The ground state construction of the two-dimensional Hubbard model on the honeycomb lattice

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In these lectures I consider the half-filled two-dimensional (2D) Hubbard model on the honeycomb lattice and I review the rigorous construction of its ground state properties by making use of constructive fermionic Renormalization Group methods.

1. INTRODUCTION

There are very few quantum interacting systems whose ground state properties (thermodynamic functions, reduced density matrices, etc.) can be computed without approximations. Among these, the Luttinger and the Thirring model (one-dimensional spinless relativistic fermions) and the BCS model (spinning fermions with infinite - mean field - interactions); the construction of the ground states of these systems is based on some remarkable exact solutions, which make use of bosonization techniques and Bethe ansatz. Besides the exact solutions, another rigorous powerful method that allowed in a few cases to fully construct the ground state properties of a system of interacting fermions is Renormalization Group (RG); most of the available results concern one-dimensional (1D) weakly interacting fermions (ultraviolet $O(N)$ models with $N \geq 2$, non-relativistic spinless and spinnig fermions). In more than one dimensions, most of the result derived by rigorous RG techniques concern finite temperature properties (two-dimensional fermionic systems above the BCS critical temperature). Two remarkable exceptions are the Fermi liquid construction by Feldman et al., which concerns zero temperature properties of a system of interacting fermions with highly asymmetric Fermi surface, and the ground state construction of the short range half-filled Hubbard model on the honeycomb lattice by Giuliani and Mastropietro, which will be reviewed here. The latter result is of interest for the physics of *graphene*, a newly discovered material consisting of a one-atom thick layer of graphite.

The goal of these lectures is to give a self-contained proof of the analyticity of the ground state energy of the Hubbard model on the two-dimensional (2D) honeycomb lattice at half filling and weak coupling. A simple extension of the proof of convergence of the series for the specific ground state energy presented below allows one to construct the whole set of reduced density matrices at weak coupling (see [1]): it turns out that the off-diagonal elements of these matrices decay to zero at infinity, with the same decay exponents as the non-interacting system; in this sense, the construction presented below rigorously exclude the presence of long range order (LRO) in the ground state, and the absence of anomalous critical exponents (in other words, the interacting system is in the same universality class as the non-interacting one).

The plan of these lectures is the following. I will first introduce the model and state the main result. Next I will: (i) review the non-interacting case; (ii) describe the formal series expansion for the ground state energy; (iii) estimate by power-counting the generic n-th order in perturbation theory and identify the potentially divergent contributions; (iv) describe an (order by order) convergent resummed perturbation theory; (v) describe a way to reorganize and estimate the perturbation theory (determinant expansion) that allows one to prove convergence of the resummed series.

2. THE MODEL AND THE MAIN RESULTS

The grandcanonical Hamiltonian of the 2D Hubbard model on the honeycomb lattice at half filling in second quantized form is given by:

$$H_\Lambda = -t \sum_{\substack{\vec x\in\Lambda_A\\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} \left(a^+_{\vec x,\sigma} b^-_{\vec x+\vec\delta_i,\sigma} + b^+_{\vec x+\vec\delta_i,\sigma} a^-_{\vec x,\sigma}\right) + \tag{2.1}$$

$$+U \sum_{\vec x\in\Lambda_A} (a^+_{\vec x,\uparrow} a^-_{\vec x,\uparrow} - \frac{1}{2})(a^+_{\vec x,\downarrow} a^-_{\vec x,\downarrow} - \frac{1}{2}) + U \sum_{\vec x\in\Lambda_B} (b^+_{\vec x,\uparrow} b^-_{\vec x,\uparrow} - \frac{1}{2})(b^+_{\vec x,\downarrow} b^-_{\vec x,\downarrow} - \frac{1}{2})$$

where:

1. $\Lambda_A = \Lambda$ is a periodic triangular lattice, defined as $\Lambda = \mathbb{B}/L\mathbb{B}$, where $L \in \mathbb{N}$ and $\mathbb{B}$ is the infinite triangular lattice with basis $\vec l_1 = \frac{1}{2}(3, \sqrt{3})$, $\vec l_2 = \frac{1}{2}(3, -\sqrt{3})$. $\Lambda_B = \Lambda_A + \vec\delta_i$ is obtained by translating Λ_A by a nearest neighbor vector $\vec\delta_i$, $i = 1, 2, 3$, where
$$\vec\delta_1 = (1, 0)\ , \quad \vec\delta_2 = \frac{1}{2}(-1, \sqrt{3})\ , \quad \vec\delta_3 = \frac{1}{2}(-1, -\sqrt{3})\ . \tag{2.2}$$
The honeycomb lattice we are interested in is the union of the two triangular sublattices Λ_A and Λ_B.

2. $a^\pm_{\vec x,\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma =\uparrow\downarrow$ and site index $\vec x \in \Lambda_A$, satisfying periodic boundary conditions in $\vec x$. Similarly, $b^\pm_{\vec x,\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma =\uparrow\downarrow$ and site index $\vec x \in \Lambda_B$, satisfying periodic boundary conditions in $\vec x$.

3. U is the strength of the on–site density–density interaction; it can be either positive or negative.

Note that the Hamiltonian (2.1) is hole-particle symmetric, i.e., it is invariant under the exchange $a^\pm_{\vec x,\sigma} \longleftrightarrow a^\mp_{\vec x,\sigma}$, $b^\pm_{\vec x+\vec\delta_1,\sigma} \longleftrightarrow -\, b^\mp_{\vec x+\vec\delta_1,\sigma}$. This invariance implies in particular that, if we define the average density of the system to be $\rho = (2|\Lambda|)^{-1}\langle N\rangle_{\beta,\Lambda}$, with $N = \sum_{\vec x,\sigma}(a^+_{\vec x,\sigma} a^-_{\vec x,\sigma} + b^+_{\vec x+\vec\delta_1,\sigma} b^-_{\vec x+\vec\delta_1,\sigma})$ the total particle number operator and $\langle\cdot\rangle_{\beta,\Lambda} = \mathrm{Tr}\{e^{-\beta H_\Lambda}\cdot\}/\mathrm{Tr}\{e^{-\beta H_\Lambda}\}$ the average with respect to the (grandcanonical) Gibbs measure at inverse temperature β, one has $\rho \equiv 1$, for any $|\Lambda|$ and any β. Let

$$f_\beta(U) = -\frac{1}{\beta} \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \log \mathrm{Tr}\{e^{-\beta H_\Lambda}\}\ . \tag{2.3}$$

be the specific free energy of the system and $e(U) = \lim_{\beta\to\infty} f_\beta(U)$ the specific ground state energy. We will prove the following Theorem.

Theorem 2.1 *There exist a constant $U_0 > 0$ such that, if $|U| \leq U_0$, the specific free energy $f_\beta(U)$ of the 2D Hubbard model on the honeycomb lattice at half filling is an analytic function of U, uniformly in β as $\beta \to \infty$, and so is the specific ground state energy $e(U)$.*

The proof is based on RG methods, which will be reviewed below. A straightforward extension of the proof of Theorem 2.1 allows one to prove that the correlation functions (i.e., the off-diagonal elements of the reduced density matrices of the system) are analytic functions of U and they decay to zero at infinity with the same decay exponents as in the non-interacting ($U = 0$) case, see [1]. This rigorously excludes the presence of LRO in the ground state and proves that the interacting system is in the same universality class as the non-interacting system.

3. THE NON-INTERACTING SYSTEM

Let us begin by reviewing the construction of the finite and zero temperature states for the non-interacting ($U = 0$) case. In this case the Hamiltonian of interest reduces to

$$H^0_\Lambda = -t \sum_{\substack{\vec x\in\Lambda \\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} \left(a^+_{\vec x,\sigma} b^-_{\vec x+\vec\delta_i,\sigma} + b^+_{\vec x+\vec\delta_i,\sigma} a^-_{\vec x,\sigma} \right) , \tag{3.1}$$

with Λ, $a^\pm_{\vec x,\sigma}$, $b^\pm_{\vec x+\vec\delta_i,\sigma}$ defined as in items (1)–(4) after (2.1). We aim at computing the spectrum of H^0_Λ by diagonalizing the right hand side (r.h.s.) of (3.1). To this purpose, we pass to Fourier space. We identify Λ with the set of vectors in a fundamental cell, and we write

$$\Lambda = \{ n_1 \vec l_1 + n_2 \vec l_2 \ : \ 0 \leq n_1, n_2 \leq L-1 \} , \tag{3.2}$$

with $\vec l_1 = \frac12(3,\sqrt3)$ and $\vec l_2 = \frac12(3,-\sqrt3)$. The reciprocal lattice Λ^* is the set of vectors $\vec K$ such that $e^{i\vec K\vec x} = 1$, if $\vec x \in \Lambda$. A basis $\vec q_1, \vec q_2$ for Λ^* can be obtained by the inversion formula:

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = 2\pi \begin{pmatrix} l_{11} & l_{21} \\ l_{12} & l_{22} \end{pmatrix}^{-1} , \tag{3.3}$$

which gives

$$\vec q_1 = \frac{2\pi}{3}(1,\sqrt3) , \qquad \vec q_2 = \frac{2\pi}{3}(1,-\sqrt3) . \tag{3.4}$$

We call $\mathcal{B}_L$ the set of quasi-momenta $\vec k$ of the form

$$\vec k = \frac{m_1}{L}\vec q_1 + \frac{m_2}{L}\vec q_2 , \qquad m_1, m_2 \in \mathbb{Z} , \tag{3.5}$$

identified modulo Λ^*; this means that $\mathcal{B}_L$ can be identified with the vectors $\vec{k}$ of the form (2.2) and restricted to the *first Brillouin zone*:

$$\mathcal{B}_L = \{\vec{k} = \frac{m_1}{L}\vec{q}_1 + \frac{m_2}{L}\vec{q}_2 \ : \ 0 \le m_1, m_2 \le L-1\} \ . \tag{3.6}$$

Given a periodic function $f : \Lambda \to \mathbb{R}$, its Fourier transform is defined as

$$f(\vec{x}) = \frac{1}{|\Lambda|} \sum_{\vec{k}\in\mathcal{B}_L} e^{i\vec{k}\vec{x}} \hat{f}(\vec{k}) \ , \tag{3.7}$$

which can be inverted into

$$\hat{f}(\vec{k}) = \sum_{\vec{x}\in\Lambda} e^{-i\vec{k}\vec{x}} f(\vec{x}) \ , \qquad \vec{k} \in \mathcal{B}_L \ , \tag{3.8}$$

where we used the identity

$$\sum_{\vec{x}\in\Lambda} e^{i\vec{k}\vec{x}} = |\Lambda|\delta_{\vec{k},\vec{0}} \tag{3.9}$$

and δ is the periodic Kronecker delta function over Λ^*.

We now associate to the set of creation/annihilation operators $a^{\pm}_{\vec{x},\sigma}$, $b^{\pm}_{\vec{x}+\vec{\delta}_i,\sigma}$ the corresponding set of operators in momentum space:

$$a^{\pm}_{\vec{x},\sigma} = \frac{1}{|\Lambda|} \sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}\vec{x}} \hat{a}^{\pm}_{\vec{k},\sigma} \ , \qquad b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma} = \frac{1}{|\Lambda|} \sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}\vec{x}} \hat{b}^{\pm}_{\vec{k},\sigma} \ . \tag{3.10}$$

Using (3.7)–(3.9), we find that

$$\hat{a}^{\pm}_{\vec{k},\sigma} = \sum_{\vec{x}\in\Lambda} e^{\mp i\vec{k}\vec{x}} a^{\pm}_{\vec{x},\sigma} \ , \qquad \hat{b}^{\pm}_{\vec{k},\sigma} = \sum_{\vec{x}\in\Lambda} e^{\mp i\vec{k}\vec{x}} b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma} \tag{3.11}$$

are fermionic creation/annihilation operators satisfying

$$\{a^{\varepsilon}_{\vec{k},\sigma}, a^{\varepsilon'}_{\vec{k}',\sigma'}\} = |\Lambda|\delta_{\vec{k},\vec{k}'}\delta_{\varepsilon,-\varepsilon'}\delta_{\sigma,\sigma'} \ , \qquad \{b^{\varepsilon}_{\vec{k},\sigma}, b^{\varepsilon'}_{\vec{k}',\sigma'}\} = |\Lambda|\delta_{\vec{k},\vec{k}'}\delta_{\varepsilon,-\varepsilon'}\delta_{\sigma,\sigma'} \tag{3.12}$$

and $\{a^{\varepsilon}_{\vec{k},\sigma}, b^{\varepsilon'}_{\vec{k}',\sigma'}\} = 0$, which are periodic over Λ^*. [Of course, there is some arbitrariness in the definition of $\hat{a}^{\pm}_{\vec{k},\sigma}, \hat{b}^{\pm}_{\vec{k},\sigma}$: we could change the two sets of operators by multiplying them by two $\vec{k}$-dependent phase factors, $e^{\pm i\gamma^a_{\vec{k},\sigma}}, e^{\pm i\gamma^b_{\vec{k},\sigma}}$, and get a completely equivalent theory in momentum space. The freedom in the choice of these phase factors corresponds to the freedom in choosing the origins of the two sublattices Λ_A, Λ_B and is reflected into the so-called Berry-gauge symmetry of the theory (a local gauge symmetry in momentum space). It must be stressed that changing the Berry phase, the boundary conditions of $\hat{a}^{\pm}_{\vec{k},\sigma}, \hat{b}^{\pm}_{\vec{k},\sigma}$ at the boundaries of the first Brillouin zone change; our explicit choice of the Berry phase has the "advantage" of making $\hat{a}^{\pm}_{\vec{k},\sigma}, \hat{b}^{\pm}_{\vec{k},\sigma}$ periodic over Λ^* rather than quasi-periodic.]

With the previous definitions, we can rewrite

$$H^0_\Lambda = -t \sum_{\substack{\vec x\in\Lambda\\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} (a^+_{\vec x,\sigma} b^-_{\vec x+\vec\delta_i,\sigma} + b^+_{\vec x+\vec\delta_i,\sigma} a^-_{\vec x,\sigma}) = \tag{3.13}$$

$$= -\frac{t}{|\Lambda|^2} \sum_{\substack{\vec x\in\Lambda\\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} \sum_{\vec k,\vec k'\in\mathcal{B}_L} (e^{+i\vec k\vec x} e^{-i\vec k'(\vec x+\vec\delta_i-\vec\delta_1)} \hat a^+_{\vec k,\sigma} \hat b^-_{\vec k',\sigma} + e^{-i\vec k\vec x} e^{+i\vec k'(\vec x+\vec\delta_i-\vec\delta_1)} \hat b^+_{\vec k',\sigma} \hat a^-_{\vec k,\sigma})$$

$$= -\frac{v}{|\Lambda|} \sum_{\vec k\in\mathcal{B}_L} \sum_{\sigma=\uparrow\downarrow} (\Omega^*(\vec k) \hat a^+_{\vec k,\sigma} \hat b^-_{\vec k,\sigma} + \Omega(\vec k) \hat b^+_{\vec k,\sigma} \hat a^-_{\vec k,\sigma}) ,$$

with $v = \frac{3}{2}t$ the unperturbed Fermi velocity (if the hopping strength t is chosen to be the one measured in real graphene, v turns out to be approximately 300 times smaller than the speed of light) and

$$\Omega(\vec k) = \frac{2}{3} \sum_{i=1}^{3} e^{i(\vec\delta_i-\vec\delta_1)\vec k} = \frac{2}{3}\Big[1 + 2e^{-i\frac{3}{2}k_1} \cos(\frac{\sqrt3}{2}k_2)\Big] \tag{3.14}$$

the complex *dispersion relation.*

The Hamiltonian H^0_Λ can be diagonalized by introducing the fermionic operators

$$\hat\alpha_{\vec k,\sigma} = \frac{1}{\sqrt2}\Big(\hat a_{\vec k,\sigma} + \frac{\Omega^*(\vec k)}{|\Omega(\vec k)|} \hat b_{\vec k,\sigma}\Big) , \qquad \hat\beta_{\vec k,\sigma} = \frac{1}{\sqrt2}\Big(\hat a_{\vec k,\sigma} - \frac{\Omega^*(\vec k)}{|\Omega(\vec k)|} \hat b_{\vec k,\sigma}\Big) , \tag{3.15}$$

in terms of which we can re-write

$$H^0_\Lambda = \frac{v}{|\Lambda|} \sum_{\vec k\in\mathcal{B}_L} \sum_{\sigma=\uparrow\downarrow} (- |\Omega(\vec k)| \hat\alpha^+_{\vec k,\sigma} \hat\alpha_{\vec k,\sigma} + |\Omega(\vec k)| \hat\beta^+_{\vec k,\sigma} \hat\beta_{\vec k,\sigma}) , \tag{3.16}$$

with

$$|\Omega(\vec k)| = \frac{2}{3}\sqrt{(1 + 2\cos(3k_1/2)\cos(\sqrt3 k_2/2))^2 + 4\sin^2(3k_1/2)\cos^2(\sqrt3 k_2/2)} , \tag{3.17}$$

which is vanishing iff $\vec k = \vec p^{\,\omega}_F$, $\omega = \pm$, with

$$\vec p^{\,\omega}_F = (\frac{2\pi}{3}, \omega\frac{2\pi}{3\sqrt3}) . \tag{3.18}$$

Close to the *Fermi points* $\vec p^{\,\omega}_F$, the complex dispersion relation vanishes linearly:

$$\Omega(\vec p^{\,\omega}_F + \vec k') = ik'_1 + \omega k'_2 + O(|\vec k'|^2) , \tag{3.19}$$

resembling in this sense the relativistic dispersion relation of $(2+1)$-dimensional Dirac fermions.

From eqn(3.16) it is apparent that the ground state of the system consists in a Fermi sea such that all the negative energy states (the "α-states") are filled and all the positive energy states (the "β-states") are empty. The specific ground state energy $e_{0,\Lambda}$ is

$$e_{0,\Lambda} = -\frac{2v}{|\Lambda|}\sum_{\vec k\in\mathcal{B}_L}|\Omega(\vec k)|\;, \tag{3.20}$$

from which

$$e(0) = -2v\int_{\mathcal{B}}\frac{d\vec k}{|\mathcal{B}|}|\Omega(\vec k)|\;, \tag{3.21}$$

where $\mathcal{B} := \{\vec k = t_1\vec q_1 + t_2\vec q_2 \;:\; t_i\in[0,1)\}$ and $|\mathcal{B}| = 8\pi^2\sqrt{3}/9$. Similarly, the finite volume specific free energy $f^\beta_{0,\Lambda} := -(\beta|\Lambda|)^{-1}\log\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}$ is

$$f^\beta_{0,\Lambda} = -\frac{2}{\beta|\Lambda|}\sum_{\vec k\in\mathcal{B}_L}\log\Big[(1+e^{\beta v|\Omega(\vec k)|})(1+e^{-\beta v|\Omega(\vec k)|})\Big]\;, \tag{3.22}$$

from which

$$f_\beta(0) = -\frac{2}{\beta}\int_{\mathcal{B}}\frac{d\vec k}{|\mathcal{B}|}\log\Big[(1+e^{\beta v|\Omega(\vec k)|})(1+e^{-\beta v|\Omega(\vec k)|})\Big]\;. \tag{3.23}$$

For the following, it is useful to compute also the Schwinger functions of the system, defined as follows. Let us introduce the two component fermionic operators $\Psi^\pm_{\vec x,\sigma} = (a^\pm_{\vec x,\sigma}, b^\pm_{\vec x+\vec\delta_1,\sigma})$ and let us write $\Psi^\pm_{\vec x,\sigma,1} = a^\pm_{\vec x,\sigma}$ and $\Psi^\pm_{\vec x,\sigma,2} = b^\pm_{\vec x+\vec\delta_1,\sigma}$. We also consider the operators $\Psi^\pm_{\mathbf{x},\sigma} = e^{Hx_0}\Psi^\pm_{\vec x,\sigma}e^{-Hx_0}$ with $\mathbf{x} = (x_0,\vec x)$ and $x_0\in[0,\beta]$, for some $\beta>0$; we call x_0 the time variable. We write $\Psi^\pm_{\mathbf{x},\sigma,1} = a^\pm_{\mathbf{x},\sigma}$ and $\Psi^\pm_{\mathbf{x},\sigma,2} = b^\pm_{\mathbf{x}+\delta_1,\sigma}$, with $\delta_1 = (0,\vec\delta_1)$. We define the n-points Schwinger functions at finite volme and finite temperature as:

$$S^{\beta,\Lambda}_n(\mathbf{x}_1,\varepsilon_1,\sigma_1,\rho_1;\ldots;\mathbf{x}_n,\varepsilon_n,\sigma_n,\rho_n) = \langle\mathbf{T}\{\Psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1}\cdots\Psi^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n}\}\rangle_{\beta,\Lambda} \tag{3.24}$$

where: $\mathbf{x}_i\in[0,\beta]\times\Lambda$, $\sigma_i=\uparrow\downarrow$, $\varepsilon_i=\pm$, $\rho_i = 1,2$ and $\mathbf{T}$ is the operator of fermionic time ordering, acting on a product of fermionic fields as:

$$\mathbf{T}(\Psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1}\cdots\Psi^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n}) = (-1)^\pi\Psi^{\varepsilon_{\pi(1)}}_{\mathbf{x}_{\pi(1)},\sigma_{\pi(1)},\rho_{\pi(1)}}\cdots\Psi^{\varepsilon_{\pi(n)}}_{\mathbf{x}_{\pi(n)},\sigma_{\pi(n)},\rho_{\pi(n)}} \tag{3.25}$$

where π is a permutation of $\{1,\ldots,n\}$, chosen in such a way that $x_{\pi(1)0}\geq\cdots\geq x_{\pi(n)0}$, and $(-1)^\pi$ is its sign. [If some of the time coordinates are equal each other, the arbitrariness of the definition is solved by ordering each set of operators with the same time coordinate so that creation operators precede the annihilation operators.] Taking the limit $\Lambda\to\infty$ in (3.24) we get the finite temperature n-point Schwinger functions, denoted by $S^\beta_n(\mathbf{x}_1,\varepsilon_1,\sigma_1,\rho_1;\ldots;\mathbf{x}_n,\varepsilon_n,\sigma_n,\rho_n)$, which describe the properties of the infinite volume system at finite temperature. Taking the $\beta\to\infty$ limit of the finite temperature Schwinger functions, we get the zero temperature Schwinger functions, simply denoted by $S_n(\mathbf{x}_1,\varepsilon_1,\sigma_1,\rho_1;\ldots;\mathbf{x}_n,\varepsilon_n,\sigma_n,\rho_n)$, which by definition characterize the properties of the *thermal ground state* of (2.1) in the thermodynamic limit.

In the non-interacting case, i.e., if $H_\Lambda = H^0_\Lambda$, the Hamiltonian is quadratic in the creation/annihilation operators. Therefore, the $2n$-point Schwinger functions satisfy the Wick rule, i.e.,

$$\begin{aligned}&\langle \mathbf{T}\{\Psi^-_{\mathbf{x}_1,\sigma_1,\rho_1}\Psi^+_{\mathbf{y}_1,\sigma'_1,\rho'_1}\cdots\Psi^-_{\mathbf{x}_n,\sigma_n,\rho_n}\Psi^+_{\mathbf{y}_n,\sigma'_n,\rho'_n}\rangle_{\beta,\Lambda} = \det G\;,\\ &G_{ij} = \delta_{\sigma_i\sigma'_j}\langle \mathbf{T}\{\Psi^-_{\mathbf{x}_i,\sigma_i,\rho_i}\Psi^+_{\mathbf{y}_j,\sigma'_j,\rho'_j}\}\rangle_{\beta,\Lambda}\;.\end{aligned} \tag{3.26}$$

Moreover, every n–point Schwinger function $S^{\beta,\Lambda}_n(\mathbf{x}_1,\varepsilon_1,\sigma_1,\rho_1;\ldots;\mathbf{x}_n,\varepsilon_n,\sigma_n,\rho_n)$ with $\sum_{i=1}^n\varepsilon_i\neq 0$ is identically zero. Therefore, in order to construct the whole set of Schwinger functions of H^0_Λ, it is enough to compute the 2–point function $S^{\beta,\Lambda}_0(\mathbf{x}-\mathbf{y}) = \langle \mathbf{T}\{\Psi^-_{\mathbf{x},\sigma,\rho}\Psi^+_{\mathbf{y},\sigma,\rho'}\}\rangle_{\beta,\Lambda}$. This can be easily reconstructed from the 2-point function of the α-fields and β-fields.

