SUSY Statistical Mechanics and Random Band Matrices

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§0. Overview The goal of these lectures is to present a mathematical approach to the spectral analysis of random band matrices $H = H^*$ using the supersymmetric formalism. The random band matrices H_{ij} we study are indexed by vertices i,j of a lattice. Their matrix elements are small for large $|i - j|$ and hence reflect the geometry of the lattice. The ultimate aim is to describe the statistical properties of their eigenvalues and eigenvectors. This information is contained in averages of Green's functions. Green's functions of random matrices may be expressed in terms of correlations of certain statistical mechanics ensembles. For each lattice site, the field or spin of such models contains both commuting and anti-commuting or Grassmann components. Because of the symmetry between the commuting and Grassman variables these systems are referred to as supersymmetric-SUSY.

The SUSY formalism has its origins in the work of Franz Wegner who formulated a bosonic sigma model with hyperbolic symmetry in the early eighties [Weg 1, Weg2]. Inspired by this work, Konstantin Efetov developed the SUSY formalism [Efe1] which gives an exact identity relating average Green's functions to correlations in SUSY field models.

For many years, I resisted the SUSY formalism in favor of renormalized perturbation theory. My first real exposure to SUSY came from reading the initial sections of Mirlin's review [Mir] which I recommend to both mathematicians and physicists. It was already clear from Mirlin's treatment of the density of states for GUE that SUSY could explain features of GUE that were very difficult to see or control perturbatively. In [DPS] we followed the GUE strategy in [Mir] to obtain results on the average Greens function for a random band matrix indexed by Z^3. For a recent overview of developments in SUSY and localization see the article by Efetov [Ef2].

The SUSY lattice field model provides a dual representation which enables us to integrate over the disorder or randomness of the Green's function. This is most conveniently achieved for Gaussian disorder. In this case the disorder average produces a lattice SUSY field model with quartic interactions. After making suitable Hubbard Stratonovich transformations, these representations have the advantage that many spectral properties of random band matrices can be formally seen from an analysis of the saddle point or saddle manifold. For example, universality of the local energy-energy correlation can be understood from the stability of a saddle manifold - a homogeneous space determined by symmetry. In addition, various symmetries and collective coordinates become evident. The main mathematical problem is to establish control of fluctuations about the saddle.

The main idea behind SUSY lattice field models is that the Green's function can be expressed as a ratio of determinants using Cramer's rule. Averages of the determinant of the Green's matrix $(E_\epsilon - H)^{-1}$ can be expressed using commuting or bosonic variables To get compute the average of the determinant of $(E_\epsilon - H)$ it is very convenient to introduce anticommuting Grassmann variables. See the appendix for a brief review of Grassmann integration.

These lectures begin with a discussion of the density of states for the $N \times N$ Gaussian Unitary ensemble - GUE. In this simple but instructive case, we shall reduce the average Green's function to an SUSY model consisting of two real variables. (The Grassmann variables can be traced out). The size of the matrix N appears only as a parameter. For large N there is a saddle point which becomes dominant. This saddle point will govern the famous Wigner semicircle distribution. There is a second saddle point for GUE which gives highly non trivial oscillatory corrections to Wigners semicircle law. See (3.6). The contributions of this second saddle cannot be seen perturbatively. We shall then generalize these methods to obtain detailed information about the density of states for random band matrices indexed by lattice sites in $\mathbb{Z}^3$ following [CFGK, DPS]. Such models can be analyzed with the help of cluster expansions. I will closely follow [Dis, Mir] for the GUE case.

To obtain information about the eigenvectors of random matrices we must analyze $< |G(E_\epsilon; j, k)|^2 >$, where G is the Green's function at energy $E - i\epsilon$, $\epsilon > 0$. The resulting lattice field model has a formal hyperbolic $SU(1,1|2)$ symmetry and is more difficult to study. This means that for the bosonic variables there exists a hyperbolic symmetry $\mathrm{U}(1,1)$ preserving an indefinite Hermitian form on $\mathbb{C}^2$, and the Grassmann variables are governed by a compact $\mathrm{U}(2)$ symmetry. Moreover, there exist odd symmetries mixing Grassmann and bosonic variables. This lattice field model may have Goldstone or zero energy modes related to the $SU(1,1|2)$ symmetry.

In a one dimensional chain of length L, the SUSY sigma model version has an appealing expression first found in [Ef3]. The Grassmann variables can be traced out and the resulting model is a nearest neighbor spin model with *positive* weights given as follows. Let h_j *and* σ_j denote spins with values in a hyperboloid and the sphere S^2 respectively. The Gibbs weight is then proportional to

$$\prod_{j=0}^{L}(h_j \cdot h_{j+1} + \sigma_j \cdot \sigma_{j+1})e^{\beta(\sigma_j \cdot \sigma_{j+1} - h_j \cdot h_{j+1})}. \tag{0.1}$$

More precisely $h_j = (x_j, y_j, z_j)$ satisfy the constraint $z_j^2 - x_j^2 - y_j^2 = 1$. The dot product for the σ spins is Euclidean and the σ is the spin for the classical Heisenberg model. On the other hand the dot product for the h spins is hyperbolic: $h \cdot h' = zz' - xx' - yy'$. It is very convenient to parameterize this hyperboloid with horospherical coordinates $s, t \in \mathbb{R}$:

$$z = \cosh t + s^2 e^t/2, \quad y = \sinh t - s^2 e^t/2, \quad x = se^t. \tag{0.2}$$

The integration measure in σ is the uniform measure over the sphere and the measure over h_j has the density $\prod e^{t_j} ds_j dt_j$. At the end points of the chain we have set $s_0 = s_L = t_0 = t_L = 0$. Thus we have nearest neighbor hyperbolic spins (Boson-Boson sector) and Heisenberg spins (Fermion- Fermion sector) coupled via the Fermion-Boson determinant. It is this coupling which is in general quite complicated. However in 1D is given by $\prod_j (h_j \cdot h_{j+1} + \sigma_j \cdot \sigma_{j+1})$. The factor of β depends on energy E, through the density of states, and on the band width defined in the next section.

In general, sigma models have spins taking values in a symmetric space. The interaction between adjacent spins respects the metric on the target space. The Ising model, and the rotator are two well known sigma models where the spin s_j takes values in $\mathbb{R}^1$, $\mathbb{R}^2$ respectively with the constraint

$|s_j^2| = 1$. Thus they take values in the groups Z_2 and S^1. It is expected that sigma models capture the qualitative physics of more complicated models with the same symmetry. For example the Ising model in 2 or 3 dimensions is expected to have the same critical exponents as the scalar ϕ^4 model with a Z_2 symmetry.

Although the SUSY sigma models are widely used in physics to make detailed predictions about eigenfunctions, energy spacings and quantum transport, there is as yet no rigorous analysis of the $SU(1, 1|2)$ models described above in 2 or more dimensions. Even in one dimension, where the problem can be reduced to a transfer matrix, rigorous results are restricted to the sigma model mentioned above. A key difficulty arises from the fact SUSY lattice field models are not expected to have positive weights, moreover, they have massless Goldstone modes.

However, in recent work with Disertori and Zirnbauer [DSZ, DS] we have established the analog of a phase transition for a simpler SUSY hyperbolic sigma model in 3 dimensions. We shall refer to this model as the $H^{2|2}$ model. The notation refers to the fact that the field takes values in hyperbolic 2 space augmented with 2 Grassmann variables to make it supersymmetric. This model, introduced by Zirnbauer in 1991, is expected to reflect the qualitative behavior of random band matrices - namely localization and diffusion - in any dimension. The great advantage of the $H^{2|2}$ model is that the Grassmann variables can be traced out producing a statistical mechanics model with positive but nonlocal weights. (The nonlocality arises from a determinant produced by integrating out the Grassmann fields.) This means that probabilistic tools can be applied. In fact we shall see that quantum localization and diffusion will be closely related to the motion of a random walk in a highly correlated random environment.

