá de Melo,[1] Mohit Randeria,[2] and Jan R. Engelbrecht[3]

- 1957, BCS Theory
- 1972, Kosterlitz-Thouless state in 2D
- 1993 Randeria, functional integral
- Since 2005 fRG efforts underway (Manchester, Heidelberg, Würzburg, Stuttgart)

Development of understanding for fermionic superfluids

- 1969 Eagles, 1980 Leggett BCS-BEC Crossover
- 1986 Nozieres, Schmitt-Rink, Crossover for lattice fermions/finite T
- 1997/2004, Castellani, di Castro: IR properties of Bose gas

Infrared Behavior of Interacting Bosons at Zero Temperature

C. Castellani,[1] C. Di Castro,[1] F. Pistolesi,[2,3] and G. C. Strinati[3]
[1]*Dipartimento di Fisica, Università "La Sapienza," Sezione INFM, I-00185 Roma, Italy*
[2]*Scuola Normale Superiore, Sezione INFM, I-56126 Pisa, Italy*
[3]*Dipartimento di Matematica e Fisica, Università di Camerino, Sezione INFM, I-62032 Camerino, Italy*
(Received 22 March 1996)

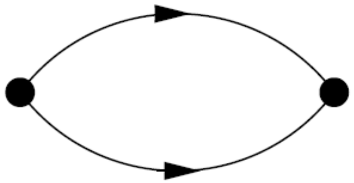
We exploit the symmetries associated with the stability of the superfluid phase to solve the long-standing problem of interacting bosons in the presence of a condensate at zero temperature.

Ordering, metastability and phase transitions in two-dimensional systems

J M Kosterlitz and D J Thouless
Department of Mathematical Physics, University of Birmingham, Birmingham B15 2TT, UK

Cooper instability

Infrared divergence in particle-particle bubble:



For vanishing total momentum (Cooper channel)
at $T = 0$

$$\text{pp-bubble} \propto \int dk_0 \int d^d k \, \frac{1}{ik_0 - \xi_{\mathbf{k}}} \, \frac{1}{-ik_0 - \xi_{-\mathbf{k}}} \overset{\xi_{-\mathbf{k}} = \xi_{\mathbf{k}}}{=}$$

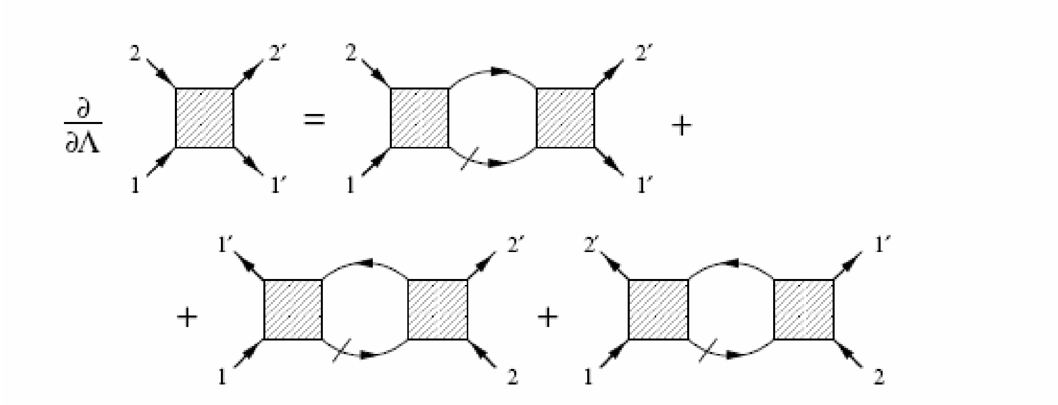
$$\int dk_0 \int d^d k \, \frac{1}{k_0^2 + \xi_{\mathbf{k}}^2} = \int dk_0 \int d\xi \, \frac{N(\xi)}{k_0^2 + \xi^2}$$

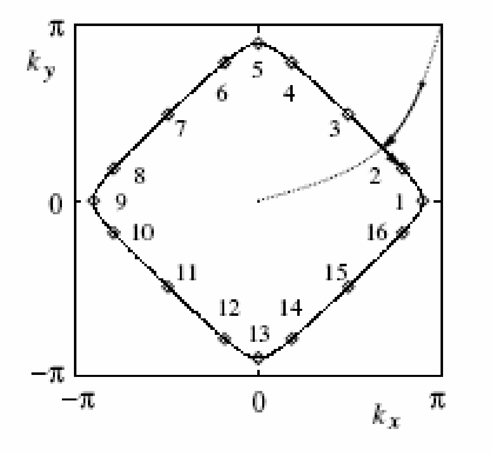
logarithmically divergent in *any* dimension if $N(0) \neq 0$

Note: Propagator divergent on $(d-1)$-dimensional manifold,
embedded in $(d+1)$-dimensional space (spanned by k_0 and $\mathbf{k}$)

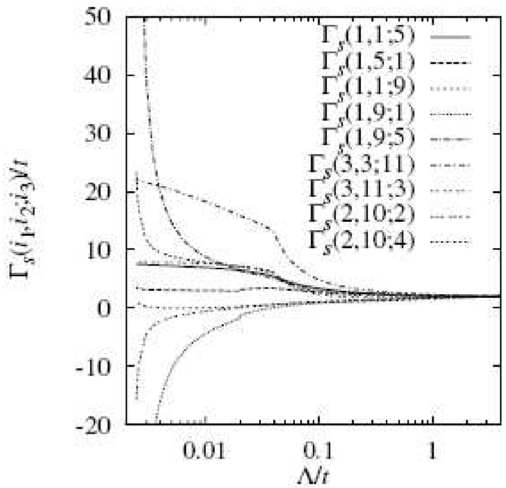
Flows to strong coupling in fermionic functional RG

- Flow of four-fermion vertex in repulsive Hubbard model,
Distinguishing particle-particle and particle-hole contributions:





- For certain parameters interaction in Cooper channel dominates:



- Hints at superconducting ground state.
- Weak-to-intermediate coupling truncation to second order in the interaction breaks down at critical scale.
- Penetration of symmetry-broken phase not possible.

→ Investigate superfluid phase with prototype model.

Agenda Fermionic superfluids I

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- **Attractive Hubbard Model**
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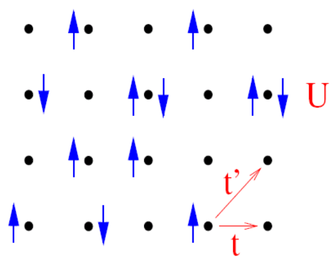
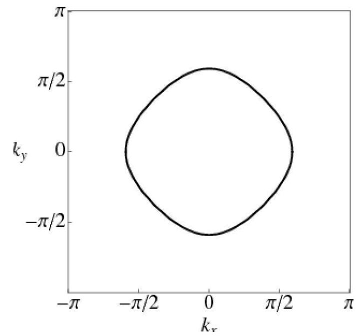
Attractive Hubbard Model as Prototype

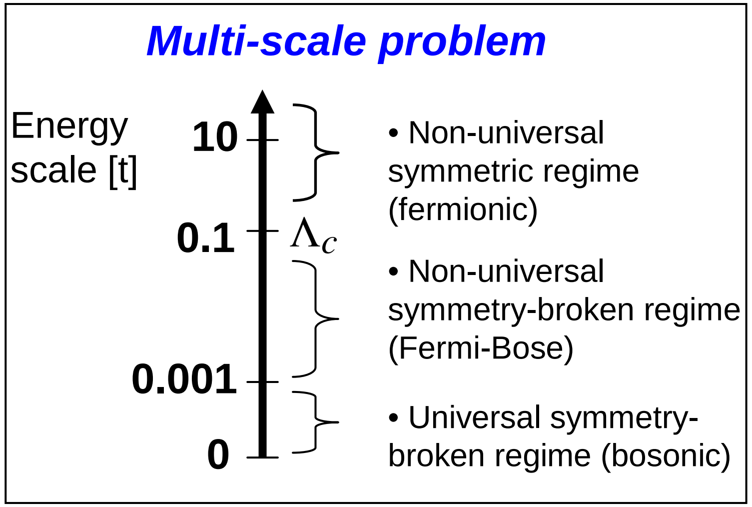
- Attractively interacting lattice fermions
- Superfluid ground state for average fermion density per lattice site 0< n < 2
- Consider quarter-filling (away from van-Hove filling)





$$\Gamma_0[\psi,\bar\psi] = -\int_{k\sigma} \bar\psi_{k\sigma}(ik_0 - \xi_{\mathbf{k}})\,\psi_{k\sigma} + \int_{k,k',q} U\,\bar\psi_{-k+\frac{q}{2}\downarrow}\bar\psi_{k+\frac{q}{2}\uparrow}\psi_{k'+\frac{q}{2}\uparrow}\psi_{-k'+\frac{q}{2}\downarrow}$$

$$\xi_{\mathbf{k}} = -2t\left(\cos k_x + \cos k_y\right) - \mu$$



- Experimentally realized in 3d optical lattice.