Let $\vec x\in\Lambda$, $\alpha^\pm_{\vec x,\sigma} = |\Lambda|^{-1}\sum_{\vec k\in\mathcal{B}_L}e^{\pm i\vec k\vec x}\hat\alpha_{\vec k,\sigma}$ and $\beta^\pm_{\vec x,\sigma} = |\Lambda|^{-1}\sum_{\vec k\in\mathcal{B}_L}e^{\pm i\vec k\vec x}\hat\alpha_{\vec k,\sigma}$; if $\mathbf{x}=(x_0,\vec x)$, let $\alpha^\pm_{\mathbf{x},\sigma} = e^{H^0_\Lambda x_0}\alpha^\pm_{\vec x,\sigma}e^{-H^0_\Lambda x_0}$ and $\beta^\pm_{\mathbf{x},\sigma} = e^{H^0_\Lambda x_0}\beta^\pm_{\vec x,\sigma}e^{-H^0_\Lambda x_0}$. A straightforward computation shows that, if $-\beta < x_0-y_0\le\beta$,

$$\begin{aligned}\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{|\Lambda|}\sum_{\vec k\in\mathcal{B}_L}e^{-i\vec k(\vec x-\vec y)}\,\cdot\\ \cdot\Big[\mathbb{1}(x_0-y_0>0)\frac{e^{v(x_0-y_0)|\Omega(\vec k)|}}{1+e^{v\beta|\Omega(\vec k)|}} - \mathbb{1}(x_0-y_0\le 0)\frac{e^{v(x_0-y_0+\beta)|\Omega(\vec k)|}}{1+e^{v\beta|\Omega(\vec k)|}}\Big]\;,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{|\Lambda|}\sum_{\vec k\in\mathcal{B}_L}e^{-i\vec k(\vec x-\vec y)}\,\cdot\\ \cdot\Big[\mathbb{1}(x_0-y_0>0)\frac{e^{-v(x_0-y_0)|\Omega(\vec k)|}}{1+e^{-v\beta|\Omega(\vec k)|}} - \mathbb{1}(x_0-y_0\le 0)\frac{e^{-v(x_0-y_0+\beta)|\Omega(\vec k)|}}{1+e^{-v\beta|\Omega(\vec k)|}}\Big]\end{aligned} \tag{3.28}$$

and $\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = 0$. A priori eqns(3.28) and (3.29) are defined only for $-\beta < x_0-y_0\le\beta$, but we can extend them periodically over the whole real axis; the periodic extension of the propagator is continuous in the time variable for $x_0-y_0\notin\beta\mathbb{Z}$, and it has jump discontinuities at the points $x_0-y_0\in\beta\mathbb{Z}$. Note that at $x_0-y_0=\beta n$, the difference between the right and left limits is equal to $(-1)^n\delta_{\vec x,\vec y}$, so that the propagator is discontinuous only at $\mathbf{x}-\mathbf{y}=\beta\mathbb{Z}\times\vec 0$. If we define $\mathcal{B}_{\beta,L} := \mathcal{B}_\beta\times\mathcal{B}_L$ with $\mathcal{B}_\beta = \{k_0 = \frac{2\pi}{\beta}(n_0+\frac12)\ :\ n_0\in\mathbb{Z}\}$, then for $\mathbf{x}-\mathbf{y}\notin\beta\mathbb{Z}\times\vec 0$ we can write

$$\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{-ik_0-v|\Omega(\vec k)|}\;, \tag{3.29}$$

$$\langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{-ik_0+v|\Omega(\vec k)|}\;. \tag{3.30}$$

If we now re-express $\alpha^{\pm}_{\mathbf{x},\sigma}$ and $\beta^{\pm}_{\mathbf{x},\sigma}$ in terms of $a^{\pm}_{\mathbf{x},\sigma}$ and $b^{\pm}_{\mathbf{x}+\delta_1,\sigma}$, using (3.15), we find that, for $\mathbf{x}-\mathbf{y}\notin\beta\mathbb{Z}\times\vec{0}$:

$$S_0^{\beta,\Lambda}(\mathbf{x}-\mathbf{y})_{\rho,\rho'} := S_2^{\beta,\Lambda}(\mathbf{x},\sigma,-,\rho;\mathbf{y},\sigma,+,\rho')\Big|_{U=0} = \\ = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k_0^2+v^2|\Omega(\vec{k})|^2}\begin{pmatrix} ik_0 & -v\Omega^*(\vec{k}) \\ -v\Omega(\vec{k}) & ik_0\end{pmatrix}_{\rho,\rho'} \tag{3.31}$$

Finally, if $\mathbf{x}-\mathbf{y}=(0^-,\vec{0})$:

$$S_0^{\beta,\Lambda}(0^-,\vec{0}) = -\frac{1}{2}+\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}}\frac{1}{k_0^2+v^2|\Omega(\vec{k})|^2}\begin{pmatrix} 0 & -v\Omega^*(\vec{k}) \\ -v\Omega(\vec{k}) & 0\end{pmatrix}. \tag{3.32}$$

4. PERTURBATION THEORY AND GRASSMANN INTEGRATION

Let us now turn to the interacting case. The first step is to derive a formal perturbation theory for the specific free energy and ground state energy. In other words, we want to find rules to compute the generic perturbative order in U of $f_{\beta,\Lambda} := -(\beta|\Lambda|)^{-1}\log \mathrm{Tr}\{e^{-\beta H_\Lambda}\}$. We write $H_\Lambda = H^0_\Lambda + V_\Lambda$, with V_Λ the operator in the second line of eqn(2.1) and we use Trotter's product formula

$$e^{-\beta H_\Lambda} = \lim_{n\to\infty}\Big[e^{-\beta H^0_\Lambda/n}(1-\frac{\beta}{n}V_\Lambda)\Big]^n \tag{4.1}$$

so that, defining $V_\Lambda(t) := e^{tH^0_\Lambda}V_\Lambda e^{-tH^0_\Lambda}$,

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}} = \tag{4.2}$$
$$= 1+\sum_{N\geq 1}(-1)^N\int_0^\beta dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{N-1}}dt_N\frac{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}V_\Lambda(t_1)\cdots V_\Lambda(t_N)\}}{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}}.$$

Using the fermionic time-ordering operator defined in eqn(3.25), we can rewrite eqn(4.2) as

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}} = 1+\sum_{N\geq1}\frac{(-1)^N}{N!}\langle \mathbf{T}\{(V_{\beta,\Lambda}(\Psi))^N\}\rangle^0_{\beta,\Lambda}, \tag{4.3}$$

where $\langle\cdot\rangle^0_{\beta,\Lambda} = \mathrm{Tr}\{e^{-\beta H^0_\Lambda}\cdot\}/\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}$,

$$V_{\beta,\Lambda}(\Psi) := U\sum_{\rho=1,2}\int_{(\beta,\Lambda)}d\mathbf{x}(\Psi^+_{\mathbf{x},\uparrow,\rho}\Psi^-_{\mathbf{x},\uparrow,\rho}-\frac{1}{2})(\Psi^+_{\mathbf{x},\downarrow,\rho}\Psi^-_{\mathbf{x},\downarrow,\rho}-\frac{1}{2}), \tag{4.4}$$

and $\int_{(\beta,\Lambda)}d\mathbf{x}$ must be interpreted as $\int_{(\beta,\Lambda)}d\mathbf{x} = \int_{-\beta/2}^{\beta/2}dx_0\sum_{\vec{x}\in\Lambda}$. Note that the N-th term in the sum in the r.h.s. of eqn(4.3) can be computed by using the Wick rule (3.26) and the explicit expression for the 2-point function eqns(3.31)-(3.32). It is straightforward to check that the "Feynman rules" needed to compute

$\langle \mathbf{T}\{(V_{\beta,\Lambda}(\Psi))^N\}\rangle^0_{\beta,\Lambda}$ are the following: (i) draw N graph elements consisting of 4-legged vertices, with the vertex associated to two labels $\mathbf{x}_i$ and ρ_i, $i = 1,\ldots,N$, and the four legs associated to two exiting fields (with labels $(\mathbf{x}_i,\uparrow,\rho_1)$ and $(\mathbf{x}_i,\downarrow,\rho_i)$) and two entering fields (with labels $(\mathbf{x}_i,\uparrow,\rho_i)$ and $(\mathbf{x}_i,\downarrow,\rho_i)$), respectively; (ii) pair the fields in all possible ways, in such a way that every pair consists of one entering and one exiting field, with the same spin index; (iii) associate to every pairing a sign, corresponding to the sign of the permutation needed to bring every pair of contracted fields next to each other; (iv) associate to every paired pair of fields $[\Psi^-_{\mathbf{x}_i,\sigma_i,\rho_i},\Psi^+_{\mathbf{x}_j,\sigma_j,\rho_j}]$ an oriented line connecting the i-th with the j-th vertex, with orientation from j to i; (v) associate to every oriented line $[j\to i]$ a value equal to

$$g_{\rho_i,\rho_j}(\mathbf{x}_i-\mathbf{x}_j) := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\frac{\chi_0(2^{-M}|k_0|)}{k_0^2+v^2|\Omega(\vec k)|^2}\begin{pmatrix} ik_0 & -v\Omega^*(\vec k)\\ -v\Omega(\vec k) & ik_0\end{pmatrix}_{\rho_i,\rho_j} \tag{4.5}$$

where $\mathcal{B}^{(M)}_{\beta,L} = \mathcal{B}^{(M)}_\beta \times \mathcal{B}_L$, $\mathcal{B}^{(M)}_\beta = \mathcal{B}_\beta \cap \{k_0 \ :\ \chi_0(2^{-M}|k_0|) > 0\}$ and $\chi_0(t)$ is a smooth compact support function that is equal to 1 for $t \le 1/3$ and equal to 0 for $t \ge 2/3$; (vi) associate to every pairing (i.e., to every Feynman graph) a value, equal to the product of the sign of the pairing times U^N times the product of the values of all the oriented lines; (vii) integrate over $\mathbf{x}_i$ and sum over ρ_i the value of each pairing, then sum over all pairings; (viii) finally, take the $M\to\infty$ limit: the result is equal to $\langle \mathbf{T}\{(V_{\beta,\Lambda}(\Psi))^N\}\rangle^0_{\beta,\Lambda}$. Note that the $M\to\infty$ limit of the *propagator* $g(\mathbf{x})$ is equal to $S_0^{\beta,\Lambda}(\mathbf{x})$ if $\mathbf{x}\neq\mathbf{0}$, while $g(\mathbf{0}) = S_0^{\beta,\Lambda}(\mathbf{0})+\frac12$: the difference between g and $S_0^{\beta,\Lambda}$ takes into account the $-\frac12$ terms in the definition of $V_{\beta,\Lambda}(\Psi)$.

An algebraically convenient way to re-express eqn(4.3) is in terms of *Grassmann integrals*. Consider the set $\{\hat\psi^\pm_{\mathbf{k},\sigma,\rho}\}^{\sigma=\uparrow\downarrow,\ \rho=1,2}_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}$, where the *Grassmann variables* $\hat\psi^\pm_{\mathbf{k},\sigma,\rho}$ satisfy by the definition the anticommutation rules $\{\hat\psi^\varepsilon_{\mathbf{k},\sigma,\rho},\hat\psi^{\varepsilon'}_{\mathbf{k}',\sigma',\rho'}\} = 0$. In particular, the square of a Grassmann variable is zero and the only non-trivial monomials in the considered Grassmann variables are at most linear in each variable. Let the Grassmann algebra generated by the considered set of Grassmann variables be the set of all polynomials obtained by linear combinations of such non-trivial monomials. Let us also define the Grassmann integration $\int[\prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\prod_{\sigma=\uparrow\downarrow}^{\rho=1,2} d\hat\psi^+_{\mathbf{k},\sigma,\rho}d\hat\psi^-_{\mathbf{k},\sigma,\rho}]$ as the linear operator on the Grassmann algebra such that, given a monomial $Q(\hat\psi^-,\hat\psi^+)$ in the variables $\hat\psi^\pm_{\mathbf{k},\sigma,\rho}$, its action on $Q(\hat\psi^-,\hat\psi^+)$ is 0 except in the case $Q(\hat\psi^-,\hat\psi^+) = \prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\prod_{\sigma=\uparrow\downarrow}^{\rho=1,2}\hat\psi^-_{\mathbf{k},\sigma,\rho}\hat\psi^+_{\mathbf{k},\sigma,\rho}$, up to a permutation of the variables. In this case the value of the integral is determined, by using the anticommutation properties of the variables, by the condition

$$\int \Big[\prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} \prod_{\sigma=\uparrow\downarrow}^{\rho=1,2} d\hat\psi^+_{\mathbf{k},\sigma,\rho} d\hat\psi^-_{\mathbf{k},\sigma,\rho} \Big] \prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} \prod_{\sigma=\uparrow\downarrow}^{\rho=1,2} \hat\psi^-_{\mathbf{k},\sigma,\rho} \hat\psi^+_{\mathbf{k},\sigma,\rho} = 1 \tag{4.6}$$

Defining the free propagator matrix $\hat g_{\mathbf{k}}$ as

$$\hat g_{\mathbf{k}} = \chi_0(2^{-M}|k_0|) \begin{pmatrix} -ik_0 & -v\Omega^*(\vec k) \\ -v\Omega(\vec k) & -ik_0 \end{pmatrix}^{-1} \tag{4.7}$$

and the "Gaussian integration" $P_M(d\psi)$ as

$$P_M(d\psi) = \Big[\prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}^{\sigma=\uparrow\downarrow} \frac{-\beta^2|\Lambda|^2 \, [\chi_0(2^{-M}|k_0|)]^2}{k_0^2 + v^2|\Omega(\vec k)|^2} d\hat\psi^+_{\mathbf{k},\sigma,1} d\hat\psi^-_{\mathbf{k},\sigma,1} d\hat\psi^+_{\mathbf{k},\sigma,2} d\hat\psi^-_{\mathbf{k},\sigma,2} \Big] \cdot$$

$$\cdot \; \exp\Big\{ -(\beta|\Lambda|)^{-1} \sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}^{\sigma=\uparrow\downarrow} \hat\psi^+_{\mathbf{k},\sigma,\cdot} \, \hat g_{\mathbf{k}}^{-1} \hat\psi^-_{\mathbf{k},\sigma,\cdot} \Big\} , \tag{4.8}$$

it turns out that

$$\int P(d\psi) \hat\psi^-_{\mathbf{k}_1,\sigma_1,\rho_1} \hat\psi^+_{\mathbf{k}_2,\sigma_2,\rho_2} = \beta|\Lambda| \delta_{\sigma_1,\sigma_2} \delta_{\mathbf{k}_1,\mathbf{k}_2} [\hat g_{\mathbf{k}_1}]_{\rho_1,\rho_2} , \tag{4.9}$$

while the average of an arbitrary monomial in the Grassmann variables with respect to $P_M(d\psi)$ is given by the fermionic Wick rule with propagator equal to the r.h.s. of eqn(4.9). Using these definitions and the Feynman rules described above, we can rewrite eqn(4.3) as

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}} = \lim_{M\to\infty} \int P_M(d\psi) e^{-\mathcal{V}(\psi)} , \tag{4.10}$$

where

$$\mathcal{V}(\psi) = U \sum_{\rho=1,2} \int_{(\beta,\Lambda)} d\mathbf{x} \, \psi^+_{\mathbf{x},\uparrow,\rho} \psi^-_{\mathbf{x},\uparrow,\rho} \psi^+_{\mathbf{x},\downarrow,\rho} \psi^-_{\mathbf{x},\downarrow,\rho} , \tag{4.11}$$

$$\psi^\pm_{\mathbf{x},\sigma,\rho} = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} e^{\pm i\mathbf{k}\mathbf{x}} \hat\psi^\pm_{\mathbf{k},\sigma,\rho} , \qquad \mathbf{x} \in (-\beta/2, \beta/2] \times \Lambda \tag{4.12}$$

and the exponential in the r.h.s. of eqn(4.10) must be identified with its Taylor series (which is finite, due to the anticommutation rules of the Grassmann variables and the fact that the Grassmann algebra is finite for every finite M).

Let now

$$F^{(M)}_{\beta,\Lambda} := -\frac{1}{\beta|\Lambda|} \log \int P_M(d\psi) (e^{-\mathcal{V}(\psi)}) . \tag{4.13}$$

Proposition 1 *Let β and $|\Lambda|$ be sufficiently large. Assume that there exists $U_0 > 0$ such that $F^{(M)}_{\beta,\Lambda}$ is analytic in the complex domain $|U| \le U_0$ and is uniformly convergent as $M \to \infty$. Then, if $|U| \le U_0$,*

$$f_{\beta,\Lambda} = -\frac{2}{\beta|\Lambda|} \sum_{\vec k\in\mathcal{B}_L} \log\left(2 + 2\cosh(\beta v|\Omega(\vec k)|)\right) + \lim_{M\to\infty} F^{(M)}_{\beta,\Lambda} . \tag{4.14}$$

Proof. We need to prove that

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\}} = \exp\{-\beta|\Lambda| \lim_{M\to\infty} F^{(M)}_{\beta,\Lambda}\}\ . \tag{4.15}$$

The first key remark is that, if β, Λ are finite, the left hand side of (4.15) is a priori well defined and analytic on the whole complex plane. In fact, by the Pauli principle, the Fock space generated by the fermion operators $a^{\pm}_{\vec x,\sigma}, b^{\pm}_{\vec x+\vec\delta_1,\sigma}$, with $\vec x \in \Lambda, \sigma =\uparrow\downarrow$, is finite dimensional. Therefore, writing $H_\Lambda = H^0_\Lambda + V_\Lambda$, with H^0_Λ and V_Λ two bounded operators, we see that $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}$ is an entire function of U, simply because $e^{-\beta H_\Lambda}$ converges in norm over the whole complex plane:

$$\begin{aligned}||e^{-\beta H_\Lambda}|| \le \sum_{n=0}^\infty \frac{\beta^n}{n!}(||H^0_\Lambda|| + ||V_\Lambda||)^n = \sum_{k=0}^\infty \frac{\beta^k ||V_\Lambda||^k}{k!} \sum_{n\ge k} \frac{\beta^{n-k}||H^0_\Lambda||^{n-k}}{(n-k)!} = \\ = e^{\beta||H^0_\Lambda|| + \beta\,||V_\Lambda||}\ ,\end{aligned} \tag{4.16}$$

where the norm $||\cdot||$ is, e.g., the Hilbert-Schmidt norm $||A|| = \sqrt{\mathrm{Tr}(A^\dagger A)}$.

On the other hand, by assumption, $F^{(M)}_{\beta,\Lambda}$ is analytic in $|U| \le U_0$, with U_0 independent of β, Λ, M, and uniformly convergent as $M \to \infty$. Hence, by Weierstrass theorem, the limit $F_{\beta,\Lambda} = \lim_{M\to\infty} F^{(M)}_{\beta,\Lambda}$ is analytic in $|U| \le U_0$ and its Taylor coefficients coincide with the limits as $M \to \infty$ of the Taylor coefficients of $F^{(M)}_{\beta,\Lambda}$. Moreover, $\lim_{M\to\infty} e^{-\beta|\Lambda|F^{(M)}_{\beta,\Lambda}} = e^{-\beta|\Lambda|F_{\beta,\Lambda}}$, again by Weierstrass theorem.

As discussed above, the Taylor coefficients of $e^{-\beta|\Lambda|F_{\beta,\Lambda}}$ coincide with the Taylor coefficients of $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}/\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\}$: therefore, $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}/\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\} = e^{-\beta|\Lambda|F_{\beta,\Lambda}}$ in the complex region $|U| \le U_0$, simply because the l.h.s. is entire in U, the r.h.s. is analytic in $|U| \le U_0$ and the Taylor coefficients at the origin of the two sides are the same. Taking logarithms at both sides proves (4.14). ∎

By Proposition 1, the Grassmann integral eqn(4.13) can be used to compute the free energy of the original Hubbard model, provided that the r.h.s. of eqn(4.13) is analytic in a domain that is uniform in M, β, Λ and that it converges to a well defined analytic function uniformly as $M \to \infty$. The rest of these notes are devoted to the proof of this fact. We start from eqn(4.13), which can be rewritten as

$$F^{(M)}_{\beta,\Lambda} := -\frac{1}{\beta|\Lambda|} \sum_{N\ge1} \frac{(-1)^N}{N!} \mathcal{E}^T(\mathcal{V}; N)\ , \tag{4.17}$$

where the *truncated expectation* $\mathcal{E}^T$ is defined as

$$\mathcal{E}^T(\mathcal{V}; N) := \frac{\partial^N}{\partial\lambda^N} \log \int P_M(d\psi) e^{\lambda\mathcal{V}(\psi)}\big|_{\lambda=0}\ . \tag{4.18}$$

More in general,

$$\mathcal{E}^T(\mathcal{V}_1,\ldots,\mathcal{V}_N) := \frac{\partial^N}{\partial\lambda_1\cdots\partial\lambda_N}\log\int P_M(d\psi)e^{\lambda_1\mathcal{V}_1(\psi)+\cdots+\lambda_N\mathcal{V}_N(\psi)}|_{\lambda_i=0} \tag{4.19}$$

and $\mathcal{E}^T(\mathcal{V}_1,\ldots,\mathcal{V}_N)|_{\mathcal{V}_i=\mathcal{V}} = \mathcal{E}^T(\mathcal{V};N)$. It can be checked by induction that the truncated expectation is related to the simple expectation $\mathcal{E}(X(\psi)) = \int P_M(d\psi)X(\psi)$ by

$$\mathcal{E}(\mathcal{V}_1\cdots\mathcal{V}_N) = \sum_{p=1}^{N}\sum^{*}_{\cup_{i=1}^p Y_i=\{1,\ldots,N\}}\mathcal{E}^T(\mathcal{V}_{j_1^1},\ldots,\mathcal{V}_{j^1_{|Y_1|}})\cdots\mathcal{E}^T(\mathcal{V}_{j_1^p},\ldots,\mathcal{V}_{j^p_{|Y_p|}})\;, \tag{4.20}$$

where the second sum in the r.h.s. runs over the partitions of $\{1,\ldots,N\}$ into disjoint sets Y_i, $i=1,\ldots,p$, such that $Y_i=\{j_1^i,\ldots,j^i_{|Y_i|}\}$. Note that $\mathcal{E}(\mathcal{V}^N) = \mathcal{E}(\mathcal{V}_1,\ldots,\mathcal{V}_N)|_{\mathcal{V}_i=\mathcal{V}}$ can be computed as a sum of Feynman diagrams whose values are determined by the same Feynman rules described after eqn(4.4) (with the exception of rule (viii): of course, since $\mathcal{E}(X) = \int P_M(d\psi)X$, M should be temporarily kept fixed in the computation); we shall write

$$\mathcal{E}(\mathcal{V}^N) = \sum_{\mathcal{G}\in\Gamma_N}\widehat{\mathrm{Val}}(\mathcal{G})\;, \tag{4.21}$$

where Γ_N is the set of all Feynman diagrams with N vertices, constructed with the rules described above; $\widehat{\mathrm{Val}}(\mathcal{G})$ includes the integration over the space-time labels $\mathbf{x}_i$ and the sum over the component labels ρ_i: if $\mathcal{G}\in\Gamma_N^T$, we shall write symbolically

$$\widehat{\mathrm{Val}}(\mathcal{G}) = \sigma_{\mathcal{G}}U^N\sum_{\rho_1,\ldots,\rho_N}\int d\mathbf{x}_1\cdots d\mathbf{x}_N\prod_{\ell\in\mathcal{G}}g_{\rho(\ell),\rho'(\ell)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell))\;, \tag{4.22}$$

where $\sigma_{\mathcal{G}}$ is the sign of the permutation associated to the graph $\mathcal{G}$ and we denoted by $(\mathbf{x}(\ell),\rho(\ell))$ and $(\mathbf{x}'(\ell),\rho'(\ell))$ the labels of the two vertices, which the line ℓ exits from and enters in, respectively. Using eqns(4.20)-(4.21), it can be proved by induction that

$$\mathcal{E}^T(\mathcal{V};N) = \sum_{\mathcal{G}\in\Gamma_N^T}\widehat{\mathrm{Val}}(\mathcal{G})\;, \tag{4.23}$$

where $\Gamma_N^T\subset\Gamma_N$ is the set of *connected* Feynman diagrams with N vertices. Combining eqn(4.17) with eqn(4.23) we finally have a formal power series expansion for the specific free energy of our model (better, of its ultraviolet regularization associated to the imaginary-time ultraviolet cutoff $\chi_0(2^{-M}|k_0|)$). The Feynman rules for computing $\widehat{\mathrm{Val}}(\mathcal{G})$ allow us to derive a first *very naive* upper bound on the N-th order contribution to $F^{(M)}_{\beta,\Lambda}$, that is to

$$F^{(M;N)}_{\beta,\Lambda} := -\frac{1}{\beta|\Lambda|}\frac{(-1)^N}{N!}\mathcal{E}^T(\mathcal{V};N)\;. \tag{4.24}$$

We have:

$$|F^{(M;N)}_{\beta,\Lambda}| \le \frac{1}{\beta|\Lambda|}\frac{1}{N!}\sum_{\mathcal{G}\in\Gamma^T_N}|\mathrm{Val}(\mathcal{G})| \le$$
$$\le \frac{|\Gamma^T_N|}{N!}2^N|U|^N||g||^{N+1}_\infty\,||g||^{N-1}_1\ , \qquad (4.25)$$

where $|\Gamma^T_N|$ is the number of connected Feynman diagrams of order N and $\beta|\Lambda|(2|U|)^N||g||^{N+1}_\infty\,||g||^{N-1}_1$ is a uniform bound on the value of a generic connected Feynman diagram of order N. The bound is obtained as follows: given $\mathcal{G}\in\Gamma^T_N$, select an arbitrary "spanning tree" $\tau\subset\mathcal{G}$, i.e. a loopless subset of $\mathcal{G}$ that connects all the N vertices; now: the integrals over the space-time coordinates of the product of the propagators on the spanning tree can be bounded by $\beta|\Lambda|\cdot||g||^{N-1}_1$; the product of the remaining propagators can be bounded by $||g||^{N+1}_\infty$; finally, the sum over the ρ_i labels is bounded by 2^N. Using eqn(4.25) and the facts that, for a suitable constant $C>0$: (i) $|\Gamma^T_N|\le C^N(N!)^2$, (ii) $||g||_\infty\le C2^{M-h^*}$, where the negative integer h^* is such that $2^{h^*-1}<\pi/\beta\le 2^{h^*}$, (iii) $||g||_1\le C2^{-h^*}$, we find:

$$|F^{(M;N)}_{\beta,\Lambda}|\le(2C^3|U|)^N N!\,2^{M(N+1)-2h^*N}\ . \qquad (4.26)$$

This pessimistic bound has two main problems: (i) a combinatorial problem, associated to the $N!$, which makes the r.h.s. of eqn(4.26) not summable over N, not even for finite M and h^*; (ii) a divergence problem, associated to the factor $2^{M(N+1)-2h^*N}$, which diverges exponentially as $M\to\infty$ (i.e., as the ultraviolet regularization is removed) and as $h^*\to-\infty$ (i.e., as the temperature is sent to 0). The combinatorial problem is solved by a smart reorganization of the perturbation theory, in the form of a determinant expansion, together with a systematic use of the Gram-Hadamard bound. The divergence problem is solved by systematic resummations of the series: we will first identify the class of contributions that produce ultraviolet or infrared divergences and then we show how to inductively resum them into a redefinition of the coupling constants of the theory; the inductive resummations are based on a multiscale integration of the theory: at the end of the construction, they will allow us to express the specific free energy in terms of modified Feynman diagrams, whose values are not affected anymore by ultraviolet or infrared divergences.