The aim of the latter part of these lectures will be to describe the $H^{2|2}$ model and establish a phase transition as $\beta(E) > 0$ goes from small to large values. Small values of β will correspond to localization - exponential decay of correlations and lack of conductance. In three dimensions, we shall see that large values of β correspond to quantum diffusion and extended states. The proof of this transition relies heavily on Ward identities arising from SUSY symmetries of the model. The simplest expression of these Ward identities is reflected by the fact that the partition function is identically one for all parameter values. The SUSY model is nevertheless highly non trivial because observables break SUSY and produce interesting correlations.

Although the $H^{2|2}$ is motivated by quantum disordered systems, eg the spectral theory of random band matrices, it appears to be related to a history dependent walk called edge reinforced random walk ERRW. This walk favors moving along edges it has visited in the past. It can also be expressed as random walk in a random environment. As in the case of the $H^{2|2}$ model the environment is highly nonlocal and it has Ward identities which reflect conservation of probability. In one dimension the ERRW is localized - that is the probability long excursions of length ℓ is exponentially small in ℓ [MR]. It is natural to conjecture that the ERRW is also localized in 2D even for weak edge reinforcement.

§1. RBM - Random Band Matrices

The goal of these lectures is to study the spectral properties of a class of Gaussian Hermitian random matrices $H = H^*$. This will include the GUE - Gaussian Unitary ensemble, Wegner's n-orbital models, and random band matrices - RBM. These models seem to be best suited to SUSY methods.

Let us start with the well known $N \times N$, Gaussian Unitary ensemble . In this case the matrix entries H_{ij} are mean zero independent random variables for $i \leq j$ and $1 \leq i, j \leq N$. Since H has a Gaussian distribution it suffices to specify its covariance:

$$< H_{ij}\bar{H}_{i'j'} >=< H_{ij}H_{j'i'} >= \delta(ii')\delta(jj')/N \tag{1.1}$$

The average over the randomness or disorder is denoted by $< \cdot >$ and $\bar{H}$ denotes the complex conjugate of H. The density for this ensemble is given by

$$1/Z_N \; e^{-NtrH^2/2} \prod dH_{ii} \prod_{i<j} dH_{ij}^{Re} dH_{ij}^{Im}.$$

The factor of $1/Z_N$ ensures that the integral is 1. It is clear that H and U^*HU have identical distrubtions for any fixed unitary matrix U. This invariance is a crucial feature in the classical analysis of such matrices via orthogonal polynomials. However, non Gaussian matrices studied by Erdos et al and RBM do not have unitarily invariant distributions and new methods are needed to obtain the desired spectral information.

Random band matrices with Gaussian distribution are defined in a similar fashion except that we shall let i and j range over a periodic box $\Lambda \subset \mathbb{Z}^d$

$$< H_{ij}\bar{H}_{i'j'} >=< H_{ij}H_{j'i'} >= \delta(ii')\delta(jj')J_{ij} \tag{1.2}$$

Here J_{ij} is a symmetric function which is small for large $|i-j|$. We shall assume that for fixed i, $\sum_j J_{ij} = 1$. With this normalization the spectrum of H is concentrated the interval $[-2, 2]$ with high probability. One especially convenient choice of J is given by the lattice Green's function

$$J_{jk} = (-W^2\Delta + 1)^{-1}(j, k) \tag{1.3}$$

where Δ is the discrete Laplacian on Λ with suitable boundary conditions

$$\Delta f(j) = \sum_{|j'-j|=1} (f(j') - f(j)).$$

Note that with this choice of J, the variance of the matrix elements is exponentially small when $|i-j| \gg W$. In fact in one dimension $J_{ij} \approx e^{-|i-j|/W}/W$ Hence W will be referred to as the width of the band.

Let us now compare discrete random Schrödinger operators on $\mathbb{Z}^d$ given by

$$H_{RS} = -\Delta + \lambda v_j$$

and RBM (1.2) of width W on $\mathbb{Z}^d$. Above, v_j are assumed to be independent identically distributed Gaussian random variables of mean 0 and variance $< v_j^2 >= 1$. The potential v acts diagonally. The parameter $\lambda > 0$ measures the strength of the disorder. Although these models look quite different, they are both local, that is their matrix elements j, k are small (or zero) if $|j-k|$ is large. The rough correspondence is expected to be that $\lambda \approx W^{-1}$. For example, eigenvectors for RS are known to decay exponentially fast in one dimension with a localization length proportional to λ^{-2}. On the other hand for 1D RBM the localization length is known to be finite [Sch] and is expected to be W^2.

Unlike the band matrices, GUE matrices have no spatial or geometric structure. They are essentially mean field models. Nevertheless, the local eigenvalue statistics of these simple models are expected to be universal in a sense to be made more precise later. In fact the local eigenvalue statistics of GUE are mysteriously connected to the statistics of zeros of the Riemann zeta function. This goes back to work by H. Montgomery, F. Dyson and A. Odylzko. The SUSY analysis of GUE also provides the foundation for the more complicated RBM models described later.

For an $N \times N$ Hermitian matrix H, define the inverse matrix:

$$G(E_\epsilon) = (E_\epsilon - H)^{-1} \qquad \textit{where} \qquad E_\epsilon = E - i\epsilon. \tag{1.4}$$

This a bounded matrix provided that E is real and $\epsilon > 0$ and the Green's function denoted, $G(E_\epsilon; k, j)$, are its matrix elements.

Let $z = (z_1, z_2, ..., z_N)$ with $z_j = x_j + iy_j$ denote an element of C^N and define the quadratic form

$$[z; Hz] = \sum_{i,j} \bar{z}_k H_{kj} z_j. \tag{1.5}$$

Then we can calculate the following Gaussian integrals:

$$\int e^{-i[z,(E_\epsilon - H)z]} \, Dz = \det(E_\epsilon - H)^{-1}, \quad Dz \equiv \prod_j dx_j dy_j / \pi \tag{1.6}$$

and

$$\int e^{-i[z,(E_\epsilon - H)z]} z_k \bar{z}_j \, Dz = \det(E_\epsilon - H)^{-1} \, G(E_\epsilon; k, j). \tag{1.7}$$

It is important to note that the integrals above are *convergent* provided that $\epsilon > 0$. The quadratic form [z;(E-H)z] is real so its contribution only oscillates. The factor of $i = \sqrt{-1}$ in the exponent is needed because the matrix $H - E$ has an indefinite signature when E is in the spectrum of H. If we had integrated over real fields then we would obtain the square root of the inverse determinant.

There is a similar identity in which the complex commuting variables z are replaced by anticommuting Grassmann variables ψ_j, $\bar{\psi}_j$, j = 1, 2 ... N. Let A be an $N \times N$ matrix

$$[\psi; A\psi] = \sum \bar{\psi}_k A_{kj} \psi_j$$

then

$$\int e^{-[\psi; A\psi]} D\psi = \det A. \tag{1.8}$$

See the appendix for a brief review of Grassmann integration. The Grassmann integral is introduced so that we can cancel the unwanted determinant in (1.7). Thus we obtain a SUSY representation for the Green's function:

$$G(E_\epsilon; k, j) = \int e^{-i[z,(E_\epsilon - H)z]} e^{-i[\psi,(E_\epsilon - H)\psi]} z_k \bar{z}_j \, Dz \, D\psi. \tag{1.9}$$

Equation (1.9) is the starting point for all SUSY formulas. We shall discuss integration over Grassmann variables in the appendix. Notice that if H has a Gaussian distribution the expectation $<\cdot>_H$, of (1.9) can be explicitly performed since H appears linearly. We obtain:

$$< G(E_\epsilon;k,j) >= \int e^{-iE_\epsilon([z,z]+[\psi,\psi])} e^{-\frac{1}{2}<\{[z;Hz]+[\psi;H\psi]\}^2>} z_k \bar{z}_j \, Dz \, D\psi. \quad (1.10)$$

The minus sign above comes from $i^2 = -1$. The resulting lattice field model will be quartic in the z and ψ fields. If the observable $z_k\bar{z}_j$ were absent, then the determinants would cancel and the integral would be 1. Thus in SUSY systems, the usual partition function is identically 1.