Objective: Connect all energy regimes with one method and understand RG in superfluid phase

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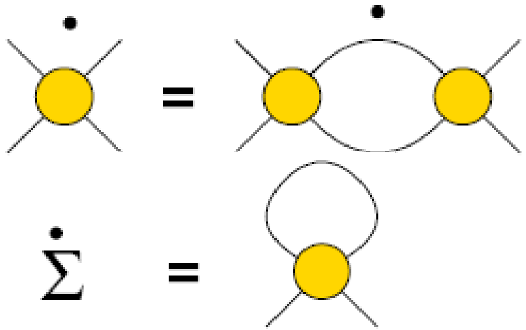
Fermionic functional RG approach

Progress of Theoretical Physics, Vol. 112, No. 6, December 2004

Renormalization Group Flows into Phases with Broken Symmetry

Manfred SALMHOFER,[1] Carsten HONERKAMP,[2] Walter METZNER[2]
and Oliver LAUSCHER[1]

- **Main idea:** small, initial symmetry-breaking field is offered to the flow.
- No decoupling of the interaction necessary.
- Flow captures all scattering channels.
- Coupled self-energy/vertex flow equations:



New Journal of Physics
The open-access journal for physics

Superconductivity in the attractive Hubbard model: functional renormalization group analysis

R Gersch[1,3], C Honerkamp[2] and W Metzner[1]
[1] Max-Planck-Institut für Festkörperforschung, Heisenbergstraße 1, 70569 Stuttgart, Germany
[2] Institut für Theoretische Physik, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany
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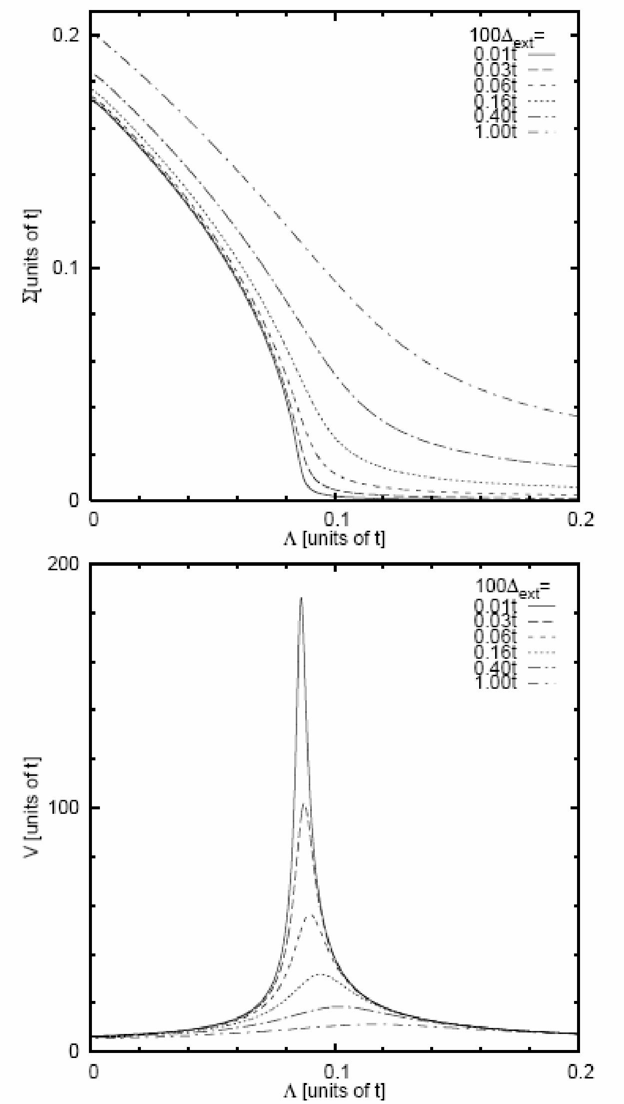
doi:10.1088/1367-2630/10/4/045003

$$\Gamma_0[\psi,\bar\psi] = - \int_{k\sigma} \bar\psi_{k\sigma}(ik_0-\xi_{\mathbf{k}})\,\psi_{k\sigma} + \int_{k\sigma}\left(\Delta_{ext}\bar\psi_{-k,\downarrow}\psi_{k,\uparrow} + \Delta^*_{ext}\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}\right) + \int_{k,k',q} U\,\bar\psi_{-k+\frac{q}{2}\downarrow}\bar\psi_{k+\frac{q}{2}\uparrow}\psi_{k'+\frac{q}{2}\uparrow}\psi_{-k'+\frac{q}{2}\downarrow}$$

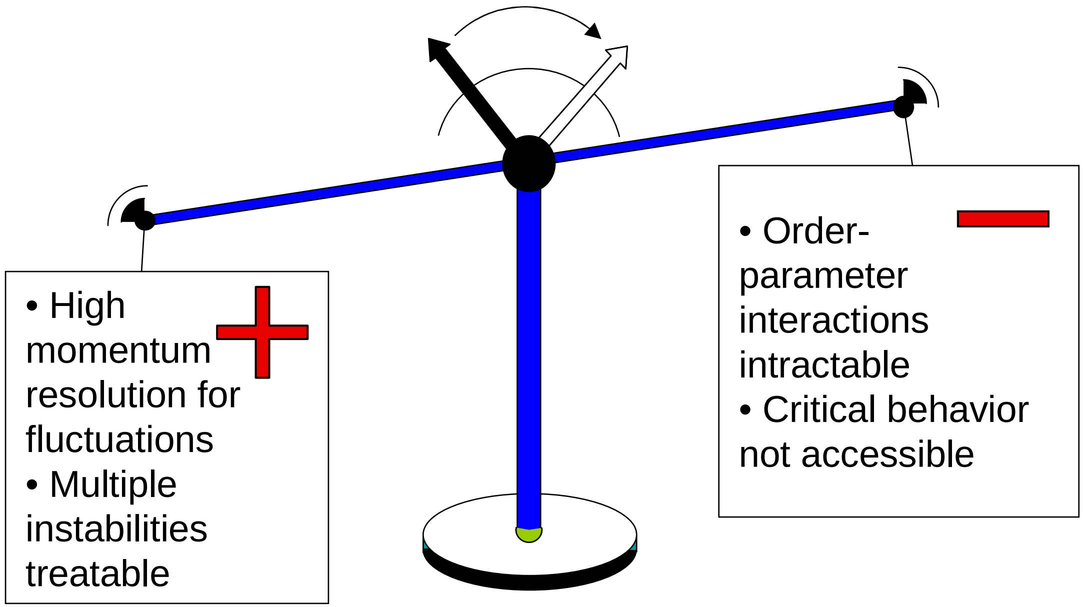
→ Flow of self-energy regularizes vertex divergence at critical scale and permits penetration of symmetry-broken phase.

Fermionic functional RG approach



- One has to augment fermionic hierarchy with terms non-local in the cutoff to fulfill Ward identities yields exact BCS mean-field solution (so what?!), but….