5. THE DETERMINANT EXPANSION

Let us now show how to attack the first of two problems that arose at the end of previous section. In other words, let us show how to solve the combinatorial problem by reorganizing the perturbative expansion discussed above into a more compact and more convenient form. In the previous section we discussed a Feynman diagram representation of the truncated expectation, see eqn(4.23). A slightly more general version of eqn(4.23) is the following. For a given set of indices $P = \{f_1, \ldots, f_{|P|}\}$, with $f_i = (\mathbf{x}_i, \sigma_i, \rho_i, \varepsilon_i)$, $\varepsilon_i \in \{+,-\}$, let

$$\widetilde{\psi}(P) := \prod_{f\in P} \psi^{\varepsilon(f)}_{\mathbf{x}(f),\sigma(f),\rho(f)} \;, \tag{5.1}$$

with obvious notation. Each field $\psi^{\varepsilon(f)}_{\mathbf{x}(f),\sigma(f),\rho(f)}$ can be represented as an oriented half-line, emerging from the point $\mathbf{x}(f)$ and carrying an arrow, pointing in the direction entering or exiting the point, depending on whether $\varepsilon(f)$ is equal to $-$ or $+$, respectively; moreover, the half-line carries two labels, $\sigma(f) \in \{\uparrow,\downarrow\}$ and $\rho(f) \in \{1,2\}$. Now, given s set of indices $P_1, \ldots, P_s$, we can enclose the points $\mathbf{x}(f)$ belonging to the set P_j, for some $j = 1, \ldots, s$, in a box: in this way, assuming that all the points $\mathbf{x}(f)$, $f \in \cup_i P_i$, are distinct, we obtain s disjoint boxes. Given $\mathcal{P} := \{P_1, \ldots, P_s\}$, we can associate to it the set $\Gamma^T(\mathcal{P})$ of connected Feynman diagrams, obtained by pairing the half-lines with consistent orientations, in such a way that the two half-lines of any connected pairs carry the same spin index, and in such a way that all the boxes are connected. Using a notation similar to eqn(4.22), we have:

$$\mathcal{E}^T(\widetilde{\psi}(P_1), \ldots, \widetilde{\psi}(P_s)) = \sum_{\mathcal{G}\in\Gamma^T(\mathcal{P})} \mathrm{Val}(\mathcal{G}) \;, \qquad \mathrm{Val}(\mathcal{G}) = \sigma_{\mathcal{G}} \prod_{\ell\in\mathcal{G}} g_{\rho(\ell),\rho'(\ell)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell)) \tag{5.2}$$

A different a more compact representation for the truncated expectation, alternative to eqn(5.2), is the following:

$$\mathcal{E}^T(\widetilde{\psi}(P_1), \ldots, \widetilde{\psi}(P_s)) = \sum_{T\in\mathbf{T}(\mathcal{P})} \alpha_T \prod_{\ell\in T} g_\ell \int dP_T(\mathbf{t}) \det G^T(\mathbf{t}) \tag{5.3}$$

where:

- any element T of the set $\mathbf{T}(\mathcal{P})$ is a set of lines forming an *anchored tree* between the boxes $P_1, \ldots, P_s$, i.e., T is a set of lines that becomes a tree if one identifies all the points in the same clusters;

- α_T is a sign (irrelevant for the subsequent bounds);

- g_ℓ is a shorthand for $g_{\rho(\ell),\rho'(\ell)}(\mathbf{x}(\ell) - \mathbf{x}'(\ell))$;

- if $\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \le i, i' \le s\}$, then $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^n$ of unit norm;

- if $2n = \sum_{i=1}^{s} |P_i|$, then $G^T(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by $G^T_{f,f'} = t_{i(f),i(f')} g_{\ell(f,f')}$, where: $f, f' \notin F_T$, $F_T \stackrel{def}{=} \cup_{\ell\in T} \{f^1_\ell, f^2_\ell\}$ and f^1_ℓ, f^2_ℓ are the two field labels associated to the two (entering and exiting) half-lines contracted into ℓ; $i(f) \in \{1,\ldots,s\}$ is s.t. $f \in P_{i(f)}$; $g_{\ell(f,f')}$ is the propagator associated to the line obtained by contracting the two half-lines with indices f and f'.

If $s = 1$ the sum over T is empty, but we can still use the above equation by interpreting the r.h.s. as 1 if P_1 is empty, and $\det G^T(\mathbf{1})$ otherwise.

The proof of the determinant representation is described in Appendix A. Using eqn(5.3) we get an alternative representation for the N-th order contribution to the specific free energy:

$$F^{(M;N)}_{\beta,\Lambda} = -\frac{1}{\beta|\Lambda|}\frac{(-1)^N}{N!}\mathcal{E}^T(\mathcal{V};N) = -\frac{1}{\beta|\Lambda|}\frac{(-1)^N}{N!}U^N \sum_{\rho_1,\ldots,\rho_N}\sum_{T\in\mathbf{T}_N}\alpha_T \int d\mathbf{x}_1\cdots d\mathbf{x}_N \cdot \prod_{\ell\in T} g_{\rho_\ell,\rho_{\ell'}}(\mathbf{x}_\ell - \mathbf{x}'_\ell)\int dP_T(\mathbf{t})\det G^T(\mathbf{t}) \;. \tag{5.4}$$

Using the fact that the number of anchored trees in $\mathbf{T}_N$ is bounded by $C^N N!$ for a suitable constant C, from eqn(5.4) we get:

$$|F^{(M;N)}_{\beta,\Lambda}| \le (\text{const.})^N |U|^N\, ||g||_1^{N-1} ||\det G^T(\cdot)||_\infty \;. \tag{5.5}$$

In order to bound $\det G^T$, we use the *Gram-Hadamard inequality*, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij} = < A_i, B_j >$, where A_i, B_j are vectors in a Hilbert space with scalar product $< \cdot,\cdot >$, then

$$|\det M| \le \prod_i ||A_i||\cdot||B_i|| \;. \tag{5.6}$$

where $||\cdot||$ is the norm induced by the scalar product.

Let $\mathcal{H} = \mathbb{R}^n \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of the functions $\mathbf{F} : [-\beta/2,\beta/2)\times\Lambda \to \mathbb{C}^2$, with scalar product $\langle \mathbf{F},\mathbf{G}\rangle = \sum_{\rho=1,2}\int d\mathbf{z}\, F^*_\rho(\mathbf{z})G_\rho(\mathbf{z})$, where $F_\rho = [\mathbf{F}]_\rho$, $G_\rho = [\mathbf{G}]_\rho$, $\rho = 1,2$, are the components of the vectors $\mathbf{F}$ and $\mathbf{G}$. It is easy to verify that

$$G^T_{f,f'} = t_{i(f),i(f')} g_{\rho(f),\rho(f')}(\mathbf{x}(f) - \mathbf{x}(f')) = \langle \mathbf{u}_{i(f)}\otimes \mathbf{A}_{\mathbf{x}(f),\rho(f)}, \mathbf{u}_{i(f')}\otimes \mathbf{B}_{\mathbf{x}(f'),\rho(f')}\rangle \;, \tag{5.7}$$

where $\mathbf{u}_i \in \mathbb{R}^n$, $i = 1,\ldots,n$, are vectors such that $t_{i,i'} = \mathbf{u}_i\cdot\mathbf{u}_{i'}$, and $\mathbf{A}_{\mathbf{x},\rho}$ and $\mathbf{B}_{\mathbf{x},\rho}$ have components:

$$[\mathbf{A}_{\mathbf{x},\rho}(\mathbf{z})]_i = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} \sqrt{\chi_0(2^{-M}|k_0|)}\,\frac{e^{-i\mathbf{k}(\mathbf{z}-\mathbf{x})}}{k_0^2 + v^2|\Omega(\vec{k})|^2}\delta_{\rho,i} \;, \tag{5.8}$$

$$[\mathbf{B}_{\mathbf{x},\rho}(\mathbf{z})]_i = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} \sqrt{\chi_0(2^{-M}|k_0|)}\, e^{-i\mathbf{k}(\mathbf{x}-\mathbf{z})}\begin{pmatrix} ik_0 & -v\Omega^*(\vec{k}) \\ -v\Omega(\vec{k}) & ik_0 \end{pmatrix}_{i,\rho} \;,$$

so that

$$||\mathbf{A}_{\mathbf{x},\rho}||^2 = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} \frac{\chi_0(2^{-M}|k_0|)}{[k_0^2+v^2|\Omega(\vec k)|^2]^2} \le C 2^{M-4h^*} , \tag{5.9}$$

$$||\mathbf{B}_{\mathbf{x},\rho}||^2 = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} \chi_0(2^{-M}|k_0|)[k_0^2+v^2|\Omega(\vec k)|^2] \le C 2^{M+2h^*} ,$$

for a suitable constant C. Using the Gram-Hadamard inequality, we find $||\det G^T||_\infty \le (\text{const.})^N 2^{(M-h^*)(N+1)}$; substituting this result into eqn(5.5), we finally get:

$$|F^{(M;N)}_{\beta,\Lambda}| \le (\text{const.})^N |U|^N\, 2^{M(N+1)-2Nh^*} , \tag{5.10}$$

which is similar to eqn(4.26), but for the fact that there is no $N!$ in the r.h.s.! In other words, using the determinant expansion, we recovered the same dimensional estimate as the one obtained by the Feynman diagram expansion and we combinatorially gained a $1/N!$. The r.h.s. of eqn(5.10) is now summable over N for $|U|$ sufficiently small, even though non uniformly in M and h^*. In the next section we will discuss how to systematically improve the dimensional bound by an iterative resummation method.

6. THE MULTISCALE INTEGRATION: THE ULTRAVIOLET REGIME

In this section we begin to illustrate the multiscale integration of the fermionic functional integral of interest. This method will later allow us to perform iterative resummations and to re-express the specific free energy in terms of a modified expansion, whose N-th order term is summable in N and uniformly convergent as $M\to\infty$ and $h^*\to-\infty$, as desired.

The first step in the computation of the partition function

$$\Xi_{M,\beta,L} := \int P_M(d\psi) e^{-\mathcal{V}(\psi)} \tag{6.1}$$

and of its logarithm is the integration of the ultraviolet degrees of freedom corresponding to the large values of k_0. We proceed in the following way. We decompose the free propagator $\hat g_{\mathbf{k}}$ into a sum of two propagators supported in the regions of k_0 “large” and “small”, respectively. The regions of k_0 large and small are defined in terms of the smooth support function $\chi_0(t)$ introduced after eqn(4.5); note that, by the very definition of χ_0, the supports of $\chi_0\Big(\sqrt{k_0^2+|\vec k-\vec p_F^{\,+}|^2}\,\Big)$ and $\chi_0\Big(\sqrt{k_0^2+|\vec k-\vec p_F^{\,-}|^2}\,\Big)$ are disjoint (here $|\cdot|$ is the euclidean norm over $\mathbb{R}^2/\Lambda^*$). We define

$$f_{u.v.}(\mathbf{k}) = 1-\chi_0\Big(\sqrt{k_0^2+|\vec k-\vec p_F^{\,+}|^2}\,\Big)-\chi_0\Big(\sqrt{k_0^2+|\vec k-\vec p_F^{\,-}|^2}\,\Big) \tag{6.2}$$

and $f_{i.r.}(\mathbf{k}) = 1 - f_{u.v.}(\mathbf{k})$, so that we can rewrite $\hat{g}_\mathbf{k}$ as:

$$\hat{g}_\mathbf{k} = f_{u.v.}(\mathbf{k})\hat{g}_\mathbf{k} + f_{i.r.}(\mathbf{k})\hat{g}_\mathbf{k} \stackrel{def}{=} \hat{g}^{(u.v.)}(\mathbf{k}) + \hat{g}^{(i.r.)}(\mathbf{k}) \; . \tag{6.3}$$

We now introduce two independent set of Grassmann fields $\{\psi^{(u.v.)\pm}_{\mathbf{k},\sigma,\rho}\}$ and $\{\psi^{(i.r.)\pm}_{\mathbf{k},\sigma,\rho}\}$, with $\mathbf{k} \in \mathcal{B}^{(M)}_{\beta,L}$, $\sigma = \uparrow\downarrow$, $\rho = 1,2$, and the Gaussian integrations $P(d\psi^{(u.v.)})$ and $P(d\psi^{(i.r.)})$ defined by

$$\begin{aligned} &\int P(d\psi^{(u.v.)})\hat{\psi}^{(u.v.)-}_{\mathbf{k}_1,\sigma_1,\rho_1}\hat{\psi}^{(u.v.)+}_{\mathbf{k}_2,\sigma_2,\rho_2} = \beta|\Lambda|\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat{g}^{(u.v.)}(\mathbf{k}_1)_{\rho_1,\rho_2} \; , \\ &\int P(d\psi^{(i.r.)})\hat{\psi}^{(u.v.)-}_{\mathbf{k}_1,\sigma_1,\rho_1}\hat{\psi}^{(i.r.)+}_{\mathbf{k}_2,\sigma_2,\rho_2} = \beta|\Lambda|\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat{g}^{(i.r.)}(\mathbf{k}_1)_{\rho_1,\rho_2} \; . \end{aligned} \tag{6.4}$$

Similarly to $P_M(d\psi)$, the Gaussian integrations $P(d\psi^{(u.v.)})$, $P(d\psi^{(i.r.)})$ also admit an explicit representation analogous to (4.8), with $\hat{g}_\mathbf{k}$ replaced by $\hat{g}^{(u.v.)}(\mathbf{k})$ or $\hat{g}^{(i.r.)}(\mathbf{k})$ and the sum over $\mathbf{k}$ restricted to the values in the support of $f_{u.v.}(\mathbf{k})$ or $f_{i.r.}(\mathbf{k})$, respectively. It easy to verify that the ultraviolet propagator $g^{(u.v.)}(\mathbf{x}-\mathbf{y}) = (\beta|\Lambda|)^{-1}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\hat{g}^{(u.v.)}(\mathbf{k})$ satisfies, for all $n \geq 0$,

$$|g^{(u.v.)}(\mathbf{x}-\mathbf{y})| \leq \frac{C_n}{1+||\mathbf{x}-\mathbf{y}||^n} \; , \tag{6.5}$$

uniformly in M; here $||\mathbf{x}|| = \sqrt{|x_0|^2_\beta + |\vec{x}|^2_\Lambda}$, with $|\cdot|_\beta$ the distance over the one-dimensional torus of length β and $|\cdot|_\Lambda$ the distance over the periodic lattice Λ. The definition of Grassmann integration implies the following identity ("addition principle"):

$$\int P(d\psi)e^{-\mathcal{V}(\psi)} = \int P(d\psi^{(i.r.)})\int P(d\psi^{(u.v.)})e^{-\mathcal{V}(\psi^{(i.r.)}+\psi^{(u.v.)})} \tag{6.6}$$

so that we can rewrite the partition function as

$$\begin{aligned} \Xi_{M,\beta,L} &= e^{-\beta|\Lambda|F^{(M)}_{\beta,\Lambda}} = \int P(d\psi^{(i.r.)})\exp\big\{\sum_{n\geq 1}\frac{1}{n!}\mathcal{E}^T_{u.v.}(-\mathcal{V}(\psi^{(i.r.)}+\cdot);n)\big\} := \\ &:= e^{-\beta|\Lambda|F_{0,M}}\int P(d\psi^{(i.r.)})e^{-\mathcal{V}_0(\psi^{(i.r.)})} \; , \end{aligned} \tag{6.7}$$

where the *truncated expectation* $\mathcal{E}^T_{u.v.}$ is defined, given any polynomial $V_1(\psi^{(u.v.)})$ with coefficients depending on $\psi^{(i.r.)}$, as

$$\mathcal{E}^T_{u.v.}(V_1(\cdot);n) = \frac{\partial^n}{\partial\lambda^n}\log\int P(d\psi^{(u.v.)})e^{\lambda V_1(\psi^{(u.v.)})}\Big|_{\lambda=0} \tag{6.8}$$

and $\mathcal{V}_0$ is fixed by the condition $\mathcal{V}_0(0) = 0$. We will prove below that $\mathcal{V}_0$ can be written as

$$\begin{aligned} \mathcal{V}_0(\psi) = \sum_{n=1}^\infty(\beta|\Lambda|)^{-2n}\sum_{\sigma_1,\ldots,\sigma_n=\uparrow\downarrow}\;\sum_{\rho_1,\ldots,\rho_{2n}=1,2}\;\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_{2n}}\Big[\prod_{j=1}^n\hat{\psi}^{(i.r.)+}_{\mathbf{k}_{2j-1},\sigma_j,\rho_{2j-1}}\hat{\psi}^{(i.r.)-}_{\mathbf{k}_{2j},\sigma_j,\rho_{2j}}\Big]\cdot \\ \cdot\hat{W}_{M,2n,\underline{\rho}}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n-1})\;\delta(\sum_{j=1}^n(\mathbf{k}_{2j-1}-\mathbf{k}_{2j}))\; , \end{aligned} \tag{6.9}$$

where $\underline{\rho} = (\rho_1, \ldots, \rho_{2n})$ and we used the notation

$$\delta(\mathbf{k}) = \delta(\vec{k})\delta(k_0) \;, \qquad \delta(\vec{k}) = |\Lambda| \sum_{n_1,n_2\in\mathbb{Z}} \delta_{\vec{k},n_1\vec{b}_1+n_2\vec{b}_2} \;, \qquad \delta(k_0) = \beta\delta_{k_0,0} \;, \quad (6.10)$$

with $\vec{b}_1, \vec{b}_2$ a basis of Λ^*. The possibility of representing $\mathcal{V}_M$ in the form (6.9), with the *kernels* $\hat{W}_{M,2n,\underline{\rho}}$ independent of the spin indices σ_i, follows from a number of remarkable symmetries, discussed in Appendix B. The regularity properties of the kernels are summarized in the following Lemma, which will be proved below.

Lemma 6.1 *The constant $F_{0,M}$ in (6.7) and the kernels $\hat{W}_{M,2n,\underline{\rho}}$ in (6.9) are given by power series in U, convergent in the complex disc $|U| \leq U_0$, for U_0 small enough and independent of β, Λ, M; after Fourier transform, the $\mathbf{x}$-space counterparts of the kernels $\hat{W}_{M,2n,\underline{\rho}}$ satisfy the following bounds:*

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} \Big[\prod_{1\leq i<j\leq 2n} ||\mathbf{x}_i - \mathbf{x}_j||^{m_{i,j}} \Big] |W_{M,2n,\underline{\rho}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n})| \leq \beta|\Lambda| C_m^n |U|^{\max\{1,n-1\}} \quad (6.11)$$

for some constant $C_m > 0$, where $m = \sum_{1\leq i<j\leq 2n} m_{i,j}$. Moreover, the limits $F_0 = \lim_{M\to\infty} F_{0,M}$ and $W_{2n,\underline{\rho}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n}) = \lim_{M\to\infty} W_{M,2n,\underline{\rho}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n})$ exist and are reached uniformly in M, so that, in particular, the limiting functions are analytic in the same domain $|U| \leq U_0$.

Remark.

Once that the ultraviolet degrees of freedom have been integrated out, the remaining infrared problem (i.e., the computation of the Grassmann integral in the second line of (6.7)) is essentially independent of M, given the fact that the limit $W_{2n,\underline{\rho}}$ of the kernles $W_{M,2n,\underline{\rho}}$ is reached uniformly and that the limiting kernels are analytic and satisfy the same bounds as (6.11). For this reason, in the infrared integration described in the next two sections, M will not play any essential role and, for this reason, from now on we shall not stress anymore the dependence on M, for notational simplicity.

Before we present the proof of Lemma 6.1, let us note that the kernels $W_{M,2n,\underline{\rho}}$ satisfy a number of non-trivial invariance properties. We will be particularly interested in the invariance properties of the quadratic part $\hat{W}_{M,2,(\rho_1,\rho_2)}(\mathbf{k})$, which will be used below to show that the structure of the quadratic part of the new effective interaction has the same symmetries as the free integration. The crucial properties that we will need are summarized in the following Lemma, which is proved in Appendix B.