§2 Averaging $Det(E_\epsilon - H)^{-1}$

Before using Grassmann variables we shall first illustrate how to use (1.6) to calculate the average of the inverse determinant over the Gaussian disorder. Although this average has no physical significance, it is a useful exercise.

First consider the simplest case: H is an $N \times N$, GUE matrix. Let us apply (1) to calculate

$$< \det(E_\epsilon - H)^{-1} >=< \int e^{-i[z,(E_\epsilon - H)z]} \, Dz > . \quad (2.1)$$

Interchange the order of integration and use (1.1) and the Gaussian identity

$$< e^{-i[z,Hz]} >_{GUE}= e^{-1/2<[z,Hz]^2>} = e^{-\frac{1}{2N}[z,z]^2}. \quad (2.2)$$

The most direct way to estimate the remaining integral over the z variables is to introduce a new coordinate $r = [z,z] = \sum |z_j|^2$. Then we have

$$< \det(E_\epsilon - H)^{-1} >= C_N \int_0^\infty e^{-\frac{1}{2N}r^2 - iE_\epsilon r} r^N \, dr$$

where C_N is an explicit constant related to the volume of the sphere in 2N dimensions. It is convenient to rescale $r \to Nr$ and obtain an integral of the form

$$\int_0^\infty e^{-N(r^2/2 - \ln r - iE_\epsilon r)} dr.$$

The method of steepest descent can now be applied. We deform our integration over r so that it passes through the saddle point. The saddle point r_s is obtained by setting the derivative of the exponent to 0: $r_s - 1/r_s - iE_\epsilon = 0$ This is a quadratic equation with a solution $r_s = iE/2 \pm \sqrt{1-(E/2)^2}$. The contour must be chosen so that the absolute value of the integrand is dominated by the saddle point.

Exercise: Derive Stirling's formula:

$$N! = \int_0^\infty e^{-t} t^N dt \approx N^N e^{-N} \sqrt{2N\pi}.$$

Let t = Ns and expand to quadratic order about the saddle point s = 1. The square root arises from the identity $N \int e^{-Ns^2/2} ds = \sqrt{2N\pi}$.

Remark: For other ensembles, radial coordinates do not suffice and one must compute the Jacobian for the new collective variables. There are several tricks for computing the Jacobian. In the mathematical literature this is done with the help of the coarea formula. It can also be computed using manipulation of delta functions. See the discuusion in §4.

An alternate way to compute $< \det(E_\epsilon - H)^{-1} >$ uses the Hubbard-Stratonovich transform. In its simplest form, introduce a real auxiliary variable a to unravel the quartic expression in z as follows:

$$e^{-\frac{1}{2N}[z,z]^2} = \sqrt{N/2\pi} \int e^{-Na^2/2} e^{ia[z,z]} da. \tag{2.3}$$

The z variables now appear quadratically and we can integrate over them. This is particularly simple because we have a product integral on the right side of (2.1). The integral over the z_j, $1 \le j \le N$ in (2.1) produces a factor $(E_\epsilon - a)^{-N}$, hence :

$$< \det(E_\epsilon - H)^{-1} >= C_N \int e^{-Na^2/2} (E_\epsilon - a)^{-N} da = C_N \int e^{-Nf(a)} da. \tag{2.4}$$

We deform our path $a \to a + a_s$ where a_s is the complex saddle point. The saddle point is obtained by setting $f'(a_s) = 0$. The gives a quadratic equation whose solution is

$$a_s = E/2 + i\sqrt{1-(E/2)^2}. \tag{2.5}$$

Note $|a_s| = 1$ and that we have chosen the + sign so that the pole of $(E_\epsilon - a)^{-N}$ has not been crossed. Along this contour one checks that for E satisfying $|E| \le 2 - \delta$ the maximum modulus of the integrand occurs at the saddle a_s. In particular this deformation of contour avoids the small denominator $E_\epsilon - a$ occurring when $a \approx E$. Note that the Hessian at the saddle is

$$f''(a_s) = 1 - a_s^2 = 1 - (E/2)^2 - i\frac{E}{2}\sqrt{1-(E/2)^2} \tag{2.6}$$

has a positive real part for $|E| \leq 2 - \delta$.

Now let us consider the more general case when H is a Gaussian random band matrix with covariance given by (1.3). Then we have

$$< e^{-i[z,Hz]} >= e^{-1/2<[z,Hz]^2>} = e^{-1/2\sum |z_i|^2 J_{ij}|z_j|^2}. \tag{2.7}$$

In order to average over the the z variables we introduce real auxiliary fields a_j with covariance J_{ij} so that

$$e^{-1/2\sum_{ij}|z_i|^2 J_{ij}|z_j|^2} =< e^{-i\sum a_j|z_j|^2} >_J \tag{2.8}$$

We can now average over the z's since they appear quadratically. By combining (15) and (16)

$$< \det(E_\epsilon - H)^{-1} >=< \int e^{-i[z,(E_\epsilon - H)z]}\, Dz >$$

$$=< \int e^{i[z;(E_\epsilon - a)z]} Dz >_J=< \prod_j (E_\epsilon - a_j)^{-1} >_J \tag{2.9}$$

By using the (3), and the definition of $< \cdot >_J$, (17) is proportional to:

$$\int e^{-\frac{1}{2}\sum_j (W^2(\nabla a)_j^2 + a_j^2)} \prod_j (E_\epsilon - a_j)^{-1}\, da_j db_j. \tag{2.10}$$

The large parameter W is related to N. It tends to make the a_j fields constant over a large range of lattice sites, hence the product in (2.10) is roughly $(E_\epsilon - a)^{-W}$. The saddle point can again be calculated as it was for GUE and we find that it is independent of the lattice site j and is given by (2.6). The Hessian at the saddle is now a large matrix which can be shown to have exponential decay. For this case a standard cluster expansion over blocks of side W controls the integral (2.10).

Thus we have transformed band matrices of the form (1.3) into a nearest neighbor interaction. We shall see that the same is true for the averages of Green's functions. However, in the SUSY formalism the variables will also have Grassmann components. In one dimension, (2.10) shows that we can calculate the $< \det(E_\epsilon - H)^{-1} >$ using a nearest neighbor transfer matrix.

§3 The average of the density of states for GUE

The average *integrated* density of states for an $N \times N$ Hermitian matrix H is denoted $n(E) = \int^E d\rho(E')$ is the fraction of eigenvalues less than E and $\rho(E)$ denotes the density of states. The average of the density of states is given by the expression

$$< \rho_\epsilon(E) >= \frac{1}{N} tr < \delta_\epsilon(H-E) >= \frac{1}{N\pi} tr\, Im \ < G(E_\epsilon) > \tag{3.1}$$

as $\epsilon \downarrow 0$. Here we are using the well known fact that

$$\delta_\epsilon(x-E) \equiv \frac{1}{\pi}\frac{\epsilon}{(x-E)^2+\epsilon^2} = \frac{1}{\pi} Im (E_\epsilon - x)^{-1}$$

is an approximate delta function at E as $\epsilon \to 0$.