- **First you walk and then you run! (J. Pawlowski)**



Extend to **multiple initial symmetry-breaking fields** like AFM and d-wave SC in repulsive Hubbard model.

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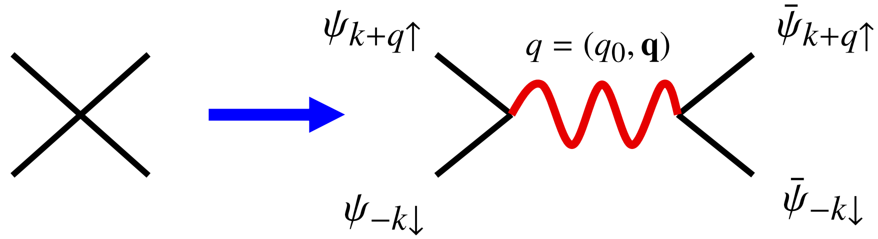
Hubbard-Stratonovich approach

- Replacing the four-fermion interaction with Hubbard-Stratonovich field in the singlet BCS pairing channel.
- Fermion-fermion interaction is mediated by collective Cooper-pairing field – all other channels neglected and hard to recover.

➡ Treating momentum-dependent (non-local) fermion-vertex is akin to propagating boson.

- Functional integral now over bosonic and fermionic fields:
- Reformulate functional integral as functional flow equation for superfields:

$$\mathcal{S}_b = \Phi\,,\quad \mathcal{S}_f = \Psi \qquad \Psi_k = \begin{pmatrix} \psi_{k\uparrow} \\ \bar\psi_{-k\downarrow} \end{pmatrix},\quad \bar\Psi_k = \left(\bar\psi_{k\uparrow}, \psi_{-k\downarrow}\right) \qquad \Phi_q = \begin{pmatrix} \phi_q \\ \phi^*_{-q} \end{pmatrix},\quad \bar\Phi_q = \left(\phi^*_q, \phi_{-q}\right)$$



„The stronger the interaction the lighter the boson."

$$\Gamma_0[\psi,\bar\psi,\phi] = -\int_{k\sigma} \bar\psi_{k\sigma}(ik_0 - \xi_{\mathbf{k}})\,\psi_{k\sigma} - \int_q \phi^*_q \frac{1}{U}\phi_q + \int_{k,q}\left(\bar\psi_{-k+\frac{q}{2}\downarrow}\bar\psi_{k+\frac{q}{2}\uparrow}\,\phi_q + \psi_{k+\frac{q}{2}\uparrow}\psi_{-k+\frac{q}{2}\downarrow}\,\phi^*_q\right)$$

$$\mathcal{Z} = \int \mathcal{D}\bar\psi\,\mathcal{D}\psi\,\mathcal{D}\phi\; e^{-\Gamma_0[\bar\psi,\psi,\phi]}$$

$$\frac{d}{d\Lambda}\Gamma^\Lambda[\mathcal{S},\bar{\mathcal{S}}] = \mathrm{Str}\,\frac{\dot{\mathbf{R}}^\Lambda}{\mathbf{\Gamma}^{(2)\,\Lambda}[\mathcal{S},\bar{\mathcal{S}}] + \mathbf{R}^\Lambda}$$

$$\mathcal{R}^\Lambda = \frac{1}{2}\int_q \bar\Phi_q\,\mathbf{R}^\Lambda_b(q)\,\Phi_q + \int_k \bar\Psi_k\,\mathbf{R}^\Lambda_f(k)\,\Psi_k$$

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Frequency Cutoff

- Implement frequency cutoff for both: fermions and bosons.
- **„Diagonal"** for both particle species; point singularity at origin of frequency axis.
- Equations free of regulator:

$$\mathcal{R}^\Lambda = \frac{1}{2}\int_q \bar{\Phi}_q\, \mathbf{R}_b^\Lambda(q)\,\Phi_q + \int_k \bar{\Psi}_k\, \mathbf{R}_f^\Lambda(k)\,\Psi_k$$

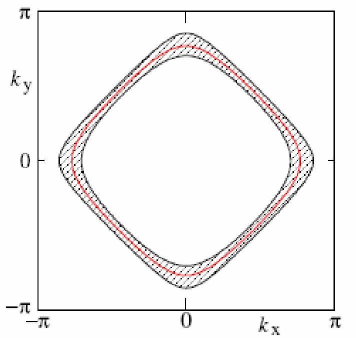
$$\mathbf{R}_s^\Lambda(k) = [\mathbf{G}_{s0}(k)]^{-1} - [\chi_s^\Lambda(k_0)\,\mathbf{G}_{s0}(k)]^{-1}$$

$$\chi_s^\Lambda(k_0) = \Theta(|k_0| - \Lambda_s)$$

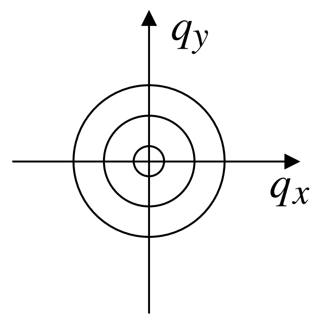
$$n\int dk_0\, \mathbf{G}'_{sR}(k_0)\,\mathbf{A}\,[\mathbf{G}_{sR}(k_0)\,\mathbf{A}]^{n-1} = \Lambda'_s \sum_{k_0=\pm\Lambda_s} [\mathbf{G}_s(k_0)\,\mathbf{A}]^n$$

- Drawback: momentum phase-space not controlled during RG flow.
- Destroys analyticity properties of Green functions in complex frequency plane.

Fermionic momentum shells:



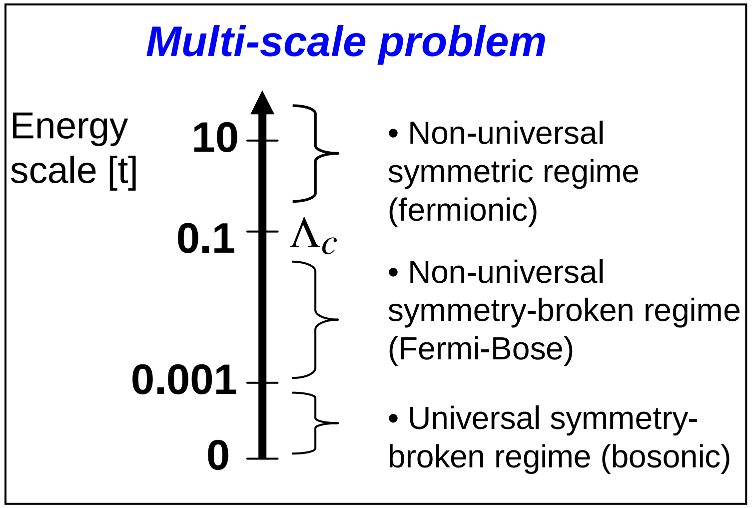
Bosonic momentum shells:



→ Exploit freedom to choose fermionic and bosonic cutoff independently.

Frequency Cutoff

• Judicious choice of relative cutoff scales:



$$\Lambda_f(\Lambda) = \Lambda_b(\Lambda) = \Lambda$$

$$\Lambda_f(\Lambda) = \frac{\Lambda^2}{\Lambda_c}$$
$$\Lambda_b(\Lambda) = \Lambda$$

• Consistent treatment of IR sector by equating:

$$\max G_f(\Lambda_f(\Lambda), \mathbf{k}) \sim \max G_b(\Lambda_b(\Lambda), \mathbf{k})$$

• 2d momentum integration over the Brillouin zone to be performed numerically.

• Results robust under cutoff changes.