Lemma 6.2. *Let* $\hat{W}_{aa}(\mathbf{k}) := \hat{W}_{M,2,(1,1)}(\mathbf{k})$*,* $\hat{W}_{bb}(\mathbf{k}) = \hat{W}_{M,2,(2,2)}(\mathbf{k})$*,* $\hat{W}_{ab}(\mathbf{k}) = \hat{W}_{M,2,(1,2)}(\mathbf{k})$ *and* $\hat{W}_{ba}(\mathbf{k}) = \hat{W}_{M,2,(2,1)}(\mathbf{k})$*. Then the following properties are valid:*
(i) $W_{aa}(\mathbf{k}) = W_{bb}(\mathbf{k})$ *and* $W_{ab}(\mathbf{k}) = W^*_{ba}(\mathbf{k})$*;*
(ii) as $\beta \to \infty$*, for* $\omega = \pm$*,* $W_{aa}(0, \vec{p}_F^{\,\omega}) = W_{ab}(0, \vec{p}_F^{\,\omega}) = 0$*;*
(iii) as $\beta, |\Lambda| \to \infty$*, for* $\omega = \pm$*,*

$$\begin{aligned}
&\partial_{\vec{k}} \hat{W}_{aa}(0, \vec{p}_F^{\,\omega}) = \vec{0} \;, \qquad \mathrm{Re}\{\partial_{k_0} \hat{W}_{aa}(0, \vec{p}_F^{\,\omega})\} = 0 \;, \qquad \partial_{k_0} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega}) = 0 \;, \\
&\mathrm{Re}\{\partial_{k_1} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega})\} = \mathrm{Im}\{\partial_{k_2} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega})\} = 0 \;, \\
&i\partial_{k_1} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega}) = \omega \partial_{k_2} \hat{W}_{ab}(0, \vec{p}_F^{\,\omega}) \;.
\end{aligned} \tag{6.12}$$

Remarks.
1) For simplicity, the properties (ii) and (iii) are spelled out only in the zero temperature limit and in the thermodynamic limit; however, as it will be clear from the proof, those properties all have a finite temperature/volume counterpart.
2) Lemma 3 implies that in the vicinity of the Fermi points the kernel $W_{M,2,(\rho,\rho')}(\mathbf{k})$ can be rewritten in the form

$$W_{M,2,(\rho,\rho')}(k_0, \vec{p}_F^{\,\omega} + \vec{k}') \simeq \begin{pmatrix} -iz_0 k_0 & \delta_0(ik_1' - \omega k_2') \\ \delta_0(-ik_1' - \omega k_2') & -iz_0 k_0 \end{pmatrix}_{\rho,\rho'} \;, \tag{6.13}$$

for some real constants z_0, δ_0, modulo higher order terms in $(k_0, \vec{k}')$. Therefore, it is apparent that its structure is the same as the one of $\hat{S}_0(\mathbf{k})$, modulo higher order terms in $(k_0, \vec{k}')$.

Let us now turn to the proof of Lemma 6.1, which illustrates the main RG strategy that will be also used below, in the more difficult infrared integration.

Proof of Lemma 6.1. Let us rewrite the Fourier transform of $\hat{g}^{(u.v.)}(\mathbf{k})$ as

$$g^{(u.v.)}(\mathbf{x}) = \sum_{h=1}^{M} g^{(h)}(\mathbf{x}) \;, \tag{6.14}$$

where

$$g^{(h)}(\mathbf{x}) = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k} \in \mathcal{B}_{\beta,L}^{(M)}} f_{u.v.}(\mathbf{k}) H_h(k_0) e^{-i\mathbf{k}\mathbf{x}} \hat{g}_{\mathbf{k}} \;, \tag{6.15}$$

with $H_1(k_0) = \chi_0(2^{-1}|k_0|)$ and, if $h \ge 1$, $H_h(k_0) = \chi_0(2^{-h}|k_0|) - \chi_0(2^{-h+1}|k_0|)$. Note that $[g^{(h)}(\mathbf{0})]_{\rho\rho} = 0$, $\rho = 1, 2$, and, for any integer $K \ge 0$, $g^{(h)}(\mathbf{x})$ satisfies the bound

$$||g^{(h)}(\mathbf{x})|| \le \frac{C_K}{1 + (2^h |x_0|_\beta + |\vec{x}|_\Lambda)^K} \;, \tag{6.16}$$

where $|\cdot|_\beta$ is the distance on the one dimensional torus of size β and $|\cdot|_\Lambda$ is the distance on the periodic lattice Λ. Moreover, $g^{(h)}(\mathbf{x})$ admits a Gram representation: $g^{(h)}_{\rho,\rho'}(\mathbf{x}-\mathbf{y}) = \int d\mathbf{z}\, [\mathbf{A}^{(h)}_\rho(\mathbf{x}-\mathbf{z})]^* \cdot \mathbf{B}^{(h)}_{\rho'}(\mathbf{y}-\mathbf{z})$, with

$$
\begin{aligned}
[\mathbf{A}^{(h)}_\rho(\mathbf{x})]_i &= \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} \sqrt{f_{u.v.}(\mathbf{k}) H_h(k_0)} \frac{e^{-i\mathbf{k}\mathbf{x}}}{k_0^2 + v^2|\Omega(\vec{k})|^2} \delta_{\rho,i}\ , \\
[\mathbf{B}^{(h)}_\rho(\mathbf{x})]_i &= \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} \sqrt{f_{u.v.}(\mathbf{k}) H_h(k_0)}\, e^{-i\mathbf{k}\mathbf{x}} \begin{pmatrix} ik_0 & -v\Omega^*(\vec{k}) \\ -v\Omega(\vec{k}) & ik_0 \end{pmatrix}_{i,\rho}
\end{aligned}
\tag{6.17}
$$

and

$$
||\mathbf{A}^{(h)}_\rho(\mathbf{x}-\cdot)||^2 \le C2^{-3h}\ , \qquad ||\mathbf{B}^{(h)}_r(\mathbf{y}-\cdot)||^2 \le C2^{3h}\ , \tag{6.18}
$$

for a suitable constant C.

Our goal is to compute

$$
\begin{aligned}
e^{-\beta|\Lambda|F_{0,M} - \mathcal{V}_0(\psi^{(i.r)})} &= \int P(d\psi^{[1,M]}) e^{-\mathcal{V}(\psi^{(i.r.)}+\psi^{[1,M]})} \\
&= \exp\Big\{ \log \int P(d\psi^{[1,M]}) e^{-\mathcal{V}(\psi^{(i.r.)}+\psi^{[1,M]})} \Big\}\ ,
\end{aligned}
\tag{6.19}
$$

where $P(d\psi^{[1,M]})$ is the fermionic "Gaussian integration" associated with the propagator $\sum_{h=1}^M \hat{g}^{(h)}(\mathbf{k})$ (i.e., it is the same as $P(d\psi^{(u.v.)})$); moreover, we want to prove that $F_{0,M}$ and $\mathcal{V}_0(\psi^{(i.r)})$ are uniformly convergent as $M\to\infty$. We perform the integration of (6.19) in an iterative fashion: in fact, we will inductively prove that

$$
e^{-\beta|\Lambda|F_{0,M} - \mathcal{V}_0(\psi^{(i.r)})} = e^{-\beta|\Lambda|F_h} \int P(d\psi^{[1,h]}) e^{-\mathcal{V}^{(h)}(\psi^{(i.r.)}+\psi^{[1,h]})} \tag{6.20}
$$

where $P(d\psi^{[1,h]})$ is the fermionic "Gaussian integration" associated with the propagator $\sum_{k=1}^h \hat{g}^{(k)}(\mathbf{k})$ and $\mathcal{V}^{(M)} = \mathcal{V}$; for $1 \le h < M$,

$$
\begin{aligned}
&\mathcal{V}^{(h)}(\psi^{[1,h]}) = \\
&\quad = \sum_{n=1}^\infty \sum_{\underline{\rho},\underline{\sigma}} \int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} \Big[\prod_{j=1}^n \psi^{[1,h]+}_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1}} \psi^{[1,h]-}_{\mathbf{x}_{2j},\sigma_j,\rho_{2j}} \Big] W^{(h)}_{M,2n,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})\ .
\end{aligned}
\tag{6.21}
$$

In order to inductively prove (6.20)-(6.21) we use the addition principle to rewrite

$$
\int P(d\psi^{[1,h]}) e^{-\mathcal{V}^{(h)}(\psi^{(i.r.)}+\psi^{[1,h]})} = \int P(d\psi^{[1,h-1]}) \int P(d\psi^{(h)}) e^{-\mathcal{V}^{(h)}(\psi^{(i.r.)}+\psi^{[1,h-1]}+\psi^{(h)})}\ , \tag{6.22}
$$

where $P(d\psi^{(h)})$ is the fermionic Gaussian integration with propagator $\hat{g}^{(h)}(\mathbf{k})$. After the integration of $\psi^{(h)})$ we define

$$
e^{-\mathcal{V}^{(h-1)}(\psi^{(i.r.)}+\psi^{[1,h-1]}) - \beta|\Lambda|\bar{e}_h} = \int P(d\psi^{(h)}) e^{-\mathcal{V}^{(h)}(\psi^{(i.r.)}+\psi^{[1,h-1]}+\psi^{(h)})}\ , \tag{6.23}
$$

which proves (6.20) with

$$F_h = \sum_{k=h+1}^{M} \bar{e}_k \; . \qquad (6.24)$$

Let $\mathcal{E}_h^T$ be the truncated expectation associated to $P(d\psi^{(h)})$: then we have

$$\bar{e}_h + \mathcal{V}^{(h-1)}(\psi) = \sum_{n\geq 1} \frac{1}{n!}(-1)^{n+1}\mathcal{E}_h^T(\mathcal{V}^{(h)}(\psi+\psi^{(h)}); n) \; . \qquad (6.25)$$

The r.h.s. of eqn(6.25) can be graphically represented as in Fig.1. The term

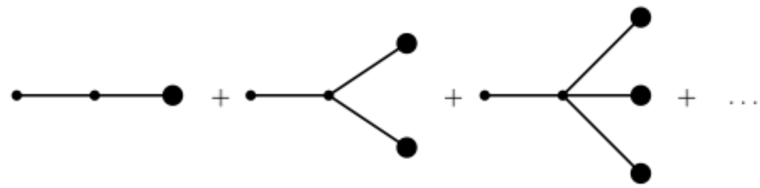

FIG. 1. The graphical representation of $\mathcal{V}^{(h-1)}$.

with n final points corresponds to the n-th term in the sum: a scale label $h-1$ should be attached to the leftmost node (called the *root*); a scale label h should be attached to the central node (corresponding to the action of $\mathcal{E}_h^T$); a scale label $h+1$ should be attached to the n rightmost nodes with the big black dots (representing $\mathcal{V}^{(h)}$). The sum of the tree graphs in Fig.1 can be represented by a simple tree consisting of a single horizontal branch, connecting the root (on scale $h-1$) with a big black dot on scale h. Iterating the graphical equation in Fig.1 up to scale M, and representing the endpoints on scale $M+1$ as simple dots (rather than big black dots), we end up with a graphical representation of $\mathcal{V}^{(h)}$ in terms of trees, see Fig.2, defined in terms of the following features.

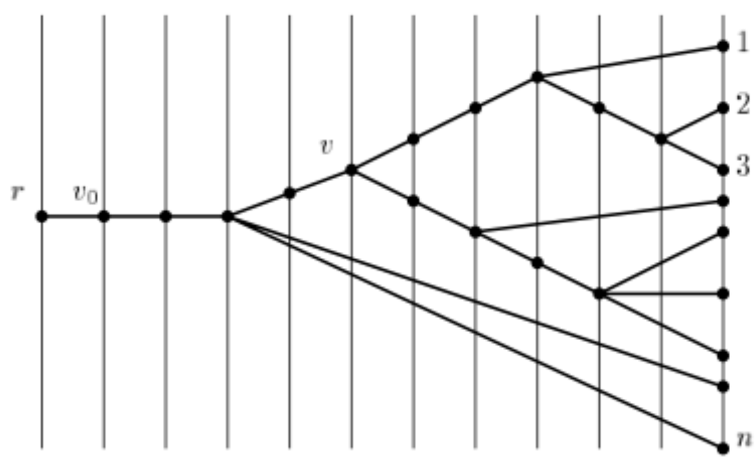

FIG. 2. A tree $\tau \in \widetilde{\mathcal{T}}_{M;n,h}$: the root is assumed to be on scale h and the endpoints to be on scale $M+1$.

1. Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n. We shall also consider the *labelled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabelled trees, as explained in the following items.

2. We associate a label $0 \leq h \leq M-1$ with the root and we denote $\widetilde{\mathcal{T}}_{M;h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, M+1]$, and we represent any tree $\tau \in \widetilde{\mathcal{T}}_{M;h,n}$ so that, if v is an endpoint, it is contained in the vertical line with index $h_v = M+1$, while if it is a non trivial vertex, it is contained in a vertical line with index $h < h_v \leq M$, to be called the *scale* of v; the root r is on the line with index h. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the *vertices* will be the union of the endpoints, of the trivial vertices and of the non trivial vertices; note that the root is not a vertex. Every vertex v of a tree will be associated to its scale label h_v, defined, as above, as the label of the vertical line whom v belongs to. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

3. There is only one vertex immediately following the root, which will be denoted v_0 and cannot be an endpoint; its scale is $h+1$.

4. Given a vertex v of $\tau \in \widetilde{\mathcal{T}}_{M;h,n}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w \geq v$. If a subtree with root v contains only v and one endpoint on scale $h_v + 1$, it will be called a *trivial subtree*.

5. With each endpoint v we associate a factor $\mathcal{V}(\psi^{[1,M]})$ and a set $\mathbf{x}_v$ of space-time points (the corresponding integration variables in the $\mathbf{x}$-space representation).

6. We introduce a *field label* f to distinguish the field variables appearing in the factors $\mathcal{V}(\psi^{[1,M]})$ associated with the endpoints; the set of field labels

associated with the endpoint v will be called I_v; note that if v is an endpoint $|I_v| = 4$. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\varepsilon(f)$, $\sigma(f)$ and $\rho(f)$ will denote the space-time point, the ε index, the σ index and the ρ index, respectively, of the Grassmann field variable with label f.

In terms of trees, the effective potential $\mathcal{V}^{(h)}$, $0 \le h \le M$ (with $\mathcal{V}^{(0)}(\psi^{(i.r.)})$ identified with $\mathcal{V}_0(\psi^{(i.r.)})$), can be written as

$$\mathcal{V}^{(h)}(\psi^{[1,h]}) + \beta|\Lambda|\bar{e}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \widetilde{\mathcal{T}}_{M;h,n}} \widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{[1,h]}) , \tag{6.26}$$

where, if v_0 is the first vertex of τ and $\tau_1, \ldots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $\widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{[1,h]})$ is defined inductively as:

$$\widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{[1,h]}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}[\widetilde{\mathcal{V}}^{(h+1)}(\tau_1, \psi^{[1,h+1]}); \ldots ; \widetilde{\mathcal{V}}^{(h+1)}(\tau_s, \psi^{[1,h+1]})] . \tag{6.27}$$

Given the constants $\bar{e}_k$, as defined by eqn(6.26), the

For what follows, it is important to specify the action of the truncated expectations on the branches connecting any endpoint v to the closest *non-trivial* vertex v' preceding it (here we are call v' non-trivial if $s_{v'} > 1$). In fact, if τ has only one end-point, it is convenient to rewrite $\widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{[1,h]}) = \mathcal{E}^T_{h+1}\mathcal{E}^T_{h+2} \cdots \mathcal{E}^T_M(\mathcal{V}(\psi^{[1,M]}))$ in telescopic series as:

$$\widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{[1,h]}) = \mathcal{V}(\psi^{[1,h]}) + \sum_{k=h+1}^{M} \mathcal{E}^T_{h+1} \cdots \mathcal{E}^T_k (\mathcal{V}(\psi^{[1,k]}) - \mathcal{V}(\psi^{[1,k-1]})) . \tag{6.28}$$

If we graphically represent the k-th term in the r.h.s. of eqn(6.28) by a subtree with only one end-point, with root on scale h and endpoint on scale $k+1$, we end up with an alternative representation of the effective potentials, which is based on a slightly modified tree expansion. The set of modified trees with n endpoints contributing to $\mathcal{V}^{(h)}$ will be denoted by $\mathcal{T}_{M;h,n}$; every $\tau \in \mathcal{T}_{M;h,n}$ is characterized in the same way as the elements of $\widetilde{\mathcal{T}}_{M;h,n}$, but for two features: (i) the endpoints of $\tau \in \mathcal{T}_{M;h,n}$ are not necessarily on scale $M+1$; (ii) if v_* is an endpoint of τ, then it is associated either to $\mathcal{V}(\psi^{[1,h_{v_*}-1]})$, if the non trivial vertex v'_* immediately preceding v_* on τ is on scale $h_{v^*} - 1$, or to $\mathcal{E}^T_{h_{v_*}-1}[\mathcal{V}(\psi^{[1,h_{v_*}-1]}) - \mathcal{V}(\psi^{[1,h_{v_*}-2]})]$, if $h_{v'_*} < h_{v_*} - 1$. See Fig.3.

In terms of these modified trees, we have:

$$\mathcal{V}^{(h)}(\psi^{[1,h]}) + \beta|\Lambda|\bar{e}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{M;h,n}} \mathcal{V}^{(h)}(\tau, \psi^{[1,h]}) , \tag{6.29}$$

where, if v_0 is the first vertex of τ and $\tau_1, \ldots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $\mathcal{V}^{(h)}(\tau, \psi^{[1,h]})$ is defined inductively as follows:

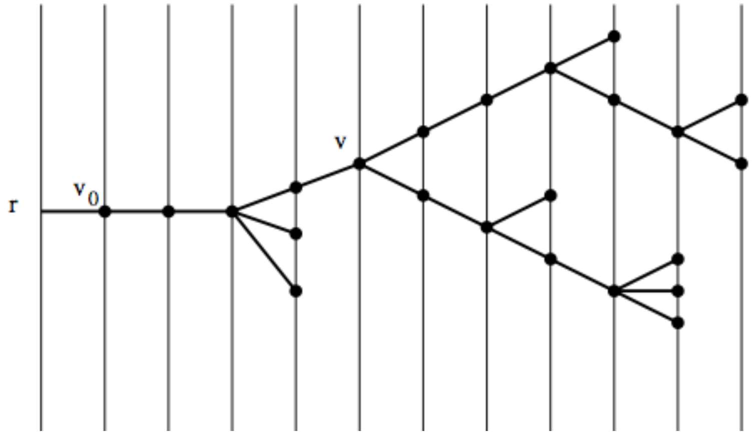

FIG. 3. A tree $\tau \in \mathcal{T}_{M;n,h}$: the root is assumed to be on scale h and the endpoints to be on scales $\leq M+1$.

i) if $s > 1$, then

$$\mathcal{V}^{(h)}(\tau, \psi^{[1,h]}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}[\bar{\mathcal{V}}^{(h+1)}(\tau_1, \psi^{[1,h+1]}); \dots ; \bar{\mathcal{V}}^{(h+1)}(\tau_s, \psi^{[1,h+1]})] \,, \tag{6.30}$$

where $\bar{\mathcal{V}}^{(h+1)}(\tau_i, \psi^{[1,h+1]})$ is equal to $\mathcal{V}^{(h+1)}(\tau_i, \psi^{[1,h+1]})$ if the subtree τ_i contains more than one end-point, or if it contains one end-point but it is not a trivial subtree; it is equal to $\mathcal{V}(\psi^{[1,h+1]})$ if τ_i is a trivial subtree;

ii) if $s = 1$, then $\mathcal{V}^{(h)}(\tau, \psi^{(\leq h)})$ is equal to $\mathcal{E}^T_{h+1}[\mathcal{V}^{(h+1)}(\tau_1, \psi^{[1,h+1]})]$ if τ_1 is not a trivial subtree; it is equal to $\mathcal{E}^T_{h+1}[\mathcal{V}(\psi^{[1,h+1]}) - \mathcal{V}(\psi^{[1,h]})]$ if τ_1 is a trivial subtree.

Note that, with $\mathcal{V}(\psi)$ defined as in eqn(4.11) and with the choice we made of the ultraviolet cutoff (such that $[g^{(h)}(\mathbf{0})]_{\rho\rho} = 0$), we get $\mathcal{E}^T_{h+1}[\mathcal{V}(\psi^{[1,h+1]}) - \mathcal{V}(\psi^{[1,h]})] = 0$ (i.e., the tadpoles are zero). This implies that, if v is not an endpoint and $n(v)$ is the number of endpoints following v on τ, and if τ has a vertex v with $n(v) = 1$, then its value vanishes: therefore, in the sum over the trees, we can freely impose the constraint that $n(v) > 1$ for all vertices $v \in \tau$ that are not endpoints. From now on we shall assume that the trees in $\mathcal{T}_{M;h,n}$ satisfy this constraint. Using its inductive definition, the right hand side of eqn(6.29) can be further expanded, and in order to describe the resulting expansion we need some more definitions.

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \dots, v_{s_v}$ are the $s_v \geq 1$ vertices immediately following it, then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union $\mathcal{I}_v$ of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the set of the *internal fields* of v, and is non empty if $s_v > 1$. Given $\tau \in \mathcal{T}_{M;h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote by $\mathcal{P}_\tau$ the family of all these choices and by $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. With these definitions, we can rewrite $\mathcal{V}^{(h)}(\tau, \psi^{[1,h]})$ as:

$$\begin{aligned}\mathcal{V}^{(h)}(\tau, \psi^{[1,h]}) &= \sum_{\mathbf{P}\in\mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}) \;, \\ \mathcal{V}^{(h)}(\tau, \mathbf{P}) &= \int d\mathbf{x}_{v_0} \widetilde{\psi}^{[1,h]}(P_{v_0}) K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0}) \;, \end{aligned} \tag{6.31}$$

where $\mathbf{x}_v = \cup_{f\in I_v}\{\mathbf{x}_v\}$,

$$\widetilde{\psi}^{[1,h]}(P_v) = \prod_{f\in P_v} \psi^{[1,h]\,\varepsilon(f)}_{\mathbf{x}(f),\sigma(f),\rho(f)} \tag{6.32}$$

and $K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$ is defined inductively by the equation, valid for any $v \in \tau$ that is not an endpoint,

$$K^{(h_v)}_{\tau,\mathbf{P}}(\mathbf{x}_v) = \frac{1}{s_v!}\prod_{i=1}^{s_v}[K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})] \;\; \mathcal{E}^T_{h_v}[\widetilde{\psi}^{(h_v)}(P_{v_1}\setminus Q_{v_1}), \ldots, \widetilde{\psi}^{(h_v)}(P_{v_{s_v}}\setminus Q_{v_{s_v}})] \;, \tag{6.33}$$

where $\widetilde{\psi}^{(h_v)}(P_{v_i} \setminus Q_{v_i})$ has a definition similar to (6.32). Moreover, if v_i is an endpoint $K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})$ is equal to the kernel of $\mathcal{V}(\psi^{[1,h]})$; if v_i is not an endpoint, $K^{(h_v+1)}_{v_i} = K^{(h_v+1)}_{\tau_i,\mathbf{P}_i}$, where $\mathbf{P}_i = \{P_w, w \in \tau_i\}$. Using in the r.h.s. of eqn(6.33) the determinant representation of the truncated expectation discussed in the previous section, we finally get:

$$\mathcal{V}^{(h)}(\tau, \mathbf{P}) = \sum_{T\in\mathbf{T}} \int d\mathbf{x}_{v_0} \widetilde{\psi}^{[1,h]}(P_{v_0}) W^{(h)}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) := \sum_{T\in\mathbf{T}} \mathcal{V}^{(h)}(\tau, \mathbf{P}, T) \;, \tag{6.34}$$

where

$$\begin{aligned} &W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) = \\ &\quad = U^n \Big\{ \prod_{\substack{v\\ \text{not e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{h_v,T_v}(\mathbf{t}_v) \Big[\prod_{l\in T_v} \delta_{\sigma^-_l,\sigma^+_l}\, g^{(h_v)}_{\rho^-_l,\rho^+_l}(\mathbf{x}_l - \mathbf{y}_l) \Big] \Big\} \end{aligned} \tag{6.35}$$

and $G^{h_v,T_v}(\mathbf{t}_v)$ is a matrix analogous to the one defined in previous section, with g replaced by $g^{(h)}$. Note that $W_{\tau,\mathbf{P},T}$ and, therefore, $\mathcal{V}^{(h)}(\tau, \mathbf{P})$ do not depend on M: $\mathcal{V}^{(h)}_M(\psi)$ depends on M only through the choice of the scale labels (i.e., the dependence on M is all encoded in $\mathcal{T}_{M;h,n}$). Using eqns(6.34)-(6.35), we finally get the bound:

$$\begin{aligned} &\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2l} |W^{(h)}_{M,2l,\underline{\rho}}(\mathbf{x}_1, \ldots, \mathbf{x}_{2l})| \le \sum_{n\ge\max\{1,l-1\}} |U|^n \sum_{\tau\in\mathcal{T}_{M;h,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}} \sum_{T\in\mathbf{T}} \cdot \\ &\cdot \int \prod_{l\in T} d(\mathbf{x}_l - \mathbf{y}_l) \Big[\prod_{v\ \text{not e.p.}} \frac{1}{s_v!} \max_{\mathbf{t}_v} |\det G^{h_v,T_v}(\mathbf{t}_v)| \prod_{l\in T_v} ||g^{(h_v)}(\mathbf{x}_l - \mathbf{y}_l)|| \Big] \;. \end{aligned} \tag{6.36}$$