Remarks: The famous Wigner semicircle distribution asserts that that the density of states of a GUE matrix is given by $\pi^{-1}\sqrt{1-(E/2)^2}$. Such results can be proved for many ensembles including RBM by first fixing ϵ and then letting N, *or* $W \to \infty$. Note that the parameter ϵ is the scale at which we can resolve different energies. So for a system of size of size N we would like to understand the case $\epsilon \approx 1/N$. On the other hand, the analysis of Green's functions becomes more difficult as ϵ gets small. In [CFGK, DPS] estimates on the density of states for a special class of band matrices are uniform in the ϵ and the size of the box for fixed $W \geq W_0$.

We now present an identity for the average GUE Green's function starting from equation (2.10). Note that

$$-\frac{1}{2} < ([z;Hz] + [\psi;H\psi])^2 >_{GUE} = -\frac{1}{2N}\{[z,z]^2 - [\psi,\psi]^2 - 2[\psi,z][z;\psi]\}.$$

Let us introduce another real auxiliary variable $b \in R^1$ and apply the Hubbard-Stratonovich transform to decouple the Grassmann variables. As in (2.3) we use the identity

$$e^{[\psi,\psi]^2/2N} = \int db\, e^{-Nb^2/2}\, e^{b[\psi,\psi]}.$$

There are also cross terms. In this case, if we expand the exponential of the cross terms, it terminates after on step because $[\psi,z]^2 = 0$. Then as in the case of the z variables we can integrate over the ψ fields and obtain an expression given by:

$$< \frac{1}{N} tr\, G(E_\epsilon) >= N/2\pi \int dadb\, (E_\epsilon - a)^{-1}\, e^{-N(a^2+b^2)/2}\, (E_\epsilon - ib)^N (E_\epsilon - a)^{-N}$$

$$\times [1 - \frac{N+1}{N}(E_\epsilon - a)^{-1}(E_\epsilon - ib)^{-1}] \equiv < (E_\epsilon - a)^{-1} >_{SUSY}\ . \qquad (3.2)$$

The first factor of $(E_\epsilon - a)^{-1}$ on the right hand side corresponds to the trace of the Greens function. Without this factor, the integral is 1 for all values of the parameter. More precisely we have

$$1 \equiv N/2\pi \int dadb\, e^{-N(a^2+b^2)/2}(E_\epsilon - ib)^N (E_\epsilon - a)^{-N}\, [1 - (E_\epsilon - a)^{-1}(E_\epsilon - ib)^{-1}]$$

for all values of E , ϵ and N. This is due to the fact that if there is no observable the determinants cancel. The last factor in (3.2) arises from the crossterms. For band matrices it is useful to introduce auxilliary dual Grassmann variables to treat the cross terms. See [Mir, Dis, DPS].

Notice that the a and the b variables are independent except for the last factor which couples them. This factor, some times referred to as the fermion boson (FB) contribution. It represents the coupling between the averaged determinants. These features are typical of many more complicated SUSY field models.

The study of $\rho(E)$ is reduces to the analysis of the saddle points of the integrand. As we have explained there is precisely one saddle point

$$a_s(E) = E/2 + i\sqrt{1 - (E/2)^2} \qquad (3.3)$$

in the a field. Note that $|a_s| = 1$. However, there are two saddle points $ib_s = a_s, and\ ib_s^{'} = \bar{a}_s$ corresponding to the b field. Hence, both saddle points (a_s, b_s) and $(a_s, b_s^{'})$ will contribute to (3.2).

Let us briefly analyze the fluctuations about the saddles as we did in (2.5-2.7). The first saddle gives the Wigner semicircle law. To see this note that the action at a_s, b_s takes the value 1. The imaginary part of the the observable gives us the Wigner semicircle law. The integral of quadratic fluctuations about the saddle,

$$N \int e^{-N(1-a_s^2)(a^2+b^2)} dadb \tag{3.4}$$

is exactly cancelled by the F-B contribution at (a_s, b_s). Thus to a high level of accuracy we can simply replace the observable in the SUSY expectation (3.2) by its value at the saddle. This gives us Wigner's semicircle law:

$$\begin{aligned}\rho(E) &= \frac{1}{\pi N} Im\, tr < G(E_\epsilon) >= \pi^{-1} < ((E_\epsilon - a)^{-1} >_{SUSY} \\ &\approx \pi^{-1} Im(E_\epsilon - a_s)^{-1} = \pi^{-1}\sqrt{1-(E/2)^2} \end{aligned} \tag{3.5}$$

It is easy to check that the second saddle vanishes when inserted into the FB factor. Thus to leading order it does not contribute to the density of states and hence (3.5) holds. However, the second saddle will contribute highly oscillatory corrections proportional to

$$\frac{1}{N}(\frac{\bar{a}_s}{a_s})^N e^{-N/2(a_s^2-\bar{a}_s^2)}. \tag{3.6}$$

If we take derivatives in E, this makes a big contribution which is not easily seen in perturbation theory. I believe this is a compelling example nonperturbative power of the SUSY method.

Remark: We have implicitly assumed that the energy E is inside the bulk ie $|E| < 2$. Near the edge of the spectrum the Hessian at the saddle point vanishes and a more delicate analysis is called for. The density of states near $E = \pm 2$ then governed by an Airy function. We refer the Disertori's review of GUE for more details.

§4. Hyperbolic symmetry

Let us analyze the average of $|\det(E_\epsilon - H)|^{-2}$ with H a GUE matrix. We study this average to illustrate the emergence of hyperbolic symmetry when analyzing the average of $|G(E_\epsilon; j, k)|^2$. To begin with let us contrast the $< G >$ and $< |G|^2 >$ for N=1. In the first case we can see that $\int e^{-H^2}(E_\epsilon - H)^{-1}dH$ is finite by shifting the contour of intergration off the real axis $H \to H + i\delta$ with $\delta > 0$ so that the pole is not crossed. On the other hand, if one takes the absolute value squared, we cannot deform the contour integral and it will diverge like ϵ^{-1}.

Let $z, w \in C^N$. As in (1.6) we can write:

$$|\det(E_\epsilon - H)|^{-2} = \det(E_\epsilon - H) \times \det(E_{-\epsilon} - H)$$

$$= \int e^{-i[z,(E_\epsilon - H)z]}\, Dz \times \int e^{i[w,(E_{-\epsilon}-H)w]}\, Dw. \qquad (4.1)$$

Note that the two factors are complex conjugates of each other. The factor of i has been reversed in the w integral to guarantee the convergence of the integral. This sign change is responsible for the hyperbolic symmetry. The Gaussian average over H is

$$< e^{-i([zHz]-[w,Hw])} >= e^{-1/2<([z,Hz]-[w,Hw])^2>} \qquad (4.2)$$

Note that

$$< ([z,Hz]-[w,Hw])^2 >=< [\sum H_{kj}(\bar{z}_k z_j - \bar{w}_k z_j)]^2 > . \qquad (4.3)$$

For GUE the right side is computed using (1)

$$< ([z,Hz]-[w,Hw])^2 >= 1/N([z,z]^2 + [w,w]^2 - 2[z,w][w,z]). \qquad (4.4)$$

Following Fyodorov, [Fy1], introduce the 2×2 **positive** matrix:

$$M(z,w) = \begin{pmatrix} [z,z] & [z,w] \\ [w,z] & [w,w] \end{pmatrix}. \qquad (4.5)$$

and let

$$L = diag(1,-1). \qquad (4.6)$$

Then we see that

$$< |\det(E_\epsilon - H)|^{-2} >= \int e^{-\frac{1}{2N}tr(ML)^2 - iEtr(ML) + \epsilon tr M} DzDw. \qquad (4.7)$$

For a positve 2×2 matrix P consider the the delta function $\delta(P - M(z,w))$ and integrate over z and w. We claim that

$$\int \delta(P - M(z,w)) DzDw = (detP)^{N-2}. \qquad (4.8)$$

Assuming this holds we can now write the right side in terms of the new collective coordinate P.