Truncation in symmetric phase

- Fermi propagator unrenormalized, qualitatively correct in low-energy regime:

$$\Gamma_{\bar\psi\psi} = -\int_{k\sigma} \bar\psi_{k\sigma}(ik_0 - \xi_{\mathbf{k}})\,\psi_{k\sigma}$$

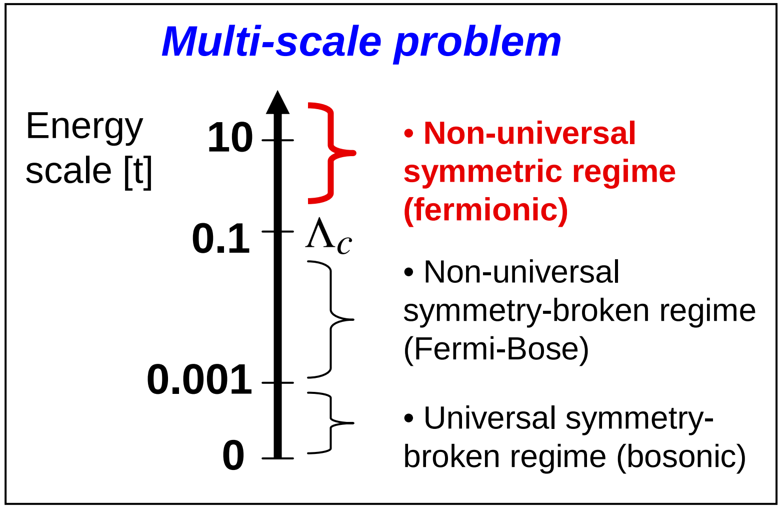
- Bosonic propagator parametrized with three flowing constants:

$$\Gamma_{\phi^*\phi} = \frac{1}{2}\int_q \phi_q^*(m_b^2 + Z_b q_0^2 + A_b \omega_{\mathbf{q}}^2)\,\phi_q$$

$$\omega_{\mathbf{q}}^2 = 2\sum_{i=1}^{d}(1-\cos q_i)$$

- One constant for most relevant bosonic self-interaction:

$$\Gamma_{|\phi|^4} = \frac{\lambda}{8}\int_{q,q',p} \phi^*_{q+p}\phi^*_{q'-p}\phi_{q'}\phi_q$$



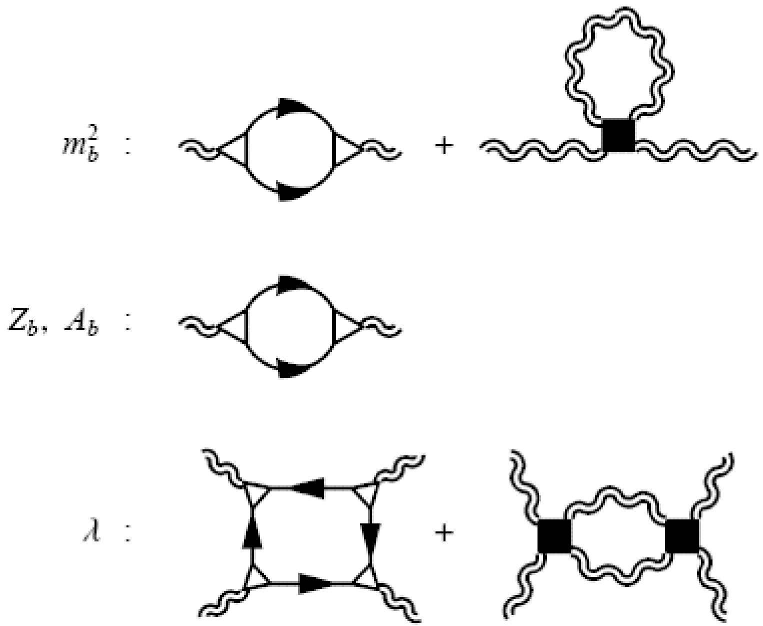
- Fermion-boson vertex not renormalized in symmetric phase:

$$\Gamma_{\psi^2\phi^*} = g\int_{k,q}\left(\bar\psi_{-k+\frac{q}{2}\downarrow}\bar\psi_{k+\frac{q}{2}\uparrow}\,\phi_q + \psi_{k+\frac{q}{2}\uparrow}\psi_{-k+\frac{q}{2}\downarrow}\,\phi_q^*\right)$$

⇒ At critical scale, transition to superfluid phase.

Flow equations in the symmetric phase

Flow equations obtained as cutoff derivatives of 1PI-diagrams:



• Particle-particle bubble at low frequencies, **non-analytic**! (see tomorrow for QCP)

$$\Pi_{pp}(\Omega_n, \mathbf{q}) \sim \ln|\Omega_n|$$

• But at intermediate energies, ***gap opens at critical scale*** and secures quadratic approximation.

• Linear term is very small close to half-filling.

→ Bosonic dynamics and self-interactions are generated by fermionic fluctuations.

Truncation in superfluid phase

- Fermi propagator gets anomalous self-energy:

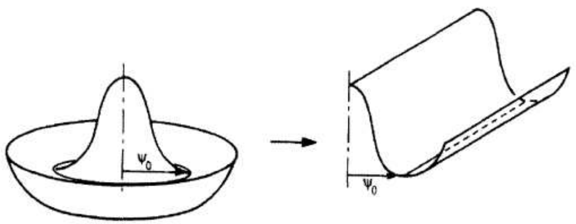
$$\Gamma_{\psi\psi} = \int_k \left(\Delta \bar{\psi}_{-k\downarrow} \bar{\psi}_{k\uparrow} + \Delta^* \psi_{k\uparrow} \psi_{-k\downarrow} \right)$$

- Local bosonic effective O(2)-potential with flowing minimum:

$$U^{\text{loc}}[\phi] = \frac{\lambda}{8} \int \left(|\phi|^2 - |\alpha|^2 \right)^2$$

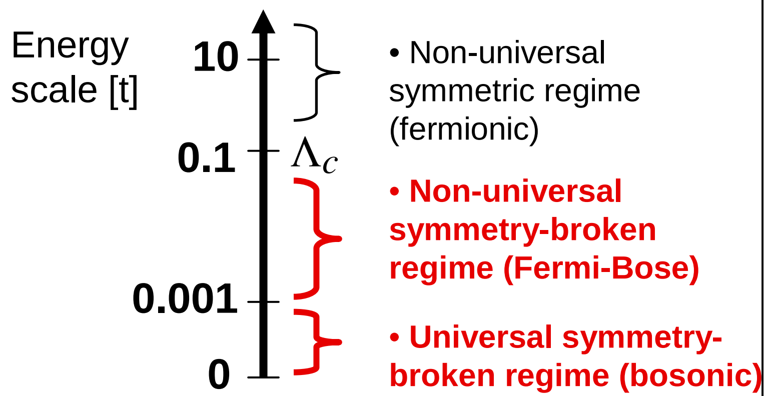
- Note that generally gap Δ need **NOT** be identical to order-parameter α .

- Field is expanded in linear basis:



$$\phi_q = \sigma_q + i\pi_q$$
$$\phi_q^* = \sigma_{-q} - i\pi_{-q}$$

Multi-scale problem



- Two distinct propagators in superfluid phase:

$$\Gamma_{\sigma\sigma} = \frac{1}{2} \int_q \sigma_{-q} (m_\sigma^2 + Z_\sigma q_0^2 + A_\sigma \omega_{\mathbf{q}}^2)\, \sigma_q$$
$$\Gamma_{\pi\pi} = \frac{1}{2} \int_q \pi_{-q} (Z_\pi q_0^2 + A_\pi \omega_{\mathbf{q}}^2)\, \pi_q$$

→ Diverging Z-factors for longitudinal mode capture momentum dependence in infrared.