Now, an application of the Gram–Hadamard inequality, combined with the dimensional bounds eqn(6.18), implies that

$$|{\rm det} G^{h_v,T_v}({\bf t}_v)| \le ({\rm const.})^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)}\;. \tag{6.37}$$

By the decay properties of $g^{(h)}({\bf x})$ given by eqn(6.16), it also follows that

$$\prod_{v\ {\rm not\ e.p.}} \frac{1}{s_v!}\int \prod_{l\in T_v} d({\bf x}_l-{\bf y}_l)\,||g^{(h_v)}({\bf x}_l-{\bf y}_l)|| \le c^n \prod_{v\ {\rm not\ e.p.}} \frac{1}{s_v!}2^{-h_v(s_v-1)}\;. \tag{6.38}$$

Plugging eqns(6.37)-(6.38) into eqn(6.36), we find that the l.h.s. of (6.36) can be bounded from above by

$$\sum_{n\ge \max\{1,l-1\}}\sum_{\tau\in\mathcal{T}_{M;h,n}}\sum_{\substack{{\bf P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\sum_{T\in{\bf T}}({\rm const.})^n|U|^n\Big[\prod_{v\ {\rm not\ e.p.}}\frac{1}{s_v!}2^{-h_v(s_v-1)}\Big]\;. \tag{6.39}$$

Using the following relations, which can be easily proved by induction,

$$\sum_{v\ {\rm not\ e.p.}}(s_v-1)=n-1\;, \tag{6.40}$$
$$\sum_{v\ {\rm not\ e.p.}}(h_v-h)(s_v-1)=\sum_{v\ {\rm not\ e.p.}}(h_v-h_{v'})(n(v)-1)\;,$$

where v' is the non trivial vertex immediately preceding v on τ, we find that eqn(6.39) can be rewritten as

$$\sum_{n\ge \max\{1,l-1\}}\sum_{\tau\in\mathcal{T}_{M;h,n}}\sum_{\substack{{\bf P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\sum_{T\in{\bf T}}({\rm const.})^n|U|^n 2^{-h(n-1)}\Big[\prod_{v\ {\rm not\ e.p.}}\frac{1}{s_v!}2^{-(h_v-h_{v'})(n(v)-1)}\Big]\;, \tag{6.41}$$

where we remind the reader that $n(v)>1$ for any $\tau\in\mathcal{T}_{M;h,n}$. Now, the number of terms in $\sum_{T\in{\bf T}}$ can be bounded by $({\rm const.})^n\prod_{v\ {\rm not\ e.p.}}s_v!$; moreover, $|P_v|\le 4n(v)$ and $n(v)-1\ge\max\{1,\frac{n(v)}{2}\}$, so that $n(v)-1\ge\frac12+\frac{|P_v|}{16}$. Therefore,

$$\frac{1}{\beta|\Lambda|}\int d{\bf x}_1\cdots d{\bf x}_{2l}|W^{(h)}_{M,2l,\underline{\rho}}({\bf x}_1,\ldots,{\bf x}_{2l})|\le\sum_{n\ge\max\{1,l-1\}}({\rm const.})^n|U|^n 2^{-h(n-1)}\,\cdot$$
$$\cdot\sum_{\tau\in\mathcal{T}_{M;h,n}}\Big(\prod_{v\ {\rm not\ e.p.}}2^{-\frac12(h_v-h_{v'})}\Big)\sum_{\substack{{\bf P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\Big(\prod_{v\ {\rm not\ e.p.}}2^{-|P_v|/16}\Big)\;. \tag{6.42}$$

Now, the sum over ${\bf P}$ can be bounded using the following combinatorial inequality: let $\{p_v, v\in\tau\}$, with $\tau\in\mathcal{T}_{M;h,n}$, a set of integers such that $p_v\le\sum_{i=1}^{s_v}p_{v_i}$ for all $v\in\tau$ that are not endpoints; then, if $\alpha>0$,

$$\prod_{\rm v\ not\ e.p.}\sum_{p_v}\gamma^{-\alpha p_v}\le C_\alpha^n\;,$$

which implies that

$$\sum_{\substack{{\bf P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\Big(\prod_{v\ {\rm not\ e.p.}}2^{-|P_v|/16}\Big)\le({\rm const.})^n\;. \tag{6.43}$$

Similarly, one can prove that

$$\sum_{\tau\in\mathcal{T}_{M;h,n}} \Big(\prod_{v \text{ not e.p.}} 2^{-\frac{1}{2}(h_v-h_{v'})}\Big) \le (\text{const.})^n \ , \tag{6.44}$$

uniformly in M as $M\to\infty$. Collecting all the previous bounds, we obtain

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{M,2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \le \sum_{n\ge\max\{1,l-1\}} (\text{const.})^n|U|^n 2^{-h(n-1)} \ , \tag{6.45}$$

which implies (6.11) with $m=0$. The proof of the general case, $m\ge 0$, is completely analogous. The constant $\overline{e}_h$ can be bounded by the r.h.s. of eqn(6.36) with $l=0$ and $n\ge\max\{2,l-1\}$ (because the contributions to $\overline{e}_h$ with $l=1$ are zero, by the condition that the tadpoles vanish), which implies

$$\overline{e}_h \le \sum_{n\ge\max\{2,l-1\}} (\text{const.})^n|U|^n 2^{-h(n-1)} \le (\text{const.})|U|^2 2^{-h} \ . \tag{6.46}$$

Therefore, $F_{0,M}=\sum_{k=1}^M \overline{e}_k$ is given by an absolutely convergent power series in U, as desired. A critical analysis of the proof shows that all the bounds are uniform in M,β,Λ and all the expressions involved admit well-defined limits as $M,\beta,|\Lambda|\to\infty$, whoch admit the same bounds. See [1] for details on these technical aspects.

7. THE MULTISCALE INTEGRATION: THE INFRARED REGIME

We are now left with computing

$$\Xi_{M,\beta,L} = e^{-\beta|\Lambda|F_0}\int P(d\psi^{(i.r.)})e^{-\mathcal{V}_0(\psi^{(i.r.)})} \ . \tag{7.1}$$

We proceed in an iterative fashion, similar to the one described in the previous section for the integratin of the large values of k_0. As a starting point, it is convenient to decompose the infrared propagator as:

$$g^{(i.r.)}(\mathbf{x},\mathbf{y}) = \sum_{\omega=\pm} e^{-i\vec p_F^{\,\omega}(\vec x-\vec y)} g^{(\le 0)}_\omega(\mathbf{x},\mathbf{y}) \ , \tag{7.2}$$

where, if $\mathbf{k}'=(k_0,\vec k')$,

$$g^{(\le 0)}_\omega(\mathbf{x},\mathbf{y}) = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{B}^\omega_{\beta,L}} \chi_0(|\mathbf{k}'|)e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})}\begin{pmatrix} -ik_0 & -v\Omega^*(\vec k'+\vec p_F^{\,\omega}) \\ -v\Omega(\vec k'+\vec p_F^{\,\omega}) & -ik_0\end{pmatrix}^{-1} \tag{7.3}$$

and $\mathcal{B}^\omega_{\beta,L}=\mathcal{B}^{(M)}_\beta\times\mathcal{B}^\omega_L$, with $\mathcal{B}^\omega_L=\{\frac{n_1}{L}\vec b_1+\frac{n_2}{L}\vec b_2-\vec p_F^{\,\omega}\ ,\ 0\le n_1,n_2\le L-1\}$. Correspondingly, we rewrite $\psi^{(i.r.)}$ as a sum of two independent Grassmann fields:

$$\psi^{(i.r.)\pm}_{\mathbf{x},\sigma,\rho} = \sum_{\omega=\pm} e^{i\vec p_F^{\,\omega}\vec x}\psi^{(\le 0)\pm}_{\mathbf{x},\sigma,\rho,\omega} \tag{7.4}$$

and we rewrite (6.7) in the form:

$$\Xi_{M,\beta,L} = e^{-\beta|\Lambda|F_0}\int P_{\chi_0,A_0}(d\psi^{(\le 0)})e^{-\mathcal{V}^{(0)}(\psi^{(\le 0)})}\ , \tag{7.5}$$

where $\mathcal{V}^{(0)}(\psi^{(\le 0)})$ is equal to $\mathcal{V}_M(\psi^{(i.r.)})$, once $\psi^{(i.r.)}$ is rewritten as in (7.4), i.e.,

$$\begin{aligned}
&\mathcal{V}^{(0)}(\psi^{(\le 0)}) = \qquad (7.6)\\
&= \sum_{n=1}^{\infty}(\beta|\Lambda|)^{-2n}\sum_{\sigma_1,\dots,\sigma_n=\uparrow\downarrow}\ \sum_{\rho_1,\dots,\rho_{2n}=1,2}^{\omega_1,\dots,\omega_{2n}=\pm}\ \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_{2n}}\Big[\prod_{j=1}^{n}\hat\psi^{(\le 0)+}_{\mathbf{k}'_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}}\hat\psi^{(\le 0)-}_{\mathbf{k}'_{2j},\sigma_j,\rho_{2j},\omega_{2j}}\Big]\ \cdot\\
&\qquad\cdot\hat W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\dots,\mathbf{k}'_{2n-1})\ \delta\Big(\sum_{j=1}^{2n}(-1)^j(\mathbf{p}_F^{\omega_j}+\mathbf{k}'_j)\Big) =\\
&= \sum_{n=1}^{\infty}\sum_{\underline{\sigma},\underline{\rho},\underline{\omega}}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2n}\Big[\prod_{j=1}^{n}\psi^{(\le 0)+}_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}}\psi^{(\le 0)-}_{\mathbf{x}_{2j},\sigma_j,\rho_{2j},\omega_{2j}}\Big]W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\dots,\mathbf{x}_{2n})
\end{aligned}$$

with:
1) $\underline{\omega}=(\omega_1,\dots,\omega_{2n})$, $\underline{\sigma}=(\sigma_1,\dots,\sigma_n)$ and $\mathbf{p}_F^\omega=(0,\vec p_F^{\ \omega})$;
2) $\hat W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\dots,\mathbf{k}'_{2n-1}) = \hat W_{M,2n,\underline{\rho}}(\mathbf{k}'_1+\mathbf{p}_F^{\omega_j},\dots,\mathbf{k}'_{2n-1}+\mathbf{p}_F^{\omega_{2n-1}})$, see (6.9);
3) the kernels $W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\dots,\mathbf{x}_{2n})$ are defined as:

$$\begin{aligned}
&W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\dots,\mathbf{x}_{2n}) = \qquad (7.7)\\
&= (\beta|\Lambda|)^{-2n}\sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_{2n}} e^{i\sum_{j=1}^{2n}(-1)^j\mathbf{k}_j\mathbf{x}_j}\hat W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\dots,\mathbf{k}'_{2n-1})\ \delta\Big(\sum_{j=1}^{2n}(-1)^j(\mathbf{p}_F^{\omega_j}+\mathbf{k}'_j)\Big)\ .
\end{aligned}$$

Moreover, $P_{\chi_0,A_0}(d\psi^{(\le 0)})$ is defined as

$$\begin{aligned}
P_{\chi_0,A_0}(d\psi^{(\le 0)}) &= \mathcal{N}_0^{-1}\Big[\prod_{\mathbf{k}'\in\mathcal{B}^\omega_{\beta,L}}^{\chi_0(|\mathbf{k}'|)>0}\ \prod_{\sigma,\omega,\rho} d\hat\psi^{(\le 0)+}_{\mathbf{k}',\sigma,\rho,\omega}d\hat\psi^{(\le 0)-}_{\mathbf{k}',\sigma,\rho,\omega}\Big]\cdot \qquad (7.8)\\
&\cdot\exp\Big\{-(\beta|\Lambda|)^{-1}\sum_{\omega=\pm,\sigma=\uparrow\downarrow}\ \sum_{\mathbf{k}'\in\mathcal{B}^\omega_{\beta,L}}^{\chi_0(|\mathbf{k}'|)>0}\chi_0^{-1}(|\mathbf{k}'|)\hat\psi^{(\le 0)+}_{\mathbf{k}',\sigma,\cdot,\omega}A_{0,\omega}(\mathbf{k}')\hat\psi^{(\le 0)-}_{\mathbf{k}',\sigma,\cdot,\omega}\Big\}\ ,
\end{aligned}$$

where:

$$\begin{aligned}
A_{0,\omega}(\mathbf{k}') &= \begin{pmatrix} -ik_0 & -v\Omega^*(\vec k'+\vec p_F^{\ \omega})\\ -v\Omega(\vec k'+\vec p_F^{\ \omega}) & -ik_0\end{pmatrix} =\\
&= \begin{pmatrix} -i\zeta_0 k_0 + s_0(\mathbf{k}') & c_0(ik'_1-\omega k'_2)+t_{0,\omega}(\mathbf{k}')\\ v_0(-ik'_1-\omega k'_2)+t^*_{0,\omega}(\mathbf{k}') & -i\zeta_0k_0+s_0(\mathbf{k}')\end{pmatrix}\ ,
\end{aligned}$$

$\mathcal{N}_0$ is chosen in such a way that $\int P_{\chi_0,A_0}(d\psi^{(\le 0)})=1$, $\zeta_0=1$, $v_0=v$, $s_0:=0$ and $|t_{0,\omega}(\mathbf{k}')|\le C|\mathbf{k}'|^2$.

It is apparent that the $\psi^{(\le 0)}$ field has zero mass (i.e., its propagator decays polynomially at large distances in $\mathbf{x}$-space). Therefore, its integration requires an infrared multiscale analysis. As in the analysis of the ultraviolet problem, we define a sequence of geometrically decreasing momentum scales 2^h, $h = 0, -1, -2, \dots$ Correspondingly we introduce compact support functions $f_h(\mathbf{k}') = \chi_0(2^{-h}|\mathbf{k}'|) - \chi_0(2^{-h+1}|\mathbf{k}'|)$ and we rewrite

$$\chi_0(|\mathbf{k}'|) = \sum_{h=-\infty}^{0} f_h(\mathbf{k}') \ . \tag{7.9}$$

The purpose is to perform the integration of (7.5) in an iterative way. We step by step decompose the propagator into a sum of two propagators, the first supported on momenta $\sim 2^h$, $h \le 0$, the second supported on momenta smaller than 2^h. Correspondingly we rewrite the Grassmann field as a sum of two independent fields: $\psi^{(\le h)} = \psi^{(h)} + \psi^{(\le h-1)}$ and we integrate the field $\psi^{(h)}$. In this way we inductively prove that, for any $h \le 0$, eqn(7.5) can be rewritten as

$$\Xi_{M,\beta,L} = e^{-\beta|\Lambda|F_h} \int P_{\chi_h, A_h}(d\psi^{(\le h)}) e^{-\mathcal{V}^{(h)}(\psi^{(\le h)})} \ , \tag{7.10}$$

where $F_h, A_h, \mathcal{V}^{(h)}$ will be defined recursively, $\chi_h(|\mathbf{k}'|) = \sum_{k=-\infty}^{h} f_k(\mathbf{k}')$ and $P_{\chi_h,A_h}(d\psi^{(\le h)})$ is defined in the same way as $P_{\chi_0,A_0}(d\psi^{(\le 0)})$ with $\psi^{(\le 0)}, \chi_0$, $A_{0,\omega}, \zeta_0, v_0, s_0, t_{0,\omega}$ replaced by $\psi^{(\le h)}, \chi_h, A_{h,\omega}, \zeta_h, v_h, s_h, t_{h,\omega}$, respectively. Moreover $\mathcal{V}^{(h)}(0) = 0$ and

$$\begin{aligned} \mathcal{V}^{(h)}(\psi) = \sum_{n=1}^{\infty} (\beta|\Lambda|)^{-2n} \sum_{\underline{\sigma},\underline{\rho},\underline{\omega}} \sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_{2n}} \Big[\prod_{j=1}^{n} \hat\psi^{(\le h)+}_{\mathbf{k}'_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \hat\psi^{(\le h)-}_{\mathbf{k}'_{2j},\sigma_j,\rho_{2j},\omega_{2j}} \Big] \ \cdot \\ \cdot \hat W^{(h)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\dots,\mathbf{k}'_{2n-1}) \ \delta\Big(\sum_{j=1}^{2n} (-1)^j (\mathbf{p}_F^{\omega_j} + \mathbf{k}'_j)\Big) = \end{aligned} \tag{7.11}$$

$$= \sum_{n=1}^{\infty} \sum_{\underline{\sigma},\underline{\rho},\underline{\omega}} \int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} \Big[\prod_{j=1}^{n} \psi^{(\le h)+}_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \psi^{(\le h)-}_{\mathbf{x}_{2j},\sigma_j,\rho_{2j},\omega_{2j}} \Big] W^{(h)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\dots,\mathbf{x}_{2n}) \ .$$

Note that the field $\psi^{(\le h)}_{\mathbf{k}',\sigma,\cdot,\omega}$, whose propagator is given by $\chi_h(|\mathbf{k}'|)[A^{(h)}_\omega(\mathbf{k}')]^{-1}$, has the same support as χ_h, that is on a neighborood of size 2^h around the singularity $\mathbf{k}' = \mathbf{0}$ (that, in the original variables, corresponds to the Dirac point $\mathbf{k} = \mathbf{p}_F^\omega$). It is important for the following to think $\hat W^{(h)}_{2n,\underline{\rho},\underline{\omega}}$, $h \le 0$, as functions of the variables $\{\zeta_k, v_k\}_{h<k\le 0}$. The iterative construction below will inductively imply that the dependence on these variables is well defined. The iteration continues up to the scale h^* and the result of the last iteration will be $\Xi_{M,\beta,L} = e^{-\beta|\Lambda|F^{(M)}_{\beta,\Lambda}}$.

Localization and renormalization. In order to inductively prove (7.10) we write

$$\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)} \tag{7.12}$$

where

$$\mathcal{L}\mathcal{V}^{(h)} = \frac{1}{\beta|\Lambda|} \sum_{\sigma=\uparrow\downarrow} \sum_{\substack{\rho_1,\rho_2=1,2 \\ \omega=\pm}} \sum_{\mathbf{k}'}^{\chi_h(|\mathbf{k}'|)>0} \hat\psi^{(\le h)+}_{\mathbf{k}',\sigma,\rho_1,\omega} \hat\psi^{(\le h)-}_{\mathbf{k}',\sigma,\rho_2,\omega} \hat W^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{k}') \ , \tag{7.13}$$

and $\mathcal{R}\mathcal{V}^{(h)}$ is given by (7.11) with $\sum_{n=1}^{\infty}$ replaced by $\sum_{n=2}^{\infty}$, that is it contains only the monomials with four or more than four fields.

Note that in (7.13) the ω-index of the ψ fields is the same; this follows from the fact that in the terms with different ω's the momenta verify $\mathbf{k}'_1-\mathbf{k}'_2+\mathbf{p}_F^{\omega}-\mathbf{p}_F^{-\omega}=n_1\vec{b}_1+n_2\vec{b}_2$, for some choice of n_1,n_2, and such a condition cannot be verified if $\mathbf{k}'_1,\mathbf{k}'_2$ are in the support of the $\psi^{(\le h)}$ fields, as one can easily check.

Remark. The fact that the quadratic terms with different ω's, i.e., the one-particle *umklapp processes*, do not contribute to the infrared effective potential is a crucial fact, which reduces the number of *relevant running coupling constants* and, in particular, tells us that the interaction does not generate *mass terms*.

The symmetries of the action, which are described in Appendix B and are preserved by the iterative integration procedure, imply that, in the zero temperature and thermodynamic limit, $\hat{W}^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{0})=0$ and

$$\mathbf{k}'\partial_{\mathbf{k}'}\hat{W}^{(h)}_{2,(\rho_1,\rho_2),(\omega,\omega)}(\mathbf{0})=\begin{pmatrix}-iz_hk_0 & \delta_h(ik'_1-\omega k'_2)\\ \delta_h(-ik'_1-\omega k'_2) & -iz_hk_0\end{pmatrix}_{\rho_1,\rho_2}, \tag{7.14}$$

for suitable real constants z_h,δ_h. The proof of (7.14) is completely analogous to the proof of Lemma 6.2 and will not be belabored here.

Once that the above definitions are given, we can describe our iterative integration procedure for $h\le 0$. We start from (7.10) and we rewrite it as

$$\int P_{\chi_h,A_h}(d\psi^{(\le h)})\,e^{-\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)})-\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h)})-\beta|\Lambda|F_h}\,, \tag{7.15}$$

with

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)})=(\beta|\Lambda|)^{-1}\sum_{\omega,\sigma}\sum_{\mathbf{k}'}^{\chi_h(|\mathbf{k}'|)>0}\cdot \tag{7.16}$$
$$\cdot\hat{\psi}^{(\le h)+}_{\mathbf{k}',\sigma,\cdot,\omega}\begin{pmatrix}-iz_hk_0+\sigma_h(\mathbf{k}') & \delta_h(ik'_1-\omega k'_2)+\tau_{h,\omega}(\mathbf{k}')\\ \delta_h(-ik'_1-\omega k_2)+\tau^*_{h,\omega}(\mathbf{k}') & -iz_hk_0+\sigma_h(\mathbf{k}')\end{pmatrix}\hat{\psi}^{(\le h)-}_{\mathbf{k}',\sigma,\cdot,\omega}\,.$$

Then we include $\mathcal{L}\mathcal{V}^{(h)}$ in the fermionic integration, so obtaining

$$\int P_{\chi_h,\overline{A}_{h-1}}(d\psi^{(\le h)})\,e^{-\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h)})-\beta|\Lambda|(F_h+e_h)}\,, \tag{7.17}$$

where

$$e_h=\frac{1}{\beta|\Lambda|}\sum_{\omega,\sigma}\sum_{\mathbf{k}'}\sum_{n\ge1}\frac{(-1)^n}{n}\mathrm{Tr}\Big\{[\chi_h(\mathbf{k}')A^{-1}_{h,\omega}(\mathbf{k}')W^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{k}')]^n\Big\} \tag{7.18}$$

is a constant taking into account the change in the normalization factor of the measure and

$$\overline{A}_{h-1,\omega}(\mathbf{k}')=\begin{pmatrix}-i\overline{\zeta}_{h-1}k_0+\overline{s}_{h-1}(\mathbf{k}') & \overline{v}_{h-1}(ik'_1-\omega k'_2)+\overline{t}_{h-1,\omega}(\mathbf{k}')\\ \overline{v}_{h-1}(-ik'_1-\omega k'_2)+\overline{t}^*_{h-1,\omega}(\mathbf{k}') & -i\overline{\zeta}_{h-1}k_0+\overline{s}_{h-1}(\mathbf{k}')\end{pmatrix} \tag{7.19}$$

with:

$$\bar\zeta_{h-1}(\mathbf{k}') = \zeta_h + z_h\chi_h(\mathbf{k}')\;, \qquad \bar v_{h-1}(\mathbf{k}') = v_h + \delta_h\chi_h(\mathbf{k}')\;,$$
$$\bar s_{h-1}(\mathbf{k}') = s_h(\mathbf{k}') + \sigma_h(\mathbf{k}')\chi_h(\mathbf{k}')\;, \quad \bar t_{h-1,\omega}(\mathbf{k}') = t_{h,\omega}(\mathbf{k}') + \tau_{h,\omega}(\mathbf{k}')\chi_h(\mathbf{k}') \tag{7.20}$$

Now we can perform the integration of the $\psi^{(h)}$ field. We rewrite the Grassmann field $\psi^{(\le h)}$ as a sum of two independent Grassmann fields $\psi^{(\le h-1)} + \psi^{(h)}$ and correspondingly we rewrite (7.17) as

$$e^{-\beta|\Lambda|(F_h+e_h)}\int P_{\chi_{h-1},A_{h-1}}(d\psi^{(\le h-1)})\int P_{f_h,\bar A_{h-1}}(d\psi^{(h)})\,e^{-\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h-1)}+\psi^{(h)})}\;, \tag{7.21}$$

where

$$A_{h-1,\omega}(\mathbf{k}') = \begin{pmatrix} -i\zeta_{h-1}k_0 + s_{h-1}(\mathbf{k}') & v_{h-1}(ik_1' - \omega k_2') + t_{h-1,\omega}(\mathbf{k}') \\ v_{h-1}(-ik_1' - \omega k_2') + t^*_{h-1,\omega}(\mathbf{k}') & -i\zeta_{h-1}k_0 + s_{h-1}(\mathbf{k}') \end{pmatrix} \tag{7.22}$$

with:

$$\zeta_{h-1} = \zeta_h + z_h\;, \qquad v_{h-1} = v_h + \delta_h\;,$$
$$s_{h-1}(\mathbf{k}') = s_h(\mathbf{k}') + \sigma_h(\mathbf{k}')\;, \quad t_{h-1,\omega}(\mathbf{k}') = t_{h,\omega}(\mathbf{k}') + \tau_{h,\omega}(\mathbf{k}')\;. \tag{7.23}$$

The single scale propagator is

$$\int P_{f_h,\bar A_{h-1}}(d\psi^{(h)})\psi^{(h)-}_{\mathbf{x}_1,\sigma_1,\rho_1,\omega_1}\psi^{(h)+}_{\mathbf{x}_2,\sigma_2,\rho_2,\omega_2} = \delta_{\sigma_1,\sigma_2}\delta_{\omega_1,\omega_2}[g^{(h)}_\omega(\mathbf{x}_1,\mathbf{x}_2)]_{\rho_1,\rho_2}\;, \tag{7.24}$$

where

$$g^{(h)}_\omega(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{D}^\omega_{\beta,L}} e^{-i\mathbf{k}'(\mathbf{x}_1-\mathbf{x}_2)}f_h(\mathbf{k}')\Big[\bar A_{h-1,\omega}(\mathbf{k}')\Big]^{-1}\;. \tag{7.25}$$

After the integration of the field on scale h we are left with an integral involving the fields $\psi^{(\le h-1)}$ and the new effective interaction $\mathcal{V}^{(h-1)}$, defined as

$$e^{-\mathcal{V}^{(h-1)}(\psi^{(\le h-1)})-\bar e_h\beta|\Lambda|} = \int P_{f_h,\bar A_{h-1}}(d\psi^{(h)})\,e^{-\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h-1)}+\psi^{(h)})}\;, \tag{7.26}$$

with $\mathcal{V}^{(h-1)}(0) = 0$. It is easy to see that $\mathcal{V}^{(h-1)}$ is of the form (7.11) and that $F_{h-1} = F_h + e_h + \bar e_h$. It is sufficient to use the identity

$$\bar e_h + \mathcal{V}^{(h-1)}(\psi^{(\le h-1)}) = \sum_{n\ge1}\frac{1}{n!}(-1)^{n+1}\mathcal{E}^T_h(\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h-1)}+\psi^{(h)});n)\;, \tag{7.27}$$

where $\mathcal{E}^T_h(X(\psi^{(h)});n)$ is the truncated expectation of order n w.r.t. the propagator $g^{(h)}_\omega$, which is the analogue of (6.8) with $\psi^{(u.v.)}$ replaced by $\psi^{(h)}$ and with $P(d\psi^{(u.v.)})$ replaced by $P_{f_h,\bar A_{h-1}}(d\psi^{(h)})$.