$$< |\det(E_\epsilon - H)|^{-2} >= C_N \int_{P>0} e^{-\frac{1}{2N}tr(PL)^2} e^{-iEtr(PL)-\epsilon trP} detP^{N-2} dP. \quad (4.9)$$

After rescaling $P \to NP$ we have

$$< |\det(E_\epsilon - H)|^{-2} >= C'_N \int_{P>0} e^{-N\{tr(PL)^2/2+iEtr(PL)+\epsilon trP\}} detP^{N-2} \quad (4.10)$$

In order to compute the integral we shall again change variables and integrate over PL. First note that for $P>0$, PL has two real eigenvalues of opposite sign. This is because PL has the same eigenvalues as $P^{1/2}LP^{1/2}$ which is self adjoint with a negative determinant. Moreover, it can be shown that

$$PL = TDT^{-1} \quad (4.11)$$

where T belongs the the non compact group $SU(1,1)$, that is

$$T^*LT = L \to T \in SU(1,1) \quad (4.12)$$

and $D = diag(p_1, -p_2)$ with p_1, p_2 positive. The proof is similar to that for Hermitian matrices. We shall regard PL as our new integration variable. All expressions can be written in terms of p_1, p_2 except for $\epsilon\, tr\, P$ which will involve the integral over $SU(1,1)$.

Converting to the new coordinate system our measure becomes

$$(p_1 + p_2)^2 dp_1 dp_2 d\mu(T) \quad (4.13)$$

where $d\mu(T)$ the Haar measure on U(1,1). For large N, the p variables are approximately given by the complex saddle point

$$p_1 = -iE/2 + \rho(E),\ p_2 = -iE/2 - \rho(E) \quad where \quad \rho(E) = \sqrt{1-(E/2)^2}. \quad (4.14)$$

However, there still remains the integral over $d\mu(T)$. The p variables fluctuate only slightly while the T matrix ranges over the symmetric space $SU(1,1)/U(1)$ and produces a singularity for small ϵ. With the p_1, p_2 set as above the only remaining integral is over $SU(1,1)$. Thus from (4.11) we have

$$Q \equiv PL \approx \rho(E)\ TLT^{-1} + iE/2I$$

This is the basis for the sigma model. The second term above is independent of T so it is dropped. The band version or N orbital version of such hyperbolic sigma models was studied in [SZ]. For each lattice site $j \in \Lambda \subset Z^d$ we define a new spin variable given by

$$S_j = T_j^{-1} L T_j \quad and \quad P_j L \approx \rho(E)\ S_j. \tag{4.15}$$

Note that $S_j^2 = 1$ and S_j naturally belongs to $SU(1,1)/U(1)$. This symmetric space is isomorphic to the hyperbolic upper half plane. In the last equation we have used the form of the p_i given as above. The imaginary part of the p_1 *and* p_2 are equal so that T and T^{-1} cancel producing only a trivial contribution. Note that the explicit dependence on E only appears through $\rho(E)$.

There is a similar picture for the average determinant using Grassmann integration. We can integrate out the Grassmann fields and in the sigma model approximation obtain an integral over the symmetric space $SU(2)/U(1) = S^2$. This is the classical Heisenberg model.

The action of the hyperbolic sigma model on the lattice is

$$A(S) = \beta \sum_{j \sim j'} tr S_j S_{j'} + \epsilon \sum_j tr L S_j. \tag{4.16}$$

The notation $j \sim j'$ denotes nearest neighbor vertices on the lattice. The Gibbs weight is proportional to $e^{-A(S)} d\mu(T)$. The integration over $SU(1,1)$ is divergent unless $\epsilon > 0$. The last term above is symmetry breaking term analogous to a magnetic field. For RBM $\beta \approx W^2 \rho(E)^2$.

To parametrize the integral over SU(1,1) we use horospherical coordinates $(s_j, t_j) \in \mathbb{R}^2$ given by (0.2). In this coordinate system, described in the first section, the action takes the form:

$$A(s,t) = \beta \sum_{j \sim j'} [cosh(t_j - t_{j'}) + \frac{1}{2}(s_j - s_{j'})^2 e^{(t_j + t_{j'})}] \tag{4.17}$$

We have ommitted the symmetry breaking term proportional to ϵ. In this coordinate system the s variables appear quadratically. Let us define the quadratic form associated to the s variables above:

$$[v\,; D_{\beta,\varepsilon}(t)\, v]_\Lambda = \beta \sum\nolimits_{(ij)} \mathrm{e}^{t_i + t_j} (v_i - v_j)^2 + \varepsilon \sum\nolimits_{k \in \Lambda} \mathrm{e}^{t_k} v_k^2 \tag{4.18}$$

Here we have included the symmetry breaking term. After integrating out the s variables, we get $Det^{-1/2}(D_{\beta,\varepsilon}(t))$.

By the matrix tree theorem, $Det(D_{\beta,\varepsilon}(t))$ is a convex function of the t variables. Thus the effective action is convex. The sigma model can now be analyzed using Brascamp-Lieb inequalities. Its convexity will imply that this model does not have a phase transition in 3 dimensions. Note that in these coordinates there is a formal symmetry $t_j \to t_j + \gamma$ and $s_j \to s_j e^{-\gamma}$ which is responsible for a Goldstone mode.

Theorem (Brascamp-Lieb) Let A(t) be a real convex function of $t_j, j \in \Lambda$ and v_j be a real vector. If the Hessian of the action A is convex $A"(t) \geq K > 0$ then

$$< e^{[v;t]} >_A \leq e^{<[v;t]>_A} e^{\frac{1}{2}[v;K^{-1}v]}. \tag{4.19}$$

Here $<>_A$ denotes the expectation with respect to $e^{-A(t)}Dt$ and K is a positive matrix independent of t. Note if A is quadratic in t this is an identity.

Thus both the hyperbolic and Heisenberg models are reasonably well understood. In 3D the Heisenberg model has a phase transition in 3D by using infrared bounds. The remaining mathematical challenge is to understand the coupling between them via the Fermion - Boson factors. The SUSY hyperbolic sigma model, $H^{2|2}$ will be a step in this direction.

§5 The Average Green's function for RBM

In this section show how to get the semicircle law and its corrections for RBM when W is large. We start with perturbation theory then compare these calculations to the mathematical results obtained using SUSY formalism. As mentioned above, the SUSY methods are closely related to the GUE analysis described earlier.

Consider Hermitian RBM with Gaussian distribution and J given by (1.3). The perturbation scheme described here is very closely related to the one used for Random Schrödinger operators.

To calculate the density of states write:

$$G(E_\epsilon) = [E_\epsilon - H]^{-1} = [\tilde{E} - H + (E_\epsilon - \tilde{E})]^{-1} \tag{5.1}$$

where $\tilde{E} = \tilde{E}(E, \epsilon)\, I$ is to be determined and I is the identity matrix. We shall perturb about $\tilde{E}^{-1} \equiv G_0$

$$\langle G(E_\epsilon)(i,i)\rangle = G_0 + \langle G_0 H_{ii} G_0\rangle +$$

$$+\sum_j \langle G_0 H_{ij} G_0 H_{ji} G_0\rangle - G_0(E_\epsilon - \tilde{E})G_0 + ... \tag{5.2}$$

We shall define $\tilde{E}$ so that the third and fourth terms on the right side of (2) cancel. The second term vanishes because $\langle H\rangle = 0$. Since

$$\sum_j \langle H_{ij} H_{ji}\rangle = \sum_j J_{ij} = 1$$

the third and fourth terms cancel when

$$G_0 = \tilde{E}^{-1} = E_\epsilon - \tilde{E} \qquad \text{hence,} \quad G_0 = a_s I. \tag{5.3}$$

The imaginary part of G_0 gives Wigner's semicircle law for the density of states. Of course, this expansion has been done only to second order in H. If we calculate to fourth order, we will see that the correction is $O(W^{-1})$. The reason that higher order averages are smaller is that adjacent factors $H_{jk}\, G_0\, H_{kj'}$ appearing in the expansion about G_0 with $j = j'$ are canceled by $E_\epsilon - \tilde{E}$ to leading order as above. Thus in Hermitian case, $\sum_j H_{ij_1} H_{j_1 j_2} H_{j_2 j_3} H_{j_3 i}$ has only terms with $(i, j_1) = (j_3, j_2)$ and $(i, j_3) = (j_1, j_2)$ which contributes after averaging. Hence we get a contribution $J_{i,i}^2 \leq W^{-2}$. Proceeding beyond fourth order one must re sum classes of graphs. This can be done by grouping together diagrams according to their "genus". Roughly speaking diagrams contributing with higher genus are suppressed by powers of W^{-1}.