Truncation in superfluid phase

- Bosonic interaction vertices from expanding the potential:

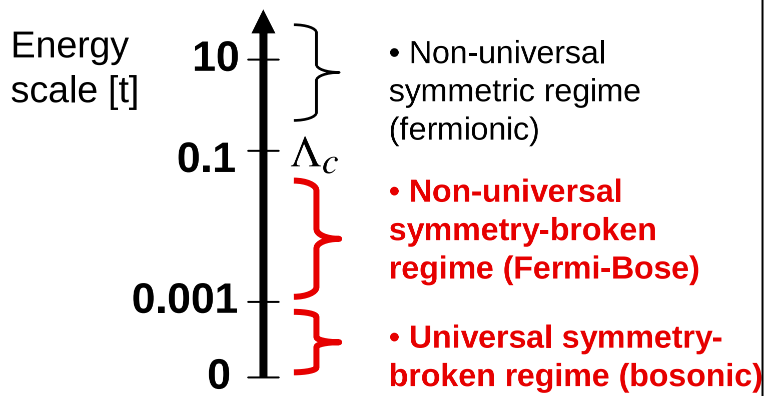
$$\begin{aligned}
\Gamma_{\sigma^4} &= \gamma_{\sigma^4} \int_{q,q',p} \sigma_{-q-p}\sigma_{-q'+p}\sigma_{q'}\sigma_q \\
\Gamma_{\pi^4} &= \gamma_{\pi^4} \int_{q,q',p} \pi_{-q-p}\pi_{-q'+p}\pi_{q'}\pi_q \\
\Gamma_{\sigma^2\pi^2} &= \gamma_{\sigma^2\pi^2} \int_{q,q',p} \sigma_{-q-p}\sigma_{-q'+p}\pi_{q'}\pi_q \\
\Gamma_{\sigma^3} &= \gamma_{\sigma^3} \int_{q,q'} \sigma_{-q-q'}\sigma_{q'}\sigma_q \\
\Gamma_{\sigma\pi^2} &= \gamma_{\sigma\pi^2} \int_{q,q'} \sigma_{-q-q'}\pi_{q'}\pi_q
\end{aligned}$$

with $\gamma_{\sigma^4} = \gamma_{\pi^4} = \lambda/8$, $\gamma_{\sigma^2\pi^2} = \lambda/4$, and $\gamma_{\sigma^3} = \gamma_{\sigma\pi^2} = \lambda\alpha/2$.

- Two different fermion-boson vertices:

$$\begin{aligned}
\Gamma_{\psi^2\sigma} &= g_\sigma \int_{k,q} \left(\bar{\psi}_{-k+\frac{q}{2}\downarrow}\bar{\psi}_{k+\frac{q}{2}\uparrow}\sigma_q + \psi_{k+\frac{q}{2}\uparrow}\psi_{-k+\frac{q}{2}\downarrow}\sigma_{-q} \right), \\
\Gamma_{\psi^2\pi} &= i g_\pi \int_{k,q} \left(\bar{\psi}_{-k+\frac{q}{2}\downarrow}\bar{\psi}_{k+\frac{q}{2}\uparrow}\pi_q - \psi_{k+\frac{q}{2}\uparrow}\psi_{-k+\frac{q}{2}\downarrow}\pi_{-q} \right)
\end{aligned}$$

Multi-scale problem



- Compute flow equations for parameters via hierarchy for vertex expansion of effective action.
- Have already truncated the effective potential extremely smartly such that few parameter truncation might suffice.
- Need external constraint to compute flow of minimum of the potential.

Vertex expansion hierarchy for symmetry-broken phases

- Recall exact flow equation: $$\frac{d}{d\Lambda}\Gamma^{\Lambda}[\phi] = \mathrm{Str}\,\frac{\dot{\mathbf{R}}^{\Lambda}}{\mathbf{\Gamma}^{(2)\,\Lambda}[\phi] + \mathbf{R}^{\Lambda}}$$

- Expand bosonic part effective action in vertices: $$\Gamma^{\Lambda}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!}\gamma^{(n)\,\Lambda}\left(\phi - \alpha^{\Lambda}\right)^{n}$$

- Execute scale-derivative on left-hand-side of flow equation: $$\partial_{\Lambda}\Gamma^{\Lambda}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!}\dot{\gamma}^{(n)\,\Lambda}\left(\phi - \alpha^{\Lambda}\right)^{n} - \frac{\dot{\alpha}^{\Lambda}}{(n-1)!}\gamma^{(n)\,\Lambda}\left(\phi - \alpha^{\Lambda}\right)^{n-1}$$

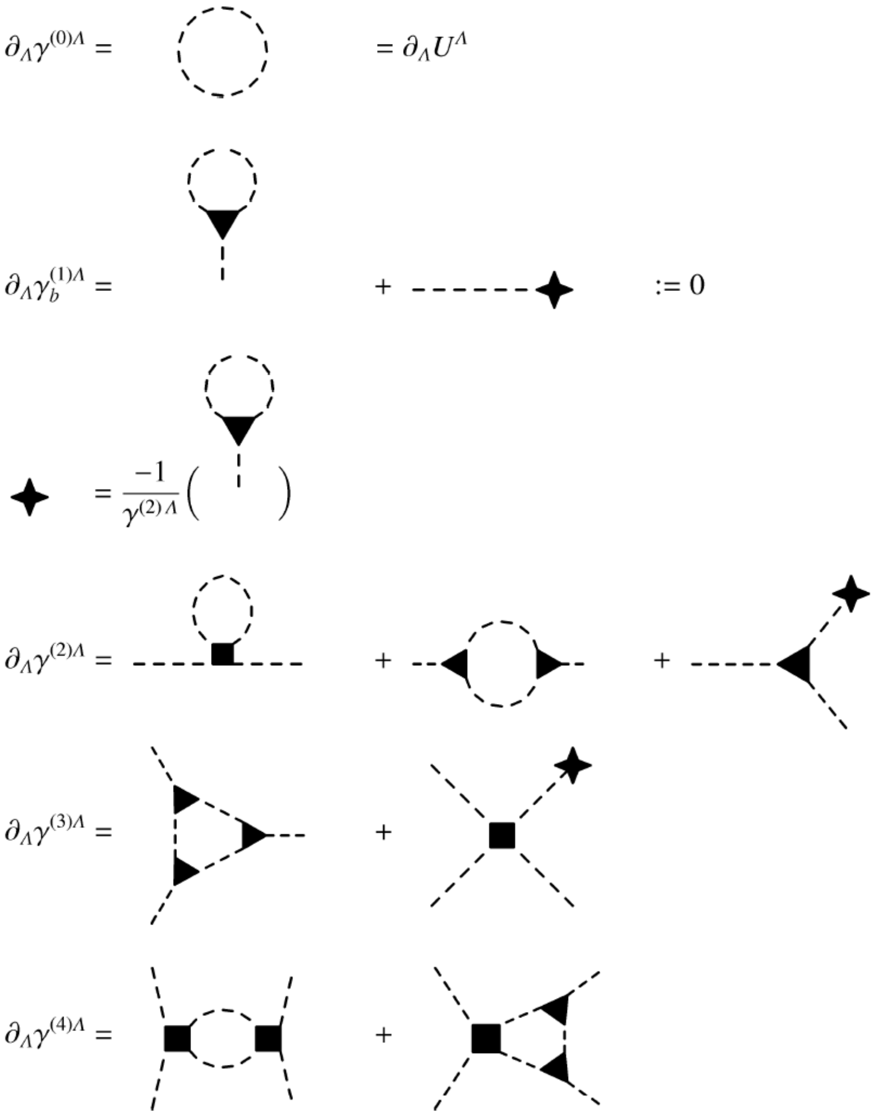
- Comparing coefficients between left- and right-hand-side yields: $$\dot{\gamma}^{(n)\,\Lambda} = \dot{\mathbf{R}}^{\Lambda}\partial_{\mathbf{R}}\left(\text{all 1-loop 1PI diagrams generated by } \mathbf{G}_{R}^{\Lambda} \text{ with n ext. legs}\right) + \dot{\alpha}^{\Lambda}\gamma^{(n+1)\,\Lambda}$$

- Need external constraint for flow of minimum: $$\partial_{\Lambda}\gamma_{b}^{(1)\Lambda} = \text{[diagram]} + \ldots + \dot{\alpha}^{\Lambda}\gamma^{(2)\,\Lambda} := 0$$

$$\partial_{\Lambda}\alpha^{\Lambda} = \frac{-1}{\gamma^{(2)\,\Lambda}}\left(\text{[diagram]} + \ldots\right).$$

Tadpole corrections are absorbed into the flow of the minimum.