Note that the above procedure allows us to write the *effective constants* (ζ_h, v_h), $h \le 0$, in terms of (ζ_k, v_k), $h < k \le 0$, namely $\zeta_{h-1} = \beta^\zeta_h((\zeta_h, v_h), \ldots, (\zeta_0, v_0))$ and

$v_{h-1} = \beta_h^v((\zeta_h, v_h), \ldots, (\zeta_0, v_0))$, where $\beta_h^{\#}$ is the so–called *Beta function*.

An iterative implementation of (7.27) leads to a representation of $\mathcal{V}^{(h)}(\psi^{(\le h)})$, $h < 0$, in terms of a new tree expansion. The set of trees of order n contributing to $\mathcal{V}^{(h)}(\psi^{(\le h)})$ is denoted by $\mathcal{T}_{h,n}$. The trees in $\mathcal{T}_{h,n}$ are defined in a way very similar to those in $\mathcal{T}_{M;h,n}$, but for the following differences: (i) the scale labels of the vertices of $\tau \in \mathcal{T}_{h,n}$ are between $h+1$ and 1; (ii) with each endpoint v we associate one of the monomials with four or more Grassmann fields contributing to $\mathcal{R}\mathcal{V}^{(0)}(\psi^{(\le h_v - 1)})$, corresponding to the terms with $n \ge 2$ in the r.h.s. of (7.6) (with $\psi^{(\le 0)}$ replaced by $\psi^{(\le h_v-1)}$). In terms of these trees, the effective potential $\mathcal{V}^{(h)}$, $h \le -1$, can be written as

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) + \beta|\Lambda|\bar{e}_{k+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) \;, \tag{7.28}$$

where, if v_0 is the first vertex of τ and $\tau_1, \ldots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$ is defined inductively as follows:
i) if $s > 1$, then

$$\mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}_{h+1}^T[\bar{\mathcal{V}}^{(h+1)}(\tau_1, \psi^{(\le h+1)}); \ldots; \bar{\mathcal{V}}^{(h+1)}(\tau_s, \psi^{(\le h+1)})] \;, \tag{7.29}$$

where $\bar{\mathcal{V}}^{(h+1)}(\tau_i, \psi^{(\le h+1)})$ is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \psi^{(\le h+1)})$ if the subtree τ_i contains more than one end-point, or if it contains one end-point but it is not a trivial subtree; it is equal to $\mathcal{R}\mathcal{V}^{(0)}(\tau_i, \psi^{(\le h+1)})$ if τ_i is a trivial subtree;
ii) if $s = 1$, then $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$ is equal to $\mathcal{E}_{h+1}^T[\mathcal{R}\mathcal{V}^{(h+1)}(\tau_1, \psi^{(\le h+1)})]$ if τ_1 is not a trivial subtree; it is equal to $\mathcal{E}_{h+1}^T[\mathcal{R}\mathcal{V}^{(0)}(\psi^{(\le h+1)}) - \mathcal{R}\mathcal{V}^{(0)}(\psi^{(\le h)})]$ if τ_1 is a trivial subtree.

Repeating step by step the discussion leading to eqns(6.31), (6.34) and (6.35), and using analogous definitions, we find that

$$\mathcal{V}^{(h)}(\tau, \mathbf{P}) = \sum_{T \in \mathbf{T}} \int d\mathbf{x}_{v_0} \widetilde{\Psi}^{(\le h)}(P_{v_0}) W_{\tau,\mathbf{P},T}^{(h)}(\mathbf{x}_{v_0}) := \sum_{T \in \mathbf{T}} \mathcal{V}^{(h)}(\tau, \mathbf{P}, T) \;, \tag{7.30}$$

where

$$W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) = \left[\prod_{i=1}^n K_{v_i^*}^{(h_i)}(\mathbf{x}_{v_i^*})\right] \cdot \tag{7.31}$$
$$\cdot \left\{ \prod_{\substack{v \\ \text{not e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{h_v,T_v}(\mathbf{t}_v) \left[\prod_{l \in T_v} \delta_{\omega_l^-,\omega_l^+} \delta_{\sigma_l^-,\sigma_l^+} \, [g_{\omega_l}^{(h_v)}(\mathbf{x}_l - \mathbf{y}_l)]_{\rho_l^-,\rho_l^+} \right] \right\} .$$

In the eqn(7.31): v_i^*, $i = 1, \ldots, n$, are the endpoints of τ; $K_{v_i^*}^{(h_i)}(\mathbf{x}_{v_i^*})$ is the kernel of one of the monomials contributing to $\mathcal{R}\mathcal{V}^{(0)}(\psi^{(\le h_v)})$; $G^{h,T}$ is a matrix with elements

$$G_{ij,i'j'}^{h,T} = t_{ii'} \delta_{\omega_l^-,\omega_l^+} \delta_{\sigma_l^-,\sigma_l^+} \, [g_{\omega_l}^{(h)}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'})]_{\rho_l^-,\rho_l^+} \;, \tag{7.32}$$

Once again, it is important to note that $G^{h,T}$ is a Gram matrix, i.e., defining $\mathbf{e}_+ = \mathbf{e}_\uparrow = (1,0)$ and $\mathbf{e}_- = \mathbf{e}_\downarrow = (0,1)$, the matrix elements in (7.32) can be written in terms of scalar products:

$$t_{ii'}\delta_{\omega_l^-,\omega_l^+}\delta_{\sigma_l^-,\sigma_l^+}\,[g^{(h)}_{\omega_l}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'})]_{\rho_l^-,\rho_l^+} = \tag{7.33}$$
$$= \Big(\mathbf{u}_i\otimes\mathbf{e}_{\omega_l^-}\otimes\mathbf{e}_{\sigma_l^-}\otimes\mathbf{A}_{\rho_l^-}(\mathbf{x}_{ij}-\cdot),\mathbf{u}_{i'}\otimes\mathbf{e}_{\omega_l^+}\otimes\mathbf{e}_{\sigma_l^+}\otimes\mathbf{B}_{\rho_l^+}(\mathbf{x}_{i'j'}-\cdot)\Big)\equiv(\mathbf{f}_\alpha,\mathbf{g}_\beta)$$

where

$$[\mathbf{A}_\rho(\mathbf{x})]_i = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{B}^\omega_{\beta,L}} e^{-i\mathbf{k}'\mathbf{x}}\sqrt{f_h(\mathbf{k}')}\,\delta_{i,\rho}\ ,$$
$$[\mathbf{B}_\rho(\mathbf{x})]_i = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{B}^\omega_{\beta,L}} e^{-i\mathbf{k}'\mathbf{x}}\sqrt{f_h(\mathbf{k}')}\Big[\overline{A}_{h-1,\omega}(\mathbf{k}')\Big]^{-1}_{i\rho}\ . \tag{7.34}$$

The symbol $(\cdot,\cdot)$ denotes the inner product, i.e.,

$$(\mathbf{u}_i\otimes\mathbf{e}_\omega\otimes\mathbf{e}_\sigma\otimes\mathbf{A}_\rho(\mathbf{x}-\cdot),\mathbf{u}_{i'}\otimes\mathbf{e}_{\omega'}\otimes\mathbf{e}_{\sigma'}\otimes\mathbf{B}_{\rho'}(\mathbf{x}'-\cdot)) =$$
$$= (\mathbf{u}_i\cdot\mathbf{u}_{i'})\,(\mathbf{e}_\omega\cdot\mathbf{e}_{\omega'})\,(\mathbf{e}_\sigma\cdot\mathbf{e}_{\sigma'})\cdot\int d\mathbf{z}\mathbf{A}^\dagger(\mathbf{x}-\mathbf{z})\mathbf{B}(\mathbf{x}'-\mathbf{z})\ , \tag{7.35}$$

and the vectors $\mathbf{f}_\alpha,\mathbf{g}_\beta$ are implicitely defined by (7.33). As we already saw in the previous sections, the usefulness of the representation (7.33) is that, by the Gram-Hadamard inequality, $|\det(\mathbf{f}_\alpha,\mathbf{g}_\beta)|\le\prod_\alpha ||f_\alpha||\,||g_\alpha||$. In our case, $||\mathbf{f}_\alpha||\le C2^{3h/2}$ and $||\mathbf{g}_\alpha||\le C2^{h/2}$. Therefore, $||f_\alpha||\,||g_\alpha||\le C2^{2h}$, uniformly in α, so that the Gram determinant can be bounded by C^D2^{2hD}, where D is the dimension of $G^{h,T}$.

The main result of this section is summarized in the following theorem, which is the analogue of Lemma 6.1 and which easily implies Theorem 2.1.

Theorem 7.1 *There exists a constants $U_0>0$, independent of M, β and L, such that the kernels $W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})$ in (7.11), $h\le-1$, are analytic functions of U in the complex domain $|U|\le U_0$, satisfying, for any $0\le\theta<1$ and a suitable constant $C_\theta>0$ (independent of M,β,L), the following estimates:*

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})|\le 2^{h(3-2l+\theta)}\,(C_\theta\,|U|)^{max(1,l-1)}\ . \tag{7.36}$$

Moreover, the constants e_h and $\overline{e}_h$ defined by (7.18) and (7.27) are analytic functions of U in the same domain $|U|\le U_0$, and there they satisfy the estimate $|e_h|+|\overline{e}_h|\le C_\theta|U|2^{h(3+\theta)}$.

Proof of Theorem 2. Let us preliminarily assume that, for $h'\le h\le-1$, and for suitable constants c,c_n, the corrections $z_h,\delta_h,\sigma_h(\mathbf{k}')$ and $\tau_h(\mathbf{k}')$ defined in (7.14) and (7.16), satisfy the following estimates:

$$\max\{|z_h|,|\delta_h|\}\le c|U|2^{\theta h}\ , \tag{7.37}$$
$$\sup_{2^{h'-1}\le|\mathbf{k}'|\le 2^{h'+1}}\{||\partial^n_{\mathbf{k}'}\sigma_h(\mathbf{k}')||,||\partial^n_{\mathbf{k}'}\tau_{h,\omega}(\mathbf{k}')||\}\le c_n|U|2^{2(h'-h)}2^{(1+\theta-n)h}\ .$$

Using (7.37) we inductively see that the running coupling functions $\zeta_h, v_h, s_h(\mathbf{k}')$ and $t_h(\mathbf{k}')$ satisfy similar estimates:

$$\max\{|\zeta_h - 1|, |v_h - v_0|\} \le c|U| \;, \tag{7.38}$$
$$\sup_{2^{h'-1}\le|\mathbf{k}'|\le 2^{h'+1}} \{||\partial^n_{\mathbf{k}'} s_h(\mathbf{k}')||, ||\partial^n_{\mathbf{k}'}(t_{h,\omega}(\mathbf{k}') - t_{0,\omega}(\mathbf{k}'))||\} \le c_n|U|2^{2(h'-h)}2^{(1+\theta-n)h} \;.$$

Now, using the definition of $g^{(h)}_\omega$, see (7.25) and (7.19), we get, after integration by parts, for any $N \ge 0$,

$$\|[g^{(h)}_\omega(\mathbf{x}_1,\mathbf{x}_2)]_{\rho,\rho'}\| \le C_N \frac{2^{2h}}{1+(2^h||\mathbf{x}_1-\mathbf{x}_2||)^N} \;, \tag{7.39}$$

where C_N is a suitable constant and $||\mathbf{x}_1 - \mathbf{x}_2||$ is the distance on the torus.

Using the tree expansion described above, we find that the l.h.s. of (7.36) can be bounded from above by

$$\sum_{n\ge1}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}\int\prod_{l\in T^*} d(\mathbf{x}_l-\mathbf{y}_l)\left[\prod_{i=1}^n |K^{(h_i)}_{v^*_i}(\mathbf{x}_{v^*_i})|\right]\cdot \tag{7.40}$$
$$\cdot\left[\prod_{v\ \text{not e.p.}}\frac{1}{s_v!}\max_{\mathbf{t}_v}|\det G^{h_v,T_v}(\mathbf{t}_v)|\prod_{l\in T_v}||g^{(h_v)}_{\omega_l}(\mathbf{x}_l-\mathbf{y}_l)||\right]$$

where $||\cdot||$ is the spectral norm and where T^* is a tree graph obtained from $T = \cup_v T_v$, by adding in a suitable (obvious) way, for each endpoint v^*_i, $i = 1,\dots,n$, one or more lines connecting the space-time points belonging to $\mathbf{x}_{v^*_i}$.

An application of the Gram–Hadamard inequality, combined with the dimensional bound on $g^{(h)}_\omega(\mathbf{x})$ given by (7.39), see the remark after (7.32), implies that

$$|\det G^{h_v,T_v}(\mathbf{t}_v)| \le (\text{const.})^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)}\cdot 2^{h_v\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)} \;. \tag{7.41}$$

By the decay properties of $g^{(h)}_\omega(\mathbf{x})$ given by (7.39), it also follows that

$$\prod_{v\ \text{not e.p.}}\frac{1}{s_v!}\int\prod_{l\in T_v} d(\mathbf{x}_l-\mathbf{y}_l)\,||g^{(h_v)}_{\omega_l}(\mathbf{x}_l-\mathbf{y}_l)|| \le c^n \prod_{v\ \text{not e.p.}}\frac{1}{s_v!}2^{-h_v(s_v-1)} \;. \tag{7.42}$$

The bound (6.11) on the kernels produced by the ultraviolet integration implies that

$$\int\prod_{l\in T^*\setminus\cup_v T_v} d(\mathbf{x}_l-\mathbf{y}_l)\prod_{i=1}^n|K^{(h_i)}_{v^*_i}(\mathbf{x}_{v^*_i})| \le \prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1} \;, \tag{7.43}$$

where $p_i = |P_{v^*_i}|$. Combining the previous bounds, we find that (7.40) can be bounded from above by

$$\sum_{n\ge1}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\sum_{T\in\mathbf{T}} C^n\Big[\prod_{v\ \text{not e.p.}}\frac{1}{s_v!}2^{h_v\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-3(s_v-1)\right)}\Big]\Big[\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\Big] \tag{7.44}$$

Let us recall that $n(v)=\sum_{i:v_i^*>v} 1$ is the number of endpoints following v on τ, that v' is the non trivial vertex immediately preceding v on τ and that $|I_v|$ is the number of field labels associated to the endpoints following v on τ (note that $|I_v|\geq 4n(v)$). Using the fact that

$$\begin{aligned}
&\sum_{v\text{ not e.p.}}\Big[(\sum_{i=1}^{s_v}|P_{v_i}|)-|P_v|\Big]=|I_{v_0}|-|\,P_{v_0}|\;,\\
&\sum_{v\text{ not e.p.}}(s_v-1)=n-1\;,\\
&\sum_{v\text{ not e.p.}}(h_v-h)\Big[(\sum_{i=1}^{s_v}|P_{v_i}|)-|P_v|\Big]=\sum_{v\text{ not e.p.}}(h_v-h_{v'})(|I_v|-|\,P_v|)\;,\\
&\sum_{v\text{ not e.p.}}(h_v-h)(s_v-1)=\sum_{v\text{ not e.p.}}(h_v-h_{v'})(n(v)-1)\;,
\end{aligned}\tag{7.45}$$

we find that (7.44) can be bounded above by

$$\begin{aligned}
&\sum_{n\geq 1}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}C^n 2^{h(3-|P_{v_0}|+|I_{v_0}|-3n)}\cdot\\
&\qquad\cdot\Big[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}2^{(h_v-h_{v'})(3-|P_v|+|I_v|-3n(v))}\Big]\Big[\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\Big]\;.
\end{aligned}\tag{7.46}$$

Using the identities

$$\begin{aligned}
&2^{hn}\prod_{v\text{ not e.p.}}2^{(h_v-h_{v'})n(v)}=\prod_{v\text{ e.p.}}2^{h_{v'}}\;,\\
&2^{h|I_{v_0}|}\prod_{v\text{ not e.p.}}2^{(h_v-h_{v'})|I_v|}=\prod_{v\text{ e.p.}}2^{h_{v'}|I_v|}\;,
\end{aligned}\tag{7.47}$$

we obtain

$$\begin{aligned}
&\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})|\leq\sum_{n\geq 1}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\sum_{T\in\mathbf{T}}C^n 2^{h(3-|P_{v_0}|)}\cdot\\
&\cdot\Big[\prod_{v\text{ not e.p.}}\frac{1}{s_v!}2^{-(h_v-h_{v'})(|P_v|-3)}\Big]\Big[\prod_{v\text{ e.p.}}2^{h_{v'}(|I_v|-3)}\Big]\Big[\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\Big]\;.
\end{aligned}\tag{7.48}$$

Note that, if v is not an endpoint, $|P_v|-3\geq 1$ by the definition of $\mathcal{R}$. Moreover, if v is an endpoint, $|I_v|-3\geq 1$; in particular, we get

$$\prod_{v\text{ e.p.}}2^{h_{v'}(|I_v|-3)}\leq 2^{\overline{h}-1}\;,\tag{7.49}$$

with $\overline{h}$ the highest scale label of the tree. Now, note that the number of terms in $\sum_{T\in\mathbf{T}}$ can be bounded by $C^n\prod_{v\text{ not e.p.}}s_v!$. Using also that $|P_v|-3\geq 1$ and $|P_v|-3\geq|P_v|/4$, we find that the l.h.s. of (7.48) can be bounded as

$$\begin{aligned}
&\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})|\leq\gamma^{h(3-|P_{v_0}|)}\sum_{n\geq 1}C^n\sum_{\tau\in\mathcal{T}_{h,n}}2^{\overline{h}-1}\cdot\\
&\cdot(\prod_{v\text{ not e.p.}}2^{-\theta(h_v-h_{v'})}2^{-(1-\theta)(h_v-h_{v'})/2})\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}(\prod_{v\text{ not e.p.}}2^{-(1-\theta)|P_v|/8})\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\;.
\end{aligned}\tag{7.50}$$

Proceeding as in the previous section, we get:

$$\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}} \Big(\prod_{v\ \text{not e.p.}} 2^{-(1-\theta)|P_v|/8}\Big)\prod_{i=1}^{n} C^{p_i}|U|^{\frac{p_i}{2}-1} \leq C_\theta^n |U|^n \;.$$

Finally, using that $\gamma^{h_*}\prod_{v\ \text{not e.p.}} 2^{-\theta(h_v-h_{v'})}\leq 2^{\theta h}$, and that, for $0<\theta<1$,

$$\sum_{\tau\in\mathcal{T}_{h,n}}\prod_{v\ \text{not e.p.}} 2^{-(1-\theta)(h_v-h_{v'})/2}\leq C^n \;,$$

as it follows by the fact that the number of non trivial vertices in τ is smaller than $n-1$ and that the number of trees in $\mathcal{T}_{h,n}$ is bounded by const^n, and collecting all the previous bounds, we obtain

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})|\leq 2^{h(3-|P_{v_0}|+\theta)}\sum_{n\geq 1}C^n|U|^n \;, \tag{7.51}$$

which is the desired result.

We now need to prove the assumption (7.37). We proceed by induction. The assumption is valid for $h=0$, as it follows by (6.11) and by the discussion in the previous section. Now, assume that (7.37) is valid for all $h\geq k+1$, and let us prove it for $k-1$. The functions $-iz_k k_0+\sigma_k(\mathbf{k}')$ and $\delta_k(ik_1'-\omega k_2')+\tau_{k,\omega}(\mathbf{k}')$ admit a representation in terms of $W^{(k)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{x},\mathbf{y})$. In particular,

$$\max\{|z_k|,|\delta_k|\}\leq\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1 d\mathbf{x}_2||\mathbf{x}-\mathbf{y}||\,|W^{(k)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{x},\mathbf{y})| \;, \tag{7.52}$$

and

$$\begin{aligned}\sup_{2^{h'-1}\leq|\mathbf{k}'|\leq 2^{h'+1}}&\{||\partial^n_{\mathbf{k}'}\sigma_k(\mathbf{k}')||,||\partial^n_{\mathbf{k}'}\tau_{k,\omega}(\mathbf{k}')||\}\leq\\ &\leq\frac{C2^{2h'}}{\beta|\Lambda|}\int d\mathbf{x}_1 d\mathbf{x}_2||\mathbf{x}-\mathbf{y}||^{n+2}|W^{(k)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{x},\mathbf{y})| \;.\end{aligned} \tag{7.53}$$

The same proof leading to (7.51) shows that the r.h.s. of (7.52) can be bounded by the r.h.s. of (7.51) times 2^{-k} (that is the dimensional estimate for $||\mathbf{x}-\mathbf{y}||$), and that the r.h.s. of (7.52) can be bounded by the r.h.s. of (7.51) times $2^{2h'}2^{-(n+2)k}$ (where $2^{-k(n+2)}$ is the dimensional estimate for $||\mathbf{x}-\mathbf{y}||^{n+2}$).

It remains to prove the estimates on $e_h,\bar{e}_h$. The bound on $\bar{e}_h$ is an immediate corollary of the discussion above, simply because $\bar{e}_h$ can be bounded by (7.40) with $l=0$. Finally, remember that e_h is given by (7.18): an explicit computation of $A^{-1}_{h,\omega}(\mathbf{k}')W^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{k}')$ and the use of (7.37)-(7.38) imply that $||A^{-1}_{h,\omega}(\mathbf{k}')W^{(h)}_{2,\underline{\rho},(\omega,\omega)}(\mathbf{k}')||\leq C|U|2^{\theta h}$, from which: $|e_h|\leq C'2^{3h}\sum_{n\geq1}(C|U|2^{\theta h})^n$, as desired. ∎

As already mentioned above, Theorem 7.1 immediately implies the analyticity of the specific free energy $f_\beta(U)$ and of its zero temperature limit $e(U)$. In fact, by construction, $f_\beta(U) = F_0 + \sum_{h=h_\beta}^{0}(e_h + \bar{e}_h)$, with F_0 an analytic function of U, see the discussion after (6.10) and in previous section. This concludes the proof of Theorem 2.1.