It is also instructive to calculate

$$\langle G(E_\epsilon; 0, x)\, G(E_\epsilon; x, 0)\rangle \tag{5.4}$$

to leading order and see how exponential decay emerges. Assume that we are in a periodic box and J is defined by (3). Let us expand $G(E_\epsilon; 0, x)\,, G(E_\epsilon; x, 0)$ about G_0 as above. This expansion will have terms of the form

$$G_0\, H_{0,j_1}\, G_0\, H_{j_1,j_2} ... H_{j_n,x}\, G_0, \quad G_0\, H_{x,k_n}\, G_0\, H_{k_n,k_{n-1}} ... H_{k_1,0}\, G_0. \tag{5.5}$$

The leading contribution to the average of this expression occurs when we pair $j_i = k_i$ producing the average

$$\sum_j G_0^2 J_{0,j_1} G_0^2 J_{j_1,j_2} ... J_{j_n,x} G_0^2 = (G_0^2 J)^n_{0,x} G_0^2 \tag{5.6}$$

endequation here J^n denotes the n-fold convolution of J. Summing the geometric series over n and defining J by (1.3) we get

$$[G_0^{-2} - (-W^2\Delta + 1)]^{-1} = -[(-W^2\Delta + 1) - a_s^2]^{-1} \tag{5.6}$$

This is the analog of the summation of ladder diagrams. If $G_0 = a_s$ were replaced by $|G_0| = 1$ we would have diffusive propagator $(W^2\Delta)^{-1}$. However since G_0^2 is complex the propagator above decays exponentially fast.

Mathematical control of the above perturbation scheme for small ϵ seems to be difficult to achieve for E inside [-2,2] unless the SUSY approach is used. Part of the reason is that the sum of all higher order contributions to the above series is divergent. If the perturbation theory is terminated at some stage, we have not been able to estimate the error. Perhaps the reason that such estimates are difficult is that the second saddle is invisible perturbatively but is estimated naturally in the SUSY approach.

The SUSY weight for RBM is expressed in terms of a_j, b_j with $j \in \Lambda \subset Z^d$ after the Grassmann variables have been integrated out. When J is given by (1.3) get :

$$\exp[-\frac{1}{2}\sum_j \{W^2(\nabla a_j)^2 + W^2(\nabla b_j)^2 + a_j^2 + b_j^2\}] \prod_j \frac{E_\epsilon - ib_j}{E_\epsilon - a_j}$$
$$\times \det\{-W^2\Delta + 1 \; - \delta_{ij}(E_\epsilon - a_j)^{-1}(E_\epsilon - ib_j)^{-1}\} DaDb \tag{5.7}$$

A rigorous analysis the perturbation theory is provided by the analysis of the above expression. One again it is shown that the dominant contribution comes from the saddle point. However as in GUE one must take into account the second saddle.

Theorem (DPS) Let d=3, J given by (3) and $|E| \leq 1.8$ For $W \geq W_0$ the average $< G(E_\epsilon, j, j) >$ for RBM is unifomly bounded in ϵ *and* Λ. Moreover we have

$$|\langle G(E_\epsilon; 0, x)\, G(E_\epsilon; x, 0)\rangle| \leq e^{-m|x|}$$

for $m \approx W^{-1}$.

Note that for random band $< G(E_\epsilon; 0, x) >= 0$ for $x \neq 0$. The resumed expression (7) above is equal to the inverse of the Hessian of the action at the saddle point a_s, b_s given by $(-W^2\Delta + 1 - a_s^2)^{-1}$ of the SUSY model for RBM. Thus the SUSY approach automatically resums the leading contribution in perturbation theory. Recall (2.6) that for GUE the Hessian is proportional to $1 - a_s^2$.

Remark The lattice random Schrödinger Green's function $< G(E_\epsilon, 0, x) >$ is expected to decay exponentially fast. This can be seen from a simple modification of perturbation theory described above. However, the corresponding SUSY model is more difficult to analyze in this case because of oscillations arising from the Laplacian.

§6 The SUSY Hyperbolic sigma model

In this section we study a simpler version of the Efetov sigma models due to Zirnbauer [Zirn91]. This model is the $\mathbb{H}^{2|2}$ model described in the introduction. This model is expected to qualitatively reflect the phenomenology of Anderson's tight binding model. The great advantage of this model is that the Grassmann degrees of freedom can be explicitly integrated out to produce a real effective action in bosonic variables. Thus probabilistic methods can be applied. In 3D we shall prove that this model has the analog of the Anderson transition.

Remark: I have copied this section from [Sp2] - some of the numbering is off.

In order to define the $\mathbb{H}^{2|2}$ sigma model, let u_j be a vector at each lattice point $j \in \Lambda \subset \mathbb{Z}^d$ with three bosonic components and two fermionic components

$$u_j = (z_j, x_j, y_j, \xi_j, \eta_j) \ ,$$

where ξ, η are odd elements and z, x, y are even elements of a real Grassmann algebra. The scalar product is defined by

$$(u, u') = -zz' + xx' + yy' + \xi\eta' - \eta\xi' \ , \qquad (u, u) = -z^2 + x^2 + y^2 + 2\xi\eta$$

and the action is obtained by summing over nearest neighbors j, j'

$$\mathcal{S}[u] = \frac{1}{2} \sum_{(j,j')\in\Lambda} \beta(u_j - u_{j'}, u_j - u_{j'}) + \sum_{j\in\Lambda} \varepsilon_j (z_j - 1) \ . \tag{6.1}$$

The sigma model constraint, $(u_j, u_j) = -1$, is imposed so that the field lies on a SUSY hyperboloid, $\mathbb{H}^{2|2}$.

We choose the branch of the hyperboloid so that $z_j \geq 1$ for each j. It is very useful to parametrize this manifold in horospherical coordinates:

$$x = \sinh t - e^t \left(\tfrac{1}{2}s^2 + \bar{\psi}\psi\right) \ , \quad y = se^t, \quad \xi = \bar{\psi}e^t \ , \quad \eta = \psi e^t,$$

and

$$z = \cosh t + e^t \left(\tfrac{1}{2}s^2 + \bar{\psi}\psi\right)$$

where t and s are even elements and $\bar{\psi}$, ψ are odd elements of a real Grassmann algebra.