Vertex expansion hierarchy for symmetry-broken phases



Flow of zero-point function yields effective potential.

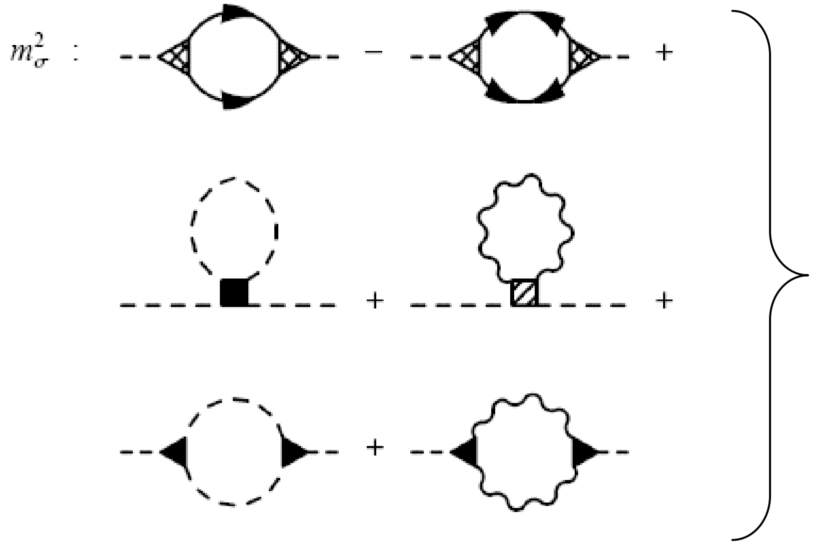
Flow of one-point function is forced to zero.

Flow of two-point function receives additional non-one-loop term.

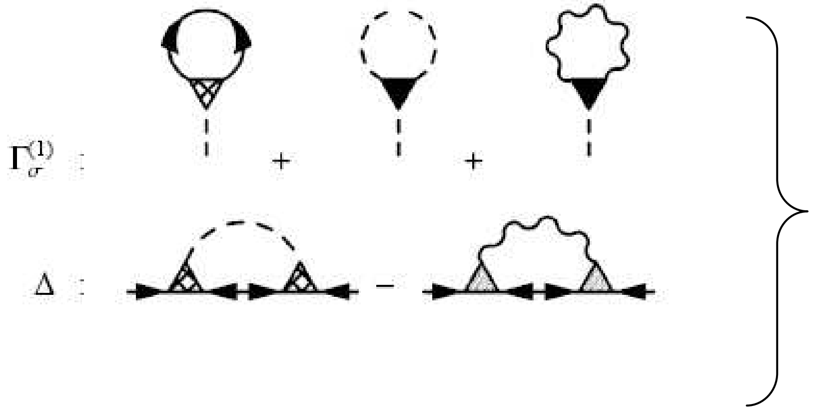
Flow of three-point function receives additional term non-one-loop term.

We have truncated n>4 to zero.

Flow equations in the superfluid phase



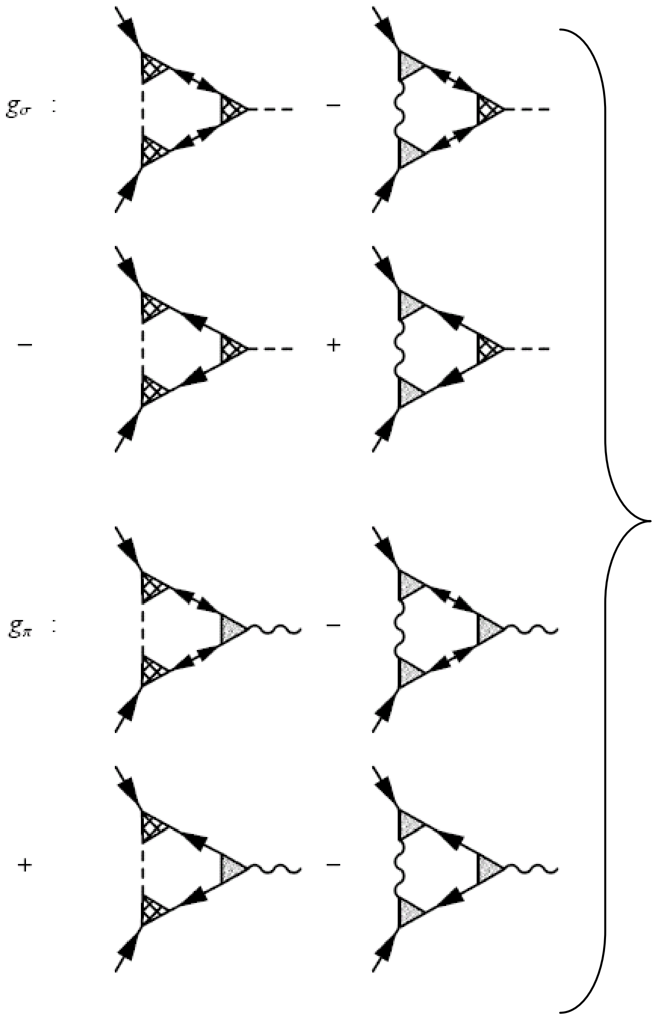
- Plus term from left-hand-side of flow equation: $3\frac{\lambda\alpha}{2}\partial_\Lambda\alpha$
- Flow of bosonic self-interaction follows from: $m_\sigma^2 = \lambda|\alpha|^2$
- Flow of Z-factors follows from frequency/momentum derivatives.



Equated to zero, determines flow of α

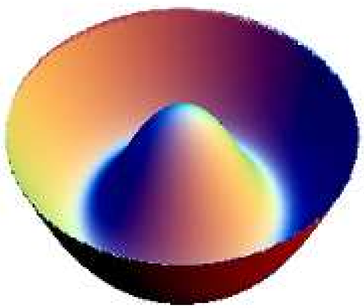
Plus term from left-hand-side which relates gap and order-parameter – without fluctuations, in MFT, gap=order-parameter. $g_\sigma\partial_\Lambda\alpha$

Flow equations in the superfluid phase



- Prime example of phase-space suppression of Goldstone singularity.
- Vertex renormalizes only very weakly as we will see later.

„We will see that fluctuations can excite collective Goldstone modes in which the system samples degenerate configurations at the bottom of the valley. Depending on the dimensionality, these collective modes can either destroy the order by equally mixing all the degenerate phase, or **have no effect because their effect is suppressed by phase space factors**.“(Negele-Orland 1987).



Still, structure of fermion-boson theory in superfluid phase clarified.

Agenda Fermionic superfluids I

- History of Fermionic Superfluids / Flows to strong coupling
- Attractive Hubbard Model
- Fermionic functional RG approach
- Hubbard-Strato-novich approach
- Truncation and cutoff choice
- **Analytical and numerical results**
- Summary Fermionic Super-fluids I

Connection of Hubbard-Stratonovich RG to MFT

- In standard MFT, the effective potential is non-polynomial in the field:

$$U^{\mathrm{MF}}(\phi) = \frac{|\phi|^2}{|U|} - \int_k \ln \frac{k_0^2 + \xi_{\mathbf{k}}^2 + |\phi|^2}{k_0^2 + \xi_{\mathbf{k}}^2}$$

- Minimizing the effective potential yields BCS gap equation and the extended MFT fulfills Goldstone's theorem with the correct collective excitation spectrum.

$$\frac{\partial U^{\mathrm{MF}}(\phi)}{\partial \phi} = 0$$

- Bosonic RG pendant, two coupled equations reproduces gap smaller than BCS.

$$\begin{aligned} \partial_\Lambda \Delta &= -\frac{2}{m_\sigma^2} \int_{k|\Lambda} F_f(k) \\ \partial_\Lambda \frac{m_\sigma^2}{2} &= \int_{k|\Lambda} \left[|G_f(k)|^2 - F_f^2(k) \right] + 3\gamma_{\sigma^3}\, \partial_\Lambda \Delta \end{aligned}$$

- If γ_{σ^3} is replaced by the value from expanding $U^{\mathrm{MF}}(\phi)$, BCS is exactly reproduced.

$$\gamma_{\sigma^3} = \frac{m_\sigma^2}{2\Delta}$$

- Congruence with fermionic RG upon identiying:

$$\frac{2}{m_\sigma^2} = V + W$$

→ ***Moral von der Geschicht':***

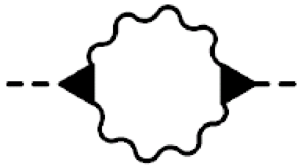
While fermionic RG displays infinities from diverging vertices when you don't know what you're doing, bosonic RG performs favorably with stupid users as mistakes result only in small gap shift.