8. CONCLUSIONS

We presented a self-contained proof of the analyticity of the specific free energy and ground state energy of the 2D Hubbard model on the honeycomb lattice. The proof is based on rigorous fermionic RG methods and can be extend to the construction of the interacting correlations, i.e., the off-diagonal elements of the reduced density matrices of the system. The construction shows that the interacting correlations decay to zero at infinity with the same decay exponents as those of the non-interacting case. The "only" effect of the interactions is to change by a finite amount the quasi-particle weight Z^{-1} at the Fermi surface and the fermi velocity v.

The example presented here is the only known example of a realistic 2D interacting Fermi system for which the ground state (including the correlations) can be constructed. The main difference with respect to other more standard 2D Fermi systems is the fact that here, at half-filling, the Fermi surface reduces to a set of two isolated points. This fact dramatically improves the infrared scaling properties of the theory: the four-fermions interaction, rather than being marginal, as in many other similar cases, is irrelevant; this is the technical fact that makes the construction of the ground state possible and "relatively easy".

It is natural to ask how the system behaves in the presence of Coulomb interactions among the fermions, which is he case of interest for applications to real graphene samples. In the latter case, the system has many analogies with $(2+1)$-dimensional QED. The four-fermions interaction, rather than being irrelevant, is marginal, and the fixed point of the theory is expected to be non-trivial. The long distance decay of correlations is expected to be described in terms of anomalous critical exponents and the effective Fermi velocity is expected to grow up to the maximal possible value, i.e., the speed of light. All these claims have been proved so far order by order in renormalized perturbation theory, but a full construction of the theory is still missing.

Appendix A: Truncated expectations and determinants

In this Appendix we prove eqn(5.3).

Given s set of indices $P_1, \ldots, P_s$, consider the quantity $\mathcal{E}^T(\widetilde{\psi}(P_1), \ldots, \widetilde{\psi}(P_s))$. Define

$$P_j^{\pm} = \{f \in P_j \ : \ \varepsilon(f) = \pm\} \tag{A.1}$$

and set $f = (j, i)$ for $f \in P_j^{\pm}$, with $i = 1, \ldots, |P_j^{\pm}|$. Note that $\sum_{j=1}^s |P_j^+| = \sum_{j=1}^s |P_j^-|$, otherwise the considered truncated expectation is vanishing.

Define

$$\mathcal{D}\psi = \prod_{j=1}^s \Big[\prod_{f \in P_j^+} \mathrm{d}\psi^+_{\mathbf{x}(f),\sigma(f),\rho(f)} \Big] \Big[\prod_{f \in P_j^-} \mathrm{d}\psi^-_{\mathbf{x}(f),\sigma(f),\rho(f)} \Big] \, , \tag{A.2}$$

$$(\psi^+, G\psi^-) = \sum_{j,j'=1}^s \sum_{i=1}^{|P_j^+|} \sum_{i'=1}^{|P_{j'}^+|} \psi^+_{(j',i')} G_{(j,i),(j',i')} \psi^-_{(j,i)} \, , \tag{A.3}$$

where $\psi^{\pm}_{(j,i)} := \psi^{\pm}_{\mathbf{x}(j,i),\sigma(j,i),\rho(j,i)}$ and, if $n = \sum_{j=1}^s \left|P_j^+\right| = \sum_{j=1}^s \left|P_j^-\right|$, then G is the $n \times n$ matrix with entries

$$G_{(j,i),(j',i')} := \delta_{\sigma(j,i),\sigma(j',i')} g_{\rho(j,i),\rho(j',i')}(\mathbf{x}(j,i) - \mathbf{x}(j',i')) \, . \tag{A.4}$$

Then one has

$$\mathcal{E}\left(\prod_{j=1}^s \widetilde{\psi}(P_j) \right) = \det G = \int \mathcal{D}\psi \, \exp\Big[- \Big(\psi^+, G\psi^- \Big) \Big] \, . \tag{A.5}$$

Setting $X := \{1, \ldots, s\}$ and

$$\overline{V}_{jj'} = \sum_{i=1}^{|P_j^-|} \sum_{i'=1}^{|P_{j'}^+|} \psi^+_{(j',i')} G_{(j,i),(j',i')} \psi^-_{(j,i)} \, , \tag{A.6}$$

we write

$$V(X) = \sum_{j,j' \in X} \overline{V}_{jj'} = \sum_{j \le j'} V_{jj'} \, , \tag{A.7}$$

so defining the quantity $V_{jj'}$ as

$$V_{jj'} = \begin{cases} \overline{V}_{jj'} \, , & \text{if } j = j' \, , \\ \overline{V}_{jj'} + \overline{V}_{j'j} \, , & \text{if } j < j' \, . \end{cases} \tag{A.8}$$

then eqn(A.5) can be written as

$$\mathcal{E}\left(\prod_{j=1}^s \widetilde{\psi}(P_j) \right) = \det G = \int \mathcal{D}\psi \, e^{-V(X)} \, . \tag{A.9}$$

We now want to express the last expression in terms of the functions W_X, defined as follows:

$$W_X(X_1, \ldots, X_r; t_1, \ldots, t_r) = \sum_{\ell} \prod_{k=1}^r t_k(\ell) \, V_\ell \, , \tag{A.10}$$

where:

1. X_k are subsets of X with $|X_k| = k$, inductively defined as:

$$\begin{cases} X_1 = \{1\} \ , \\ X_{k+1} \supset X_k \ , \end{cases} \tag{A.11}$$

2. $\ell = (jj')$ is a pair of elements $j, j' \in X$ and the sum in eqn(A.10) is over all the possible pairs (jj'),

3. the functions $t_k(\ell)$ are defined as follows:

$$t_k(\ell) = \begin{cases} t_k \ , & \text{if } \ell \sim \partial X_k \ , \\ 1 \ , & \text{otherwise} \ , \end{cases} \tag{A.12}$$

   where $\ell \sim X_k$ means that $\ell = (jj')$ "intersects the boundary" of X_k, *i.e.*connects a point in P_j, $j \in X_k$, to a point in $P_{j'}$, $j' \notin X_k$. See Fig. 4.

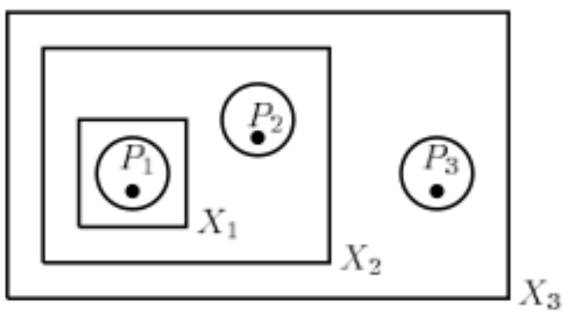

FIG. 4. Graphical representation of the sets X_k, k=1,2,3. In the example X_1={1}, X_2={1,2} and X_3={1,2,3}. The ℓ=(1 3) intersects the boundaries of X_1 and of X_2.

From definition eqn(A.10 it follows:

$$W_X(X_1; t_1) = \sum_{j=2}^{s} t_1 V_{1j} + V_{11} + \sum_{1<j'\leq j} V_{j'j} = (1-t_1)\,[V(X_1) + V(X \setminus X_1)] + t_1 V(X) \tag{A.13}$$

so that

$$\begin{aligned} e^{-V(X)} &= \int_0^1 \mathrm{d}t_1 \left[\frac{\partial}{\partial t_1}\, e^{-W_X(X_1;t_1)}\right] + e^{-W_X(X_1;0)} \\ &= -\sum_{\ell_1 \sim \partial X_1} V_{\ell_1} \int_0^1 \mathrm{d}t_1\, e^{-W_X(X_1;t_1)} + e^{-W_X(X_1;0)} \ . \end{aligned} \tag{A.14}$$

Again by definition we have:

$W_X(X_1, X_2; t_1, t_2) =$

$$= V_{11} + t_1 V_{12} + t_1 t_2 \sum_{j=3}^{s} V_{1j} + V_{22} + t_2 \sum_{j=3}^{s} V_{2j} + \sum_{2<j'\leq j} V_{j'j} = \tag{A.15}$$

$$= t_1 t_2 \sum_{j=2}^{s} V_{1j} + t_2 V_{11} + t_2 \sum_{1<j'\leq j} V_{j'j} + (1-t_2)\left[V_{11} + t_1 V_{12} + V_{22} + \sum_{2<j'\leq j} V_{j'j}\right] =$$

$$= t_2 W_X(X_1;t_1) + (1-t_2)\left[W_{X_2}(X_1;t_1) + V(X\setminus X_2)\right]$$

If we define $X_2 := X_1 \cup \ell_1$, i.e., $X_2 = \{1, \text{point connected by } \ell_1 \text{ with } 1\}$, then:

$$e^{-W_X(X_1;t_1)} = \int_0^1 \mathrm{d}t_2 \left[\frac{\partial}{\partial t_2} e^{-W_X(X_1,X_2;t_1,t_2)}\right] + e^{-W_X(X_1,X_2;t_1,0)} \tag{A.16}$$

$$= -\sum_{\ell_2\sim\partial X_2} V_{\ell_2} \int_0^1 \mathrm{d}t_2\, t_1(\ell_2)\, e^{-W_X(X_1,X_2;t_1,t_2)} + e^{-W_X(X_1,X_2;t_1,0)} \,.$$

Substituting (A.16) into (A.14) we get:

$$e^{-V(X)} = \sum_{\ell_1\sim\partial X_1} \sum_{\ell_2\sim\partial X_2} \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2\, (-1)^2\, V_{\ell_1} V_{\ell_2}\, t_1(\ell_2)\, e^{-W_X(X_1,X_2;t_1,t_2)}$$

$$+ \sum_{\ell_1\sim\partial X_1} \int_0^1 \mathrm{d}t_1\, (-1)\, V_{\ell_1}\, e^{-W_X(X_1,X_2;t_1,0)} + e^{-W_X(X_1;0)} \,. \tag{A.17}$$

A relation generalizing eqn(A.15) holds:

$$W_X(X_1,\ldots,X_{p+1};t_1,\ldots,t_{p+1}) = t_{p+1} W_X(X_1,\ldots,X_p;t_1,\ldots,t_p) +$$
$$(1-t_{p+1})\left[W_{X_{p+1}}(X_1,\ldots,X_p;t_1,\ldots,t_p) + V(X\setminus X_{p+1})\right] \tag{A.18}$$

where $p < s$. In fact in the sum over ℓ in eqn(A.10 we can distinguish two cases: either $\ell \sim X_{p+1}$ or $\ell \not\sim X_{p+1}$. In the former case V_ℓ is necessarily multiplied by t_{p+1} and, if $\ell = (j'j)$, $j' \leq p+1$, $j > p+1$; in the latter case V_ℓ is not multiplied by t_{p+1} and either $j', j \leq p+1$ or $j', j > p+1$. Then, clearly:

$$W_X(X_1,\ldots,X_{p+1};t_1,\ldots,t_{p+1}) = t_{p+1}\Big[W_X(X_1,\ldots,X_p;t_1,\ldots,t_p)$$
$$-W_{X_{p+1}}(X_1,\ldots,X_p;t_1,\ldots,t_p) - W_{X\setminus X_{p+1}}(X_1,\ldots,X_p;t_1,\ldots,t_p)\Big] \tag{A.19}$$
$$+W_{X_{p+1}}(X_1,\ldots,X_p;t_1,\ldots,t_p) + W_{X\setminus X_{p+1}}(X_1,\ldots,X_p;t_1,\ldots,t_p)$$

that is equivalent to eqn(A.18). We can iterate the procedure followed to get eqn(A.14) and eqn(A.17). In the general case we find:

$$e^{-V(X)} = \sum_{r=0}^{s-1} \sum_{\ell_1\sim\partial X_1} \cdots \sum_{\ell_r\sim\partial X_r} \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_r\, (-1)^r\, V_{\ell_1}\ldots V_{\ell_r}$$
$$\left(\prod_{k=1}^{r-1} t_1(\ell_{k+1})\ldots t_k(\ell_{k+1})\right) e^{-W_X(X_1,\ldots,X_{r+1};t_1,\ldots,t_r,0)} \,, \tag{A.20}$$

where the meaningless factors must be replaced by 1. Moreover, from eqn(A.18) we soon realize that

$$W_X(X_1,\ldots,X_s;t_1,\ldots,t_{s-1},0) = W_X(X_1,\ldots,X_{s-1};t_1,\ldots,t_{s-1}) \tag{A.21}$$
$$W_X(X_1,\ldots,X_r;t_1,\ldots,t_{r-1},0) = W_{X_r}(X_1,\ldots,X_{r-1};t_1,\ldots,t_{r-1}) + V(X\setminus X_r) \,.$$

The last equation holds for $r > 1$. If $r = 1$:

$$W_X(X_1; 0) = V(X_1) + V(X \setminus X_1) \tag{A.22}$$

Let T be a tree graph connecting $X_1, \ldots, X_r$, such that:

1. for all $k = 1, \ldots, r$, T is "anchored" to some point (j, i), *i.e.* T contains a line incident with (j, i), where $j \in X_k$ and $i \in \{1, \ldots, |P_j^{\pm}|\}$,
2. each line $\ell \in T$ intersects at least one boundary ∂X_k,
3. the lines $\ell_1, \ell_2, \ldots$ are ordered in such a way that $\ell_1 \sim \partial X_1, \ell_2 \sim \partial X_2, \ldots$,
4. for each $\ell \in T$ there exist two indexes $n(\ell)$ and $n'(\ell)$ defined as follows:

$$\begin{aligned} n(\ell) &= \max\{k : \ell \sim \partial X_k\} \ , \\ n'(\ell) &= \min\{k : \ell \sim \partial X_k\} \ . \end{aligned} \tag{A.23}$$

We shall say that T is an *anchored tree*.

Using the definitions above, we can rewrite eqn(A.20) as:

$$\begin{aligned} e^{-V(X)} = & \sum_{r=1}^{s} \sum_{X_r \subset X} \sum_{X_2 \ldots X_{r-1}} \sum_{T \text{ on } X_r} (-1)^{r-1} \prod_{\ell \in T} V_\ell \\ & \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_{r-1} \left(\prod_{\ell \in T} \frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_{n(\ell)}} \right) e^{-W_{X_r}(X_1, \ldots, X_{r-1}; t_1, \ldots, t_{r-1})} \, e^{-V(X \setminus X_r)} \end{aligned} \tag{A.24}$$

where "T on X_r" means that T is an anchored tree for the clusters P_j with $j \in X_r$.

Let us define

$$\begin{aligned} K(X_r) = & \sum_{X_2 \ldots X_{r-1}} \sum_{T \text{ on } X_r} \prod_{\ell \in T} V_\ell \\ & \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_{r-1} \left(\prod_{\ell \in T} \frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_{n(\ell)}} \right) e^{-W_{X_r}(X_1, \ldots, X_{r-1}; t_1, \ldots, t_{r-1})} \ , \end{aligned} \tag{A.25}$$

so that eqn(A.24) can be rewritten as

$$e^{-V(X)} = \sum_{\substack{Y \subset X \\ Y \ni \{1\}}} (-1)^{|Y|-1} K(Y) \, e^{-V(X \setminus Y)} \ , \tag{A.26}$$

and, iterating,

$$e^{-V(X)} = \sum_{Q_1, \ldots, Q_m} (-1)^{|X|} (-1)^m \prod_{q=1}^{m} K(Q_q) \ . \tag{A.27}$$

The sets $Q_1, \ldots, Q_m$ in eqn(A.27) are disjoint subsets of X, such that $\cup_{i=1}^m Q_i = X$.

Substituting eqn(A.27) into eqn(A.9), we find

$$\mathcal{E}\left(\prod_{j=1}^{s}\widetilde{\psi}(P_j)\right)=\int\mathcal{D}\psi\sum_{(Q_1,\ldots,Q_m)}(-1)^{s+m}\prod_{q=1}^{m}K(Q_q)\;,\tag{A.28}$$

where the sum is over the partitions $(Q_1,\ldots,Q_m)$ of X. It is easy to realize that in the last equation $K(Q_q)$ (already defined in eqn(A.25)) can be rewritten as

$$K(Q)=\sum_{T\,\mathrm{on}\,Q}\sum_{\substack{X_2,\ldots,X_{|Q|-1}\\ \text{fixed }T}}\prod_{\ell\in T}V_\ell\int_0^1 dt_1\cdots\int_0^1 dt_{|Q|-1}\cdot$$
$$\cdot\prod_{\ell\in T}(t_{n'(\ell)}\cdots t_{n(\ell)-1})e^{-\sum_{\ell\in Q\times Q}t_{n'(\ell)}\cdots t_{n(\ell)}V_\ell}\;.\tag{A.29}$$

Moreover, we can also rewrite eqn(A.28) as:

$$\mathcal{E}\left(\prod_{j=1}^{s}\widetilde{\psi}(P_j)\right)=\sum_{(Q_1,\ldots,Q_m)}(-1)^{s+m}(-1)^{\sigma}\prod_{q=1}^{m}\int\mathcal{D}\psi_{Q_q}K(Q_q)\;,\tag{A.30}$$

where $\mathcal{D}\psi_{Q_q}=\prod_{j\in Q_q}\Big[\prod_{f\in P_j^+}\mathrm{d}\psi^+_{\mathbf{x}(f),\sigma(f),\rho(f)}\Big]\Big[\prod_{f\in P_j^-}\mathrm{d}\psi^-_{\mathbf{x}(f),\sigma(f),\rho(f)}\Big]$ and $(-1)^\sigma$ is the sign of the permutation leading from the ordering of the fields in $\mathcal{D}\psi$ to the ones in $\prod_q\mathcal{D}\psi_{Q_q}$.

Let us now consider the well known relation:

$$\mathcal{E}\left(\prod_{j=1}^{s}\widetilde{\psi}(P_j)\right)=\tag{A.31}$$
$$=\sum_{(Q_1,\ldots,Q_m)}(-1)^{\sigma}\mathcal{E}^T\left(\widetilde{\psi}(P_{j_{11}}),\ldots,\widetilde{\psi}(P_{j_{1|Q_1|}})\right)\ldots\mathcal{E}^T\left(\widetilde{\psi}(P_{j_{m1}}),\ldots,\widetilde{\psi}(P_{j_{m|Q_m|}})\right)\;,$$

where the sum is over the partitions of $\{1,\ldots s\}$, $Q_q=\{j_{q1},\ldots,j_{q|Q_q|}\}$ and $(-1)^\sigma$ is the parity of the permutation leading to the ordering on the r.h.s. from the one on the l.h.s. (note that σ is the same as in eqn(A.30)). Comparing eqn(A.31) with eqn(A.30) we get:

$$\mathcal{E}^T(\widetilde{\psi}(P_1),\ldots,\widetilde{\psi}(P_s))=(-1)^{s+1}\sum_{T\,\mathrm{on}\,X}\int\mathcal{D}\psi\prod_{\ell\in T}V_\ell\int dP_T(\mathbf{t})e^{-V(\mathbf{t})}\;,\tag{A.32}$$

where we defined:

$$dP_T(\mathbf{t})=\sum_{\substack{X_2\ldots X_{s-1}\\ \text{fixed }T}}\prod_{\ell\in T}\left(t_{n'(\ell)}\ldots t_{n(\ell)-1}\right)\prod_{q=1}^{s-1}\mathrm{d}t_q\tag{A.33}$$

and

$$V(\mathbf{t})\equiv\sum_{\ell\in X\times X}t_{n'(\ell)}\ldots t_{n(\ell)}\,V_\ell\;.\tag{A.34}$$

If in eqn(A.32) we integrate the Grassman fields appearing in the product

$$\prod_{\ell\in T} V_\ell = \prod_{(jj')\in T} \left(\overline{V}_{jj'} + \overline{V}_{jj'}\right) \ , \tag{A.35}$$

we obtain

$$\mathcal{E}^T\left(\widetilde{\psi}(P_1),\ldots,\widetilde{\psi}(P_s)\right) = (-1)^{s+1}\sum_{T\,\text{on}\,P}\alpha_T\prod_{\ell\in T}G_{f^1_\ell,f^2_\ell}\int\mathcal{D}^*(\mathrm{d}\psi)\int\mathrm{d}P_T(\mathbf{t})\,e^{-V^*(\mathbf{t})}\ , \tag{A.36}$$

where $P=\cup_i P_i$, the sum $\sum_{T\,\text{on}\,P}$ denotes the sum over the graphs whose elements are lines connecting pairs of distinct points $\mathbf{x}(f)$, $f\in P$ such that, if we identify all the points in the clusters P_j, $j=1,\ldots,s$, then T is a tree graph on X; moreover α_T is a suitable sign,

$$\mathcal{D}^*(\mathrm{d}\psi) = \prod_{\substack{f\in P\\ f\notin T}} d\psi^{\varepsilon(f)}_{\mathbf{x}(f),\sigma(f),\rho(f)}\ , \qquad V^*(\mathbf{t}) = \sum_{\ell\notin T} t_{n'(\ell)}\ldots t_{n(\ell)}\,V_\ell \tag{A.37}$$

and f^1_ℓ, f^2_ℓ are the two field labels associated to the two (entering and exiting) half-lines contracted into ℓ. The term

$$\int\mathcal{D}^*(\mathrm{d}\psi)\int\mathrm{d}P_T(\mathbf{t})\,e^{-V^*(\mathbf{t})} \tag{A.38}$$

in eqn(A.36) is (modulo a sign) the determinant of a suitable matrix $G^T(\mathbf{t})$, with elements

$$G^T_{f,f'}(\mathbf{t}) = t_{n'(\ell)}\ldots t_{n(\ell)}G_{f,f'}\ , \tag{A.39}$$

where $\ell=(j(f)j(f'))$, $j(f)\in X$ is s.t. $f\in P_{j(f)}$ and $G_{f,f'}$ was defined in eqn(A.4). So eqn(5.3) is proven, with $t_{j,j'}=t_{n'(jj')}\ldots t_{n(jj')}$.

In order to complete the proof of the claims following eqn(5.3) we must prove that $dP_T(\mathbf{t})$ is a normalized, positive and σ–additive measure, so it can be interpreted as a probability measure in $\mathbf{t}=(t_1,\ldots,t_{s-1})$; and that, moreover, we can find a family of unit vectors $\mathbf{u}_j\in\mathbb{R}^s$ such that $t_{j,j'}=\mathbf{u}_j\cdot\mathbf{u}_{j'}$.

So, let us conclude this Appendix by proving the following Lemma.

Lemma A1.1 *$dP_T(\mathbf{t})$ is a normalized, positive and σ–additive measure on the natural σ–algebra of $[0,1]^{s-1}$. Moreover there exists a set of unit vectors $\mathbf{u}_j\in\mathbb{R}^s$, $j=1,\ldots,s$, such that $t_{j,j'}=\mathbf{u}_j\cdot\mathbf{u}_{j'}$.*

Proof. Let us denote by b_k the number of lines $\ell\in T$ exiting from the points $x(j,i)$, $j\in X_k$, such that $\ell\sim X_k$. Let us consider the integral

$$\sum_{\substack{X_2\ldots X_{s-1}\\ T\text{ fixed}}}\int_0^1\mathrm{d}t_1\ldots\int_0^1\mathrm{d}t_{s-1}\prod_{\ell\in T}\left(t_{n'(\ell)}\ldots t_{n(\ell)-1}\right) = 1\ , \tag{A.40}$$

and note that, by construction, the parameter t_k inside the integral in the l.h.s. appears at the power $b_k - 1$. In fact any line intersecting ∂X_k contributes by a factor t_k, except for the line connecting X_k with the point in $X_{k+1} \setminus X_k$. See Fig. 5.