In these coordinates, the sigma model action is given by

$$\mathcal{S}[t, s, \psi, \bar{\psi}] = \sum_{(ij)\in\Lambda} \beta(\cosh(t_i - t_j) - 1)$$

$$+\tfrac{1}{2}[s; D_{\beta,\varepsilon}s] + [\bar{\psi}D_{\beta,\varepsilon}\psi] + \sum_{j\in\Lambda} \varepsilon_j(\cosh t_j - 1) \tag{6.2}$$

Note that the action is quadratic in the Grassmann and s variables. Here $D_{\beta,\varepsilon} = D_{\beta,\varepsilon}(t)$ is the generator of a random walk in random environment, given by the quadratic form

$$[v\,; D_{\beta,\varepsilon}(t)\, v]_\Lambda \equiv \beta \sum\nolimits_{(jj')} \mathrm{e}^{t_j + t_{j'}} (v_j - v_{j'})^2 + \sum\nolimits_{k\in\Lambda} \varepsilon_k\, \mathrm{e}^{t_k} v_k^2\,. \tag{6.3}$$

The weights, $e^{t_j + t_{j'}}$, are the local conductances across an nearest neighbor edge j, j'. The $\varepsilon_j\, e^{t_j}$ term is a killing rate for the walk at j.

After integrating over the Grassmann variables ψ, $\bar{\psi}$ and the variables $s_j \in \mathbb{R}$ we get the effective bosonic field theory with action $\mathcal{S}_{\beta,\varepsilon}(t)$ and partition function

$$Z_\Lambda(\beta, \epsilon) = \int e^{-\mathcal{S}_{\beta,\epsilon}(t)} \prod e^{-t_j} dt_j$$

$$= \int e^{-\beta\mathcal{L}(t)} \cdot [\det\ D_{\beta,\varepsilon}(t)]^{1/2} \prod_j e^{-t_j} \frac{dt_j}{\sqrt{2\pi}}. \tag{6.4}$$

where

$$\mathcal{L}(t) = \sum_{j\sim j'} [\, cosh(t_j - t_{j'}) - 1] + \sum_j \frac{\varepsilon_j}{\beta} [(\cosh(t_j - 1)) - 1]. \tag{6.5}$$

Note that the determinant is a positive but nonlocal functional of the t_j hence the effective action, $\mathcal{S}$, is also nonlocal. The additional factor of e^{-t_j} in (6.4) arises from a Jacobian. Because of the internal supersymmetry, we know that for all values of β, ε the partition function

$$Z(\beta, \varepsilon) \equiv 1. \tag{6.6}$$

This identity holds even if β is edge dependent.

The analog of the Green's function $< |G(E_\epsilon; 0, x)|^2 >$ of the Anderson model is the average of the Green's function of $D_{\beta,\varepsilon}$,

$$< s_0 e^{t_0} s_x e^{t_x} > (\beta, \varepsilon) =< e^{(t_0+t_x)} D_{\beta,\varepsilon}(t)^{-1}(0, x) > (\beta, \varepsilon) \equiv \mathcal{G}_{\beta,\varepsilon}(0, x) \tag{6.7}$$

where the expectation is with respect to the SUSY statistical mechanics weight defined above. The parameter $\beta = \beta(E)$ is roughly the bare conductance across an edge and we shall usually set $\varepsilon = \varepsilon_j$ for all j. In addition to the identity $Z(\beta, \varepsilon) \equiv 1$ there are additional Ward identities

$$< e^{t_j} >\equiv 1, \qquad \varepsilon \sum_x \mathcal{G}_{\beta,\varepsilon}(0, x) = 1 \tag{6.8}$$

which hold for all values of β and ε.

Note that if the $|t_j| \leq Const$, then the conductances are uniformly bounded from above and below and

$$D_{\beta,\varepsilon}(t)^{-1}(0, x) \approx (-\beta\Delta + \varepsilon)^{-1}(0, x)$$

is the diffusion propagator. Thus the Anderson transition can only occur due to the large deviations of the t field.

An alternative Schrödinger like representation of (6.7) is given by

$$\mathcal{G}_{\beta,\varepsilon}(0, x) =< \tilde{D}^{-1}_{\beta,\varepsilon}(t)(0, x) > \tag{6.9}$$

where

$$e^{-t} D_{\beta,\varepsilon}(t) e^{-t} \equiv \tilde{D}_{\beta,\varepsilon}(t) = -\beta\Delta + \beta V(t) + \varepsilon\, \mathrm{e}^{-t} \ , \tag{6.10}$$

and $V(t)$ is a diagonal matrix (or 'potential') given by

$$V_{jj}(t) = \sum\nolimits_{|i-j|=1} (\mathrm{e}^{t_i - t_j} - 1).$$

In this representation, the potential is highly correlated and $\tilde{D} \geq 0$ as a quadratic form.

Some insight into the transition for the $\mathbb{H}^{2|2}$ model can be obtained by finding the configuration $t_j = t^*$ which minimizes that action $\mathcal{S}_{\beta,\varepsilon}(t)$ appearing in (6.4). It is shown in [DSZ] that this configuration is unique and does not depend on j. For large β

$$\text{1D:} \quad \varepsilon\, \mathrm{e}^{-t^*} \simeq \beta^{-1}, \qquad \text{2D:} \quad \varepsilon\, \mathrm{e}^{-t^*} \simeq \mathrm{e}^{-\beta} \qquad \text{3D:} \qquad t^* \simeq 0\ . \tag{6.11}$$

Note that in one and two dimensions, t^* depends sensitively on ϵ and that negative values of t_j are favored as $\varepsilon \to 0$. This means that at t^* a mass εe^{-t^*} in (6.11) appears even as $\varepsilon \to 0$. Another interpretation is that the classical conductance $e^{t_j+t_{j'}}$ should be small in *some sense.* This is a somewhat subtle point. Due to large deviations of the t field in 1D and 2D, $< e^{t_j+t_{j'}} >$ is expected to diverge, whereas $< e^{t_j/2} >$ should become small as $\varepsilon \to 0$.

When β is small, $\varepsilon e^{-t^*} \simeq 1$ in any dimension. Thus the saddle point t^* suggests localization occurs in both 1D and 2D for all β and in 3D for small β. In 2D, this agrees with the predictions of localization by Abrahams, Anderson, Licciardello and Ramakrishnan [Abra79] at any nonzero disorder. Although the saddle point analysis has some appeal, it does not account for the large deviations away from t^* and seems incompatible with the sum rule $< e^{t_j} >= 1$. In 3D, large deviations away from $t^* = 0$ are controlled for large β. See the discussion below.

For later discussion it is interesting to consider the case in which $\varepsilon_0 = 1$ but $\varepsilon_j = 0$ otherwise. This corresponds to a random walk starting at O with no killing. In this case the saddle point is not translation invariant. In one and two dimensions we have $e^{t_j^*}$ goes to 0 exponentially fast for large $|j|$. Thus the conductance becomes small as we move away from 0. We expect that this implies $< e^{t_j/2} > \to 0$ exponentially fast in 1D and 2D producing localization.

The main theorem established in [DSZ] states that in 3D fluctuations around $t^* = 0$ are rare. Let $G_0 = (-\beta\Delta + \epsilon)^{-1}$ be the Green's function for the Laplacian.

Theorem 3 If $d \geq 3$, and the volume $\Lambda \to \mathbb{Z}^d$, there is a $\bar{\beta} \geq 0$ such that for $\beta \geq \bar{\beta}$ then for all j,

$$< cosh^8(t_j) > \leq Const \tag{6.12}$$

where the constant is uniform in ϵ. This implies quasi-diffusion: Let $\mathcal{G}$ be given by (6.7) or (6.9). There is a constant K so that we have the quadratic form bound

$$\frac{1}{K}[\tilde{f};G_0\tilde{f}] \le \sum_{x,y} \mathcal{G}_{\beta,\varepsilon}(x,y)\, f(x)f(y) \le K[f;G_0 f]\ , \tag{6.13}$$

where f(x) is nonnegative function and $\tilde{f}(x) = (1+|x|)^{-\alpha} f(x)$. The constant $\alpha > 0$ is small for large β.