Analytical results for infrared regime

• For scales smaller than fermionic gap, fermions decouple from flow.

• When longitudinal mass is larger than scale, longitudinal excitations decouple from the flow.

• Universal Goldstone regime determines infrared result.

$m_\sigma^2,\ Z_\sigma,\ A_\sigma\ :$ 

$$G_\sigma(sq) \propto s^{-1} \quad \text{for } d=2$$
$$G_\sigma(sq) \propto \log s \quad \text{for } d=3$$

• Not position, but existence of fixed-point for rescaled bosonic self-interaction enough for exact proof.

•Bogoliubov fixed-point with finite longitudinal mass unstable for d<3 at T=0 (proof in Castellani, et. al. PRB **69**, 024513, 2004)

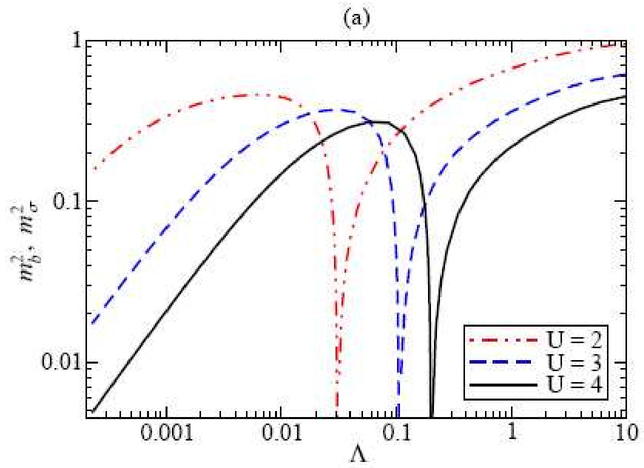
• In d<2 distinction between longitudinal and transversal modes breaks down, in d=1 longitudinal is identical to Goldstone mode (Kosterlitz-Thouless).

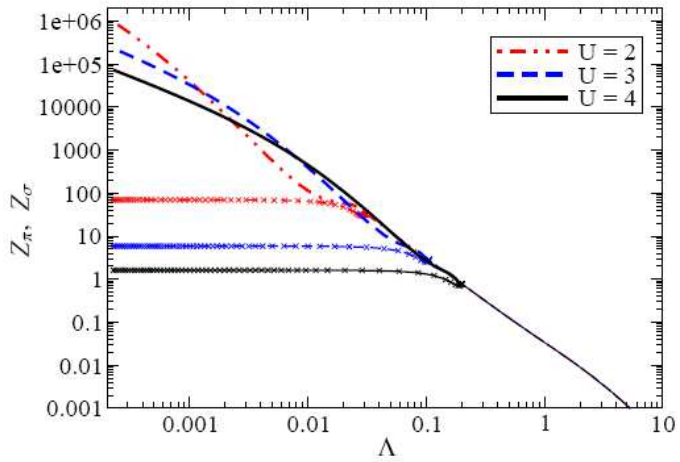
$$m_\sigma^2 = \lambda\alpha^2$$
$$\partial_\Lambda \lambda = \lambda^2 \int_{q|\Lambda} G_\pi^2(q)$$
$$\tilde{\lambda} = \frac{\lambda}{\Lambda} \quad \text{in d=2}$$
$$\frac{d\tilde{\lambda}}{d\log\Lambda} = -\tilde{\lambda} + \frac{\tilde{\lambda}^2}{4\pi^2 A_\pi Z_\pi}$$
$$\lambda \rightarrow 4\pi^2 A_\pi Z_\pi \Lambda \quad \text{for} \quad \Lambda \rightarrow 0$$

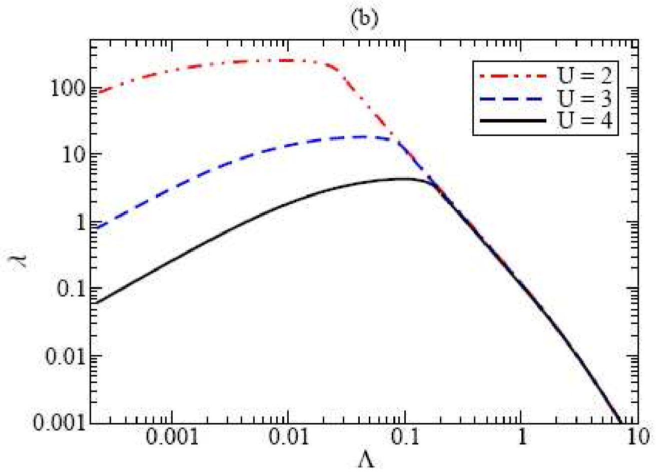
Functional RG flow delivers exact IR scaling, confirmed by numerical solution.

Numerical results for 2d-attractive Hubbard model

• Bosonic sector flows:







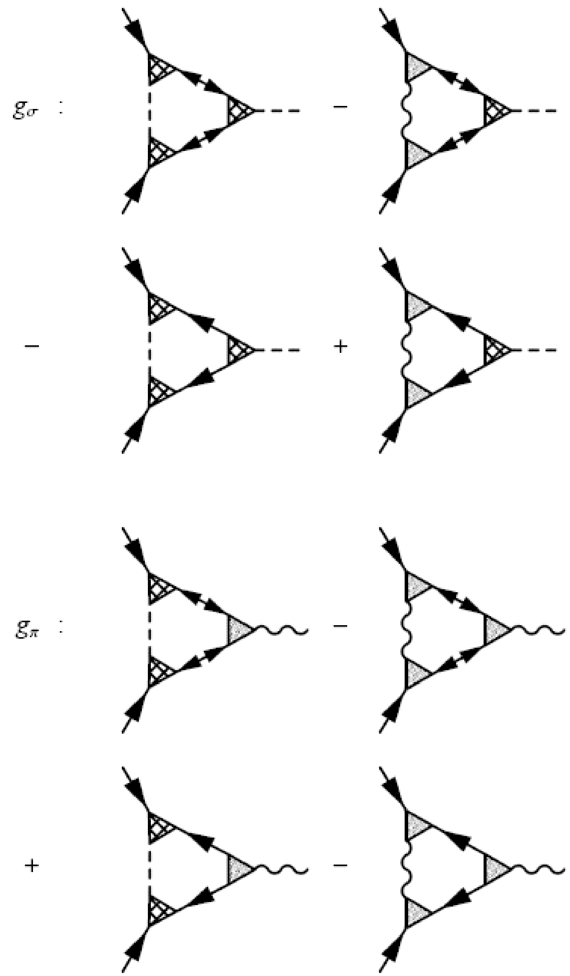
• At critical scale, „boson splits into longitudinal and Goldstone mode“