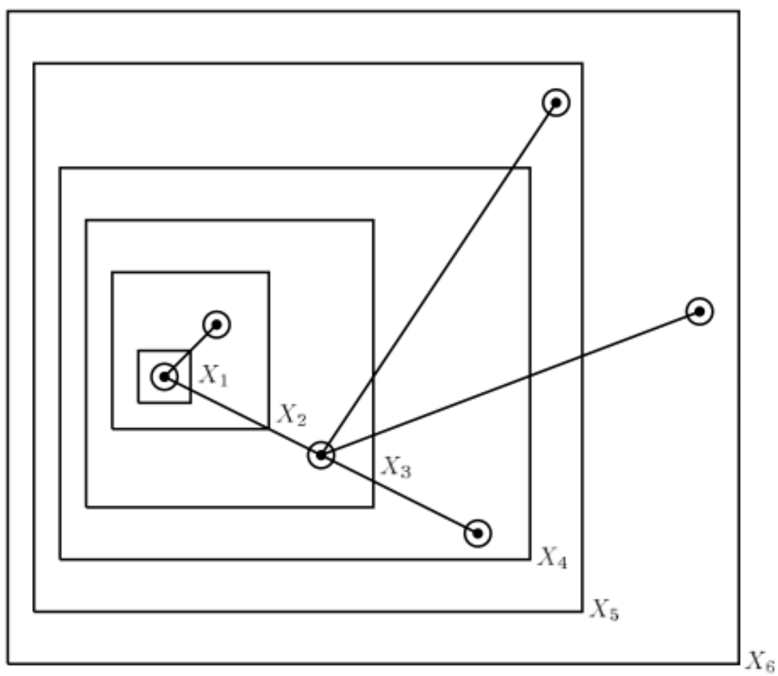

FIG. 5. The sets $X_1,...,X_6$, the anchored tree T and the lines $\ell_1,...,\ell_5$ belonging to T. In the example, the coefficients $b_1,...,b_5$ are respectively equal to: 2,1,3,2,1.

Then

$$\prod_{\ell \in T} \left(t_{n'(\ell)} \dots t_{n(\ell)-1} \right) = \prod_{k=1}^{s-1} t_k^{b_k - 1} , \tag{A.41}$$

and in eqn(A.40) the $s-1$ integrations are independent. One has:

$$\int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_{s-1} \prod_{\ell\in T} \left(t_{n'(\ell)} \ldots t_{n(\ell)-1}\right) = \prod_{k=1}^{s-1} \left(\int_0^1 \mathrm{d}t_k\, t_k^{b_k-1}\right) = \prod_{k=1}^{s-1} \frac{1}{b_k} \;, \quad \text{(A.42)}$$

which is well defined, since $b_k \geq 1$, $k = 1, \ldots, m-2$. Moreover we can write:

$$\sum_{\substack{X_2 \ldots X_{s-1} \\ T \text{ fixed}}} = \sum_{\substack{X_2 \\ T, X_1 \text{ fixed}}} \;\; \sum_{\substack{X_3 \\ T, X_1, X_2 \text{ fixed}}} \cdots \sum_{\substack{X_{s-1} \\ T, X_1, \ldots, X_{s-2} \text{ fixed}}} \;, \quad \text{(A.43)}$$

where the number of possible choices in summing over X_k, once that T and the sets $X_1, \ldots, X_{k-1}$ are fixed, is exactly b_{k-1}. In fact, if from X_{k-1} there are b_{k-1} exiting lines, then X_k is obtained by adding to X_{k-1} one of the b_{k-1} points connected to X_{k-1} through the tree lines. Then:

$$\sum_{\substack{X_2 \ldots X_{s-1} \\ T \text{ fixed}}} 1 = b_1 \ldots b_{s-2} \;, \quad \text{(A.44)}$$

and, recalling that $b_{s-1} = 1$,

$$\sum_{\substack{X_2 \ldots X_{s-1} \\ T \text{ fixed}}} \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_{s-1} \prod_{\ell\in T} \left(t_{n'(\ell)} \ldots t_{n(\ell)-1}\right) = \prod_{k=1}^{s-2} \frac{b_k}{b_k} \;, \quad \text{(A.45)}$$

yielding to $\int dP_T(\mathbf{t}) = 1$. The positivity and σ–addivity of $dP_T(\mathbf{t})$ is obvious by definition.

We are left with proving that we can find unit vectors $\mathbf{u}_j \in \mathbb{R}^s$ such that $t_{j,j'} = \mathbf{u}_j \cdot \mathbf{u}_{j'}$.

To this aim, let us introduce a family of unit vectors in $\mathbb{R}^s$ defined as follows:

$$\begin{cases} \mathbf{u}_1 = \mathbf{v}_1 \;, \\ \mathbf{u}_j = t_{j-1}\mathbf{u}_{j-1} + \mathbf{v}_j \sqrt{1 - t_{j-1}^2} \;, \quad j = 2, \ldots, s \;, \end{cases} \quad \text{(A.46)}$$

where $\{\mathbf{v}_i\}_{i=1}^s$ is an orthonormal basis. Let us rename the sets P_i, $\quad i = 1, \ldots, s$ in such a way that $X_1 = \{1\}$, $X_2\{1,2\}$, …, $X_{s-1} = \{1, \ldots, s-1\}$. Then, for a given line (jj'), we have:

$$t_{j,j'} = t_{n'(jj')} \ldots t_{n(jj')} = t_j \ldots t_{j'-1} \quad \text{(A.47)}$$

From eqn(A.46) it follows that

$$\mathbf{u}_j \cdot \mathbf{u}_{j'} = t_j \ldots t_{j'-1} \;, \quad \text{(A.48)}$$

as desired. ∎

Appendix B: Symmetry properties

Consider the partition function $\Xi_{M,\beta,L} := \int P_M(d\psi)e^{-\mathcal{V}(\psi)}$. It is important to note that both the Gaussian integration $P_M(d\psi)$ and the interaction $\mathcal{V}(\psi)$ are invariant under the action of a number of remarkable symmetry transformations, which will be preserved by the subsequent iterative integration procedure and will guarantee the vanishing of some running coupling constants (see below for details). Let us collect in the following lemma all the symmetry properties we will need in the following.

Lemma B.1. *For any choice of M,β,Λ, both the quadratic Grassmann measure $P_M(d\psi)$ and the quartic Grassmann interaction $\mathcal{V}(\psi)$ are invariant under the following transformations:*
(1) spin exchange: $\hat\psi^\varepsilon_{\mathbf{k},\sigma,\rho} \longleftrightarrow \hat\psi^\varepsilon_{\mathbf{k},-\sigma,\rho}$;
(2) global $U(1)$: $\hat\psi^\varepsilon_{\mathbf{k},\sigma,\rho} \to e^{i\varepsilon\alpha_\sigma}\hat\psi^\varepsilon_{\mathbf{k},\sigma,\rho}$, with $\alpha_\sigma \in \mathbb{R}$ independent of $\mathbf{k}$;
(3) spin $SO(2)$: $\begin{pmatrix}\hat\psi^\varepsilon_{\mathbf{k},\uparrow,\rho}\\ \hat\psi^\varepsilon_{\mathbf{k},\downarrow,\rho}\end{pmatrix} \to R_\theta \begin{pmatrix}\hat\psi^\varepsilon_{\mathbf{k},\uparrow,\rho}\\ \hat\psi^\varepsilon_{\mathbf{k},\downarrow,\rho}\end{pmatrix}$, with $R_\theta = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ and $\theta \in \mathbb{T}$ independent of $\mathbf{k}$;
(4) discrete spatial rotations: $\hat\psi^\pm_{(k_0,\vec k),\sigma,\rho} \to e^{\mp i\vec k(\vec\delta_3-\vec\delta_1)(\rho-1)}\hat\psi^\pm_{(k_0,T_1\vec k),\sigma,\rho}$, with $T_1\vec x := R_{2\pi/3}\vec x$; note that in real space this transformation simply reads $a^\pm_{(x_0,\vec x),\sigma} \to a^\pm_{(x_0,T_1\vec x),\sigma}$ and $b^\pm_{(x_0,\vec x),\sigma} \to b^\pm_{(x_0,T_1\vec x),\sigma}$;
(5) complex conjugation: $\hat\psi^\pm_{\mathbf{k},\sigma,\rho} \to \hat\psi^\pm_{-\mathbf{k},\sigma,\rho}$, $c \to c^*$, where c is a generic constant appearing in $P_M(d\psi)$ and/or in $\mathcal{V}(\psi)$;
(6.a) horizontal reflections: $\hat\psi^\pm_{(k_0,k_1,k_2),\sigma,1} \longleftrightarrow \hat\psi^\pm_{(k_0,-k_1,k_2),\sigma,2}$;
(6.b) vertical reflections: $\hat\psi^\pm_{(k_0,k_1,k_2),\sigma,\rho} \to \hat\psi^\pm_{(k_0,k_1,-k_2),\sigma,\rho}$;
(7) particle-hole: $\hat\psi^\pm_{(k_0,\vec k),\sigma,\rho} \to i\hat\psi^\mp_{(k_0,-\vec k),\sigma,\rho}$.
(8) inversion: $\hat\psi^\pm_{(k_0,\vec k),\sigma,\rho} \to i(-1)^\rho\hat\psi^\pm_{(-k_0,\vec k),\sigma,\rho}$.

Proof. A moment's thought shows that the invariance of $\mathcal{V}(\psi)$ under the above symmetries is obvious, and so is the invariance of $P_M(d\psi)$ under (1)-(2)-(3). Let us then prove the invariance of $P_M(d\psi)$ under (4)-(5)-(6.a)-(6.b)-(7)-(8). More precisely, let us consider the term

$$\begin{aligned}\sum_{\mathbf{k}} \hat\psi^+_{\mathbf{k},\sigma,\cdot}\, \hat g_{\mathbf{k}}^{-1}\hat\psi^-_{\mathbf{k},\sigma,\cdot} &= -i\sum_{\mathbf{k}}\hat\psi^+_{\mathbf{k},\sigma,1}k_0\hat\psi^-_{\mathbf{k},\sigma,1}\\ &-\sum_{\mathbf{k}}\hat\psi^+_{\mathbf{k},\sigma,1}v\Omega^*(\vec k)\hat\psi^-_{\mathbf{k},\sigma,2} - \sum_{\mathbf{k}}\hat\psi^+_{\mathbf{k},\sigma,2}v\Omega(\vec k)\hat\psi^-_{\mathbf{k},\sigma,1} - i\sum_{\mathbf{k}}\hat\psi^+_{\mathbf{k},\sigma,2}k_0\hat\psi^-_{\mathbf{k},\sigma,2}\end{aligned} \tag{B.1}$$

in (4.8), and let us prove its invariance under the transformations (4)-(5)-(6.a)-(6.b)-(7)-(8).

Under the transformation (4), the first and fourth term in the second line of (B.1) are obviously invariant, while the sum of the second and third is changed into

$$-\sum_{\mathbf{k}}\Big[\hat\psi^+_{(k_0,T_1\vec k),\sigma,1}v\Omega^*(\vec k)e^{+i\vec k(\vec\delta_3-\vec\delta_1)}\hat\psi^-_{(k_0,T_1\vec k),\sigma,2}+ \tag{B.2}$$
$$+\hat\psi^+_{(k_0,T_1\vec k),\sigma,2}e^{-i\vec k(\vec\delta_3-\vec\delta_1)}v\Omega(\vec k)\hat\psi^-_{(k_0,T_1\vec k),\sigma,1}\Big]=$$
$$=-\sum_{\mathbf{k}}\Big[\hat\psi^+_{\mathbf{k},\sigma,1}v\Omega^*(T_1^{-1}\vec k)e^{+i\vec k(\vec\delta_1-\vec\delta_2)}\hat\psi^-_{\mathbf{k},\sigma,2}+\hat\psi^+_{\mathbf{k},\sigma,2}e^{-i\vec k(\vec\delta_1-\vec\delta_2)}v\Omega(T_1^{-1}\vec k)\hat\psi^-_{\mathbf{k},\sigma,1}\Big]\,.$$

Using that $\Omega(T_1^{-1}\vec k)=e^{i\vec k(\vec\delta_1-\vec\delta_2)}\Omega(\vec k)$, as it follows by the definition $\Omega(\vec k)=(2/3)\sum_{i=1,2,3}e^{i\vec k(\vec\delta_i-\vec\delta_1)}$, we find that the last line of (B.2) is equal to the sum of the second and third term in (B.1), as desired.

The invariance of (B.1) under the transformation (5) is very simple, if one notes that $\Omega(-\vec k)=\Omega^*(\vec k)$, as it follows by the definition of $\Omega(\vec k)$.

Under the transformation (6.a), the sum of the first and fourth term in the second line of (B.1) is obviously invariant, while the sum of the second and third is changed into

$$-\sum_{\mathbf{k}}\hat\psi^+_{(k_0,-k_1,k_2),\sigma,2}v\Omega^*(\vec k)\hat\psi^-_{(k_0,-k_1,k_2),\sigma,1}-\sum_{\mathbf{k}}\hat\psi^+_{(k_0,-k_1,k_2),\sigma,1}v\Omega(\vec k)\hat\psi^-_{(k_0,-k_1,k_2),\sigma,2}=$$
$$=-\sum_{\mathbf{k}}\hat\psi^+_{\mathbf{k},\sigma,2}v\Omega^*((-k_1,k_2))\hat\psi^-_{\mathbf{k},\sigma,1}-\sum_{\mathbf{k}}\hat\psi^+_{\mathbf{k},\sigma,1}v\Omega((-k_1,k_2))\hat\psi^-_{\mathbf{k},\sigma,2}\,. \tag{B.3}$$

Noting that $\Omega((-k_1,k_2))=\Omega^*(\mathbf{k})$, one sees that this is the same as the sum of the second and third term in (B.1), as desired.

Similarly, noting that $\Omega((k_1,-k_2))=\Omega(\mathbf{k})$, one finds that (B.1) is invariant under the transformation (6.b).

Under the transformation (7), the sum of the first and fourth term in (B.1) is obviously invariant, while the sum of the second and third term is changed into

$$+\sum_{\mathbf{k}}\hat\psi^-_{(k_0,-\vec k),\sigma,1}v\Omega^*(\vec k)\hat\psi^+_{(k_0,-\vec k),\sigma,2}+\sum_{\mathbf{k}}\hat\psi^-_{(k_0,-\vec k),\sigma,2}v\Omega(\vec k)\hat\psi^+_{(k_0,-\vec k),\sigma,1}=$$
$$=-\sum_{\mathbf{k}}\hat\psi^+_{\mathbf{k},\sigma,2}v\Omega^*(-\vec k)\hat\psi^-_{\mathbf{k},\sigma,1}-\sum_{\mathbf{k}}\hat\psi^+_{\mathbf{k},\sigma,1}v\Omega(-\vec k)\hat\psi^-_{\mathbf{k},\sigma,2}\,. \tag{B.4}$$

Using, again, that $\Omega(-\vec k)=\Omega^*(\vec k)$, we see that the latter sum is the same as the sum of the second and third term in (B.1), as desired.

Finally, under the transformation (8), all the terms in the right hand side of (B.1) are separately invariant, and the proof of Lemma B.1 is concluded. ■

Let us now discuss the proof of Lemma 6.2.

Proof of Lemma 6.2 As remarked after (6.10), $P(d\psi^{(u.v.)})$ and $P(d\psi^{(i.r.)})$ are separately invariant under the symmetry properties listed in Lemma 1. Therefore $\mathcal{V}(\psi)$ is also invariant under the same symmetries, and so is the quadratic part of $\mathcal{V}(\psi)$, that is

$$(\beta|\Lambda|)^{-2}\sum_{\sigma}\sum_{\mathbf{k},\mathbf{p}}\delta(\mathbf{p})\Big[\hat\psi^{(i.r.)+}_{\mathbf{k},\sigma,1}\hat\psi^{(i.r.)-}_{\mathbf{k}+\mathbf{p},\sigma,1}W_{aa}(\mathbf{k})+\hat\psi^{(i.r.)+}_{\mathbf{k},\sigma,1}\hat\psi^{(i.r.)-}_{\mathbf{k}+\mathbf{p},\sigma,2}W_{ab}(\mathbf{k})+ \\ +\hat\psi^{(i.r.)+}_{\mathbf{k},\sigma,2}\hat\psi^{(i.r.)-}_{\mathbf{k}+\mathbf{p},\sigma,1}W_{ba}(\mathbf{k})+\hat\psi^{(i.r.)+}_{\mathbf{k},\sigma,2}\hat\psi^{(i.r.)-}_{\mathbf{k}+\mathbf{p},\sigma,2}W_{bb}(\mathbf{k})\Big]\ . \tag{B.5}$$

Recall that, as assumed in the lines preceding (6.2), the support of $\hat\psi^{(i.r.)}$ consists of two disjoint regions around $\vec p_F^{\,+}$ and $\vec p_F^{\,-}$, respectively; in particular, we assumed that $2a_0\gamma < 4\pi/3 - 4\pi/(3\sqrt{3})$. Under this condition, it is easy to realize that if both $\mathbf{k}$ and $\mathbf{p}+\mathbf{k}$ belong to the support of $\hat\psi^{(i.r.)}$, then $|\mathbf{p}| < 4\pi/3$. As a consequence, in (6.11), the only non zero contributions correspond to the terms with $\mathbf{p}=\mathbf{0}$ (in fact, if $\mathbf{p}$ is $\neq \mathbf{0}$ and belongs to the support of $\delta(\mathbf{p})$, then $|\mathbf{p}| \geq 4\pi/3$, which means that either $\mathbf{k}$ or $\mathbf{k}+\mathbf{p}$ is outside the support of $\hat\psi^{(i.r.)}$, and the corresponding term in the sum is identically zero). This means that the sum

$$\sum_{\sigma,\mathbf{k}}\Big[\hat\psi^{(i.r.)+}_{\mathbf{k},\sigma,1}\hat\psi^{(i.r.)-}_{\mathbf{k},\sigma,1}W_{aa}(\mathbf{k})+\hat\psi^{(i.r.)+}_{\mathbf{k},\sigma,1}\hat\psi^{(i.r.)-}_{\mathbf{k},\sigma,2}W_{ab}(\mathbf{k})+ \\ +\hat\psi^{(i.r.)+}_{\mathbf{k},\sigma,2}\hat\psi^{(i.r.)-}_{\mathbf{k},\sigma,1}W_{ba}(\mathbf{k})+\hat\psi^{(i.r.)+}_{\mathbf{k},\sigma,2}\hat\psi^{(i.r.)-}_{\mathbf{k},\sigma,2}W_{bb}(\mathbf{k})\Big]\ . \tag{B.6}$$

is invariant under the symmetries (1)–(7) listed in Lemma 1.

Invariance under symmetry (4) implies that:

$$W_{aa}(k_0,\vec k)=W_{aa}(k_0,T_1^{-1}\vec k)\ ,\qquad W_{bb}(k_0,\vec k)=W_{bb}(k_0,T_1^{-1}\vec k)\ , \tag{B.7}$$
$$W_{ab}(k_0,\vec k)=e^{i\vec k(\vec\delta_1-\vec\delta_2)}W_{ab}(k_0,T_1^{-1}\vec k)\ ,\qquad W_{ba}(k_0,\vec k)=e^{-i\vec k(\vec\delta_1-\vec\delta_2)}W_{ab}(k_0,T_1^{-1}\vec k)\ ;$$

invariance under (5) implies that:

$$W_{aa}(\mathbf{k})=W_{aa}(-\mathbf{k})^*\ ,\qquad W_{bb}(\mathbf{k})=W_{bb}(-\mathbf{k})^*\ , \tag{B.8}$$
$$W_{ab}(\mathbf{k})=W_{ab}(-\mathbf{k})^*\ ,\qquad W_{ba}(\mathbf{k})=W_{ba}(-\mathbf{k})^*\ ;$$

invariance under (6.a) implies that:

$$W_{aa}(k_0,k_1,k_2)=W_{bb}(k_0,-k_1,k_2)\ ,\ W_{ab}(k_0,k_1,k_2)=W_{ba}(k_0,-k_1,k_2)\ ; \tag{B.9}$$

invariance under (6.b) implies that:

$$W_{aa}(k_0,k_1,k_2)=W_{aa}(k_0,k_1,-k_2)\ ,\ W_{bb}(k_0,k_1,k_2)=W_{bb}(k_0,k_1,-k_2) \tag{B.10}$$
$$W_{ab}(k_0,k_1,k_2)=W_{ab}(k_0,k_1,-k_2)\ ,\ W_{ba}(k_0,k_1,k_2)=W_{ba}(k_0,k_1,-k_2)\ ;$$

invariance under (7) implies that:

$$W_{aa}(k_0,\vec k)=W_{aa}(k_0,-\vec k)\ ,\qquad W_{bb}(k_0,\vec k)=W_{bb}(k_0,-\vec k)\ , \tag{B.11}$$
$$W_{ab}(k_0,\vec k)=W_{ba}(k_0,-\vec k)\ ;$$

Finally, invariance under (8) implies that:

$$W_{aa}(k_0,\vec k) = -W_{aa}(-k_0,\vec k)\;, \qquad W_{bb}(k_0,\vec k) = -W_{bb}(-k_0,\vec k)\;, \tag{B.12}$$
$$W_{ab}(k_0,\vec k) = W_{ab}(-k_0,\vec k)\;, \qquad W_{ba}(k_0,\vec k) = W_{ba}(-k_0,\vec k)\;;$$

Now, combining the first of (B.9), the second of (B.10) and the second of (B.11), we find that $W_{aa}(\mathbf{k}) = W_{bb}(\mathbf{k})$. Combining the third of (B.8), the third of (B.11) and the last of (B.12), we find that $W_{ab}(\mathbf{k}) = W_{ba}(\mathbf{k})^*$. This concludes the proof of item (i).

The first of (B.12) implies that, as $\beta\to\infty$, $W_{aa}(0,\vec k)=0$, and this proves, in particular, that $W_{aa}(0,\vec p_F^{\,\omega})=0$ and that, in the limit $|\Lambda|\to\infty$, $\partial_{\vec k}W_{aa}(0,\vec p_F^{\,\omega})=\vec 0$.

Using that $\vec p_F^{\,\omega}$ is invariant under the action of T_1, we see that the third of (B.7) implies that $(1-e^{i\vec p_F^{\,\omega}(\vec\delta_1-\vec\delta_2)})W_{ab}(k_0,\vec p_F^{\,\omega})=0$. Since $e^{i\vec p_F^{\,\omega}(\vec\delta_1-\vec\delta_2)} = -e^{i\omega\pi/3}\neq 1$, this identity proves, in particular, that $W_{ab}(0,\vec p_F^{\,\omega})=0$, and $\partial_{k_0}W_{ab}(0,\vec p_F^{\,\omega})=0$. This concludes the proof of item (ii).

Now, combining the first of (B.8) with the first of (B.11), we find that $W_{aa}(k_0,\vec k)=W_{aa}(-k_0,\vec k)^*$, which implies, in particular, that

$$\mathrm{Re}\{\partial_{k_0}\hat W_{aa}(0,\vec p_F^{\,\omega})\}=0\;.$$

Finally, let $W_{ab}(0,\vec p_F^{\,\omega}+\vec k')\simeq \alpha_1^\omega k_1'+\alpha_2^\omega k_2'$, modulo higher order terms in $\vec k'$. Using that $T_1^{-1}=\begin{pmatrix}-1/2 & \sqrt3/2\\ -\sqrt3/2 & -1/2\end{pmatrix}$ in the third of (B.7), we find that

$$\alpha_1^\omega k_1'+\alpha_2^\omega k_2' = e^{-i\omega\pi/3}\Big[\alpha_1^\omega(k_1'/2-\sqrt3 k_2'/2)+\alpha_2^\omega(\sqrt3 k_1'/2+k_2'/2)\Big]\;, \tag{B.13}$$

which implies $\alpha_1^\omega=-i\omega\alpha_2^\omega$. Moreover, using the third of (B.8) we find that $\alpha_i^\omega=-(\alpha_i^{-\omega})^*$, and using the third of (B.10) we find that $\alpha_2^\omega=-\alpha_2^{-\omega}$. Therefore, $\alpha_2^\omega=-\alpha_2^{-\omega}=-(\alpha_2^{-\omega})^*$, and we see that α_2^ω is real and odd in ω, that is $\alpha_2^\omega=\omega a$, for some real constant a. Therefore, $\alpha_1^\omega=-i\omega\alpha_2^\omega=-ia$, and this concludes the proof of item (iii). ∎

[1] A. Giuliani and V. Mastropietro: The two-dimensional Hubbard model on the honeycomb lattice. Comm. Math. Phys. **293**, 301-346 (2010).