Remarks: The power 8 can be increased by making β larger. The lower bound is not sharp, (α should be 0), and one expects point wise diffusive bounds on $\mathcal{G}_{\beta,\varepsilon}(x,y)$ to hold. However, in order to prove this one needs to show that the set $(j : |t_j| \ge M >> 0)$ does not percolate. This is expected to be true but has not yet been mathematically established partly because of the high degree of correlation in the t field.

The next theorem establishes localization for small β in any dimension. See [DS].

Theorem 4 Let $\varepsilon_x > 0$, $\varepsilon_y > 0$ and $\sum_{j\in\Lambda} \varepsilon_j \le 1$. Then for all $0 < \beta < \beta_c$ (β_c defined below) the correlation function $\mathcal{G}_{\beta,\varepsilon}(x,y)$, (6.9), decays exponentially with the distance $|x-y|$. More precisely:

$$\mathcal{G}_{\beta,\varepsilon}(x,y) = \langle \tilde{D}^{-1}_{\beta,\varepsilon}(t)(x,y)\rangle \le\ C_0\left(\varepsilon_x^{-1} + \varepsilon_y^{-1}\right)\, \left[I_\beta\, e^{\beta(c_d-1)}\, c_d\right]^{|x-y|} \tag{6.14}$$

where $c_d = 2d-1$, C_0 is a constant and

$$I_\beta = \sqrt{\beta}\int_{-\infty}^{\infty}\frac{dt}{\sqrt{2\pi}}e^{-\beta(\cosh t-1)}\ .$$

Finally β_c is defined so that:

$$\left[I_\beta\, e^{\beta(c_d-1)}\, c_d\right]\ <\ \left[I_{\beta_c} e^{\beta_c(c_d-1)} c_d\right]\ =\ 1 \quad \forall \beta < \beta_c.$$

These estimates hold uniformly in the volume.

Remarks: The first proof of localization for the $\mathbb{H}^{2|2}$ model in 1D was given by Zirnbauer in [Zirn91]. Note that in 1D, $c_d - 1 = 0$ and inequality holds for any $\beta_c \ge 0$. The above estimate is sharp in 1D. Thus the decay for small β is proportional to $|\sqrt{\beta}\, ln\beta|^{|x-y|}$ rather than $\beta^{|x-y|}$ which is typical for lattice sigma models with compact targets. The divergence of ε^{-1} is compatible with the sum rule (1.27) and is a signal of localization.

The proof of the above theorem relies heavily on the supersymmetric nature of the action. It is known that a purely hyperbolic sigma model of the kind studied in [SZ] cannot have a phase transition. The action for the purely hyperbolic case looks like that of the $\mathbb{H}^{2|2}$ model except that $[DetD_{\beta,\varepsilon}(t)]^{1/2}$ is replaced by $[DetD_{\beta,\varepsilon}(t)]^{-1/2}$. D. Brydges has pointed out that since the logarithm of $DetD_{\beta,\varepsilon}(t)$ is convex as a functional of t, the action for the hyperbolic sigma model is always convex and therefore no transition can occur. See [DSZ] for details. In Wegner's hyperbolic model the replica number must be 0 in order to see localization.

Role of Ward identities in the Proof.

The proof of Theorems 3 and 4 above rely heavily on Ward identities. For Theorem 3 we use Ward identities to bound fluctuations of the t field by getting bounds in 3D on $< cosh^m(t_i - t_j) >$. One these bounds or established we can control This is done by induction on the distance $|i-j|$. For Theorem 4 we use the fact that for any region Λ the partition function $Z_\Lambda = 1$.

If a function S of the variables $x, y, z, \psi, \bar{\psi}$ is supersymmetric, i.e., it is invariant under transformations preserving

$$x_i x_j + y_i y_j + \bar{\psi}_i \psi_j - \psi_i \bar{\psi}_j$$

then $\int S = S(0)$. In horospherical coordinates the function S_{ij} given by

$$S_{ij} = B_{ij} + e^{t_i+t_j}(\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j)) \tag{6.15}$$

where

$$B_{ij} = cosh(t_i - t_j) + \frac{1}{2} e^{t_i+t_j}(s_i - s_j)^2 \tag{6.16}$$

is supersymmetric. If i and j are nearest neighbors, $S_{ij} - 1$ is a term in the action and it follows that the partition function $Z_\Lambda(\beta, \epsilon) \equiv 1$. More generally for each m we have

(1) $$< S_{ij}^m >=< B_{ij}^m \, [1 - m B_{ij}^{-1} e^{t_i+t_j}(\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j)] > \equiv 1.$$

The integration over the Grassmann variables above is explicitly given by

(2) $$G_{ij} = \frac{\mathrm{e}^{t_i+t_j}}{B_{xy}} \left[(\delta_i - \delta_j);\; D_{\beta,\varepsilon}(t)^{-1}(\delta_i - \delta_j)\right]_\Lambda$$

since the action is quadratic in $\bar{\psi}, \psi$. Thus we have the identity

(3) $$< B_{ij}^m (1 - m G_{ij}) > \equiv 1.$$

Note that $0 \le cosh^m(t_i - t_j) \le B_{ij}^m$. From the definition of $D_{\beta,\varepsilon}$ given in (1.24) we see that for large β, G is typically proportional to $1/\beta$ in 3D. However, there are rare configurations where $t_k \approx -\infty$ for k on a closed surface $\subset \mathbb{Z}^3$ separating i and j for which G_{ij} can diverge as $\epsilon \to 0$. If this surface is of finite volume enclosing i, then there is a finite volume 0 mode producing a divergence in $D_{\beta,\varepsilon}(t)^{-1}(i,i)$. If i, j are nearest neighbors then it is easy to show that G_{ij} is less than β^{-1} for all t configurations. Thus if $m/\beta \le 1/2$ then (1.39) implies that $0 \le cosh^m(t_i - t_j) \le 2$. In general, there is no uniform bound on G_{ij} and we must use induction on $|i-j|$ to prove that configurations for which $1/2 \le m\,G_{ij}$ are rare for large β in 3D. In this way fluctuations of the t field can be controlled and quasi-diffusion is established, see [DSZ].

The proof of the localized phase is technically simpler than the proof of Theorem 3. Nevertheless, it is of some interest because it shows that $\mathbb{H}^{2|2}$ sigma model reflects the localized as well as the extended states phase in 3D. The main idea relies on the following lemma. Let M be an invertible matrix indexed by sites of Λ and let γ denote a self avoiding path starting at i and ending at j. Let M_{ij}^{-1} be matrix elements of the inverse and let M_{γ^c} be the matrix obtained from M by striking out all rows and columns indexed by the vertices covered by γ.

Lemma Let M and M_{γ^c} be as above, then

$$\frac{\partial}{\partial M_{ji}} \det M = [M_{ij}^{-1} \det M] \;=\; \sum_{\gamma_{ij}} [(-M_{ij_1})(-M_{j_1 j_2}) \cdots (-M_{j_m j})] \;\det M_{\gamma^c}$$

where the sum ranges over all self-avoiding paths γ connecting i and j, $\gamma_{ij} = (i, j_1, \ldots j_m, j)$, with $m \ge 0$.

Apply this lemma to

$$M = e^{-t} D_{\beta,\varepsilon}(t) e^{-t} \equiv \tilde{D}_{\beta,\varepsilon}(t) = -\beta\Delta + \beta V(t) + \varepsilon\, \mathrm{e}^{-t} \tag{4}$$

and notice that with this choice of M, for all non-zero contributions, γ are nearest neighbor self-avoiding paths and that each step contributes a factor of β. The proof of (1.40) comes from the fact the determinant of M can be expressed as a gas of non overlapping cycles covering Λ. The derivative with respect to M_{ji} selects the cycle containing j and i and produces the path γ_{ij}.

The other loops contribute to