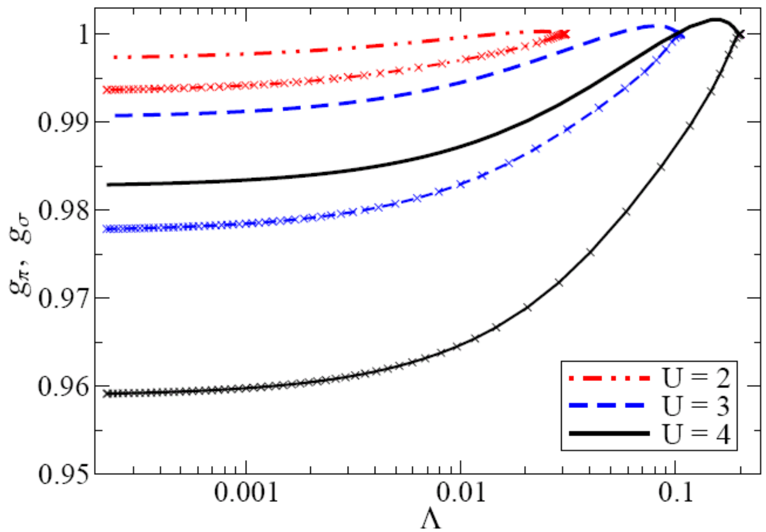
• Flow continuous across critical scale

• The scale at which the infrared asymptotics sets in can be determined

• IR scaling confirms analytical calculation

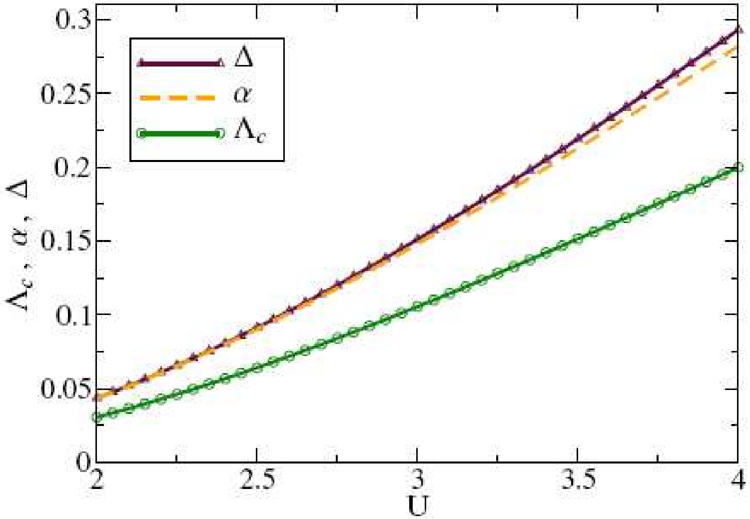
Numerical results for 2d-attractive Hubbard model

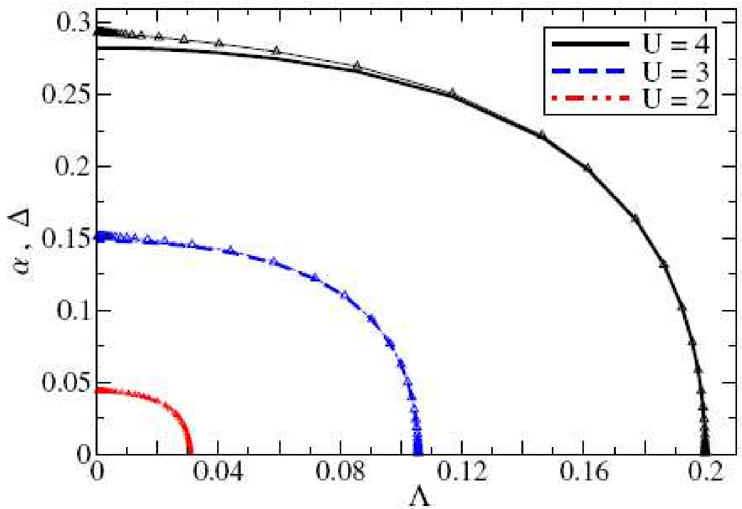




- Vertex renormalization relatively weak.
- Might change at finite-T then single Goldstone propagator is potentially logarithmically singular.
- Here, phase-space suppressed.

Numerical results for 2d-attractive Hubbard model





- Gap and order parameter split when increasing attraction, more pronounced in study of full BCS-Bose crossover. $\Delta \neq \alpha$

- Gap is reduced compared to MFT, **not from Goldstone mode**, however. Mostly from symmetric phase fluctuations: $\frac{\Delta}{\Delta_{\mathrm{BCS}}} \approx 0.25$

- Ratio of critical scale and gap: $\frac{\Delta}{\Lambda_c} \approx 1.4$

- Compare with BCS-theory at finite temperature: $\frac{\Delta}{T_c} \approx 1.7$

- Flow below critical scale: $\alpha, \Delta \propto (\Lambda_c - \Lambda)^{1/2}$

→ Compare with fermionic functional RG.

Numerical results for 2d-attractive Hubbard model

New Journal of Physics
The open-access journal for physics

Superconductivity in the attractive Hubbard model: functional renormalization group analysis

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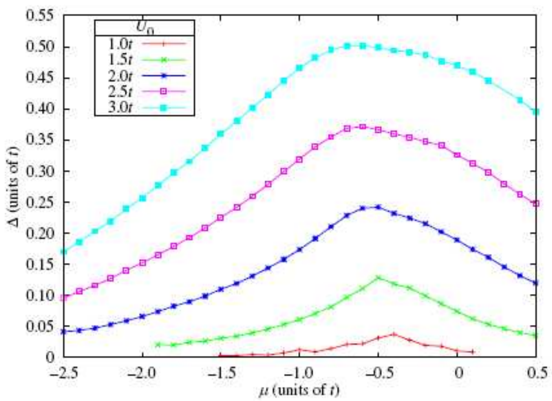

Figure 15. Magnitude of the superconducting order parameter Δ as calculated via the interaction flow procedure for varying interaction strength U_0 and chemical potential μ, $t' = -0.1t$.

$$\frac{\Delta}{\Delta_{BCS}} \approx 0.5$$

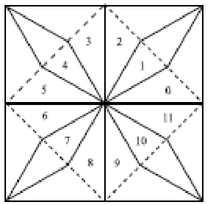

Figure 10. Patching of the Brillouin zone used with varying total number of patches in the numerical calculations. Every patch covers the same angle.

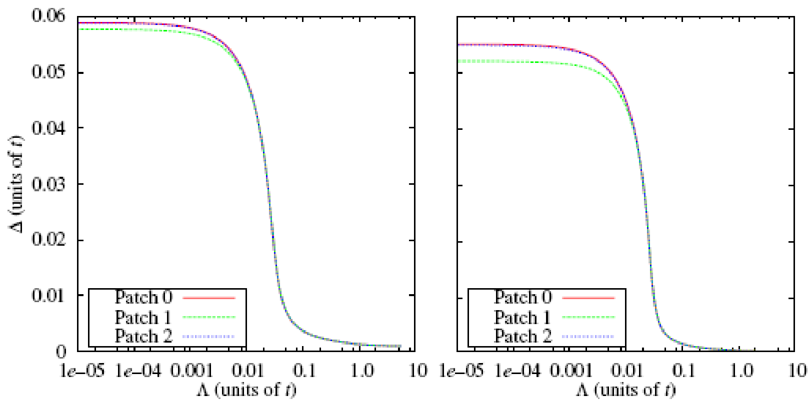

Figure 11. Momentum-shell flow of the order parameter. Twelve patches, quarter filling: $\mu = -1.41t$, $t' = -0.1t$; $U_0 = 1.5t$, $\Delta_{ext} = 10^{-3}t$ (left) and $\Delta_{ext} = 5 \times 10^{-4}t$ (right).

- Captured particle-hole fluctuations.
- Influence of 3+1 vertices only modest.

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Summary Fermionic superfluids I

Fermionic functional RG approach

+ Initial symmetry-breaking field regularizes diverging vertices at critical scale
+ Requires no decoupling of the interaction
+ Multiple instabilities treatable

Hubbard-Stratonovich approach

+ Convenient for collective excitations in symmetry-broken phase
+ Careful with hierarchy in SSB, terms from the right-hand-side
+ Hard to recover multiple channels

Results in superfluid-phase

+ Gap reduced from symmetric phase fluctuations
+ Goldstone mode suppressed by small phase-space for vertex and gap correction
+ Exact IR scaling of longitudinal sector

Outlook/tomorrow

+ Outlook for fermion self-energy studies
+ Normal-to-superfluid quantum phase transition

→ Thanks for your attention!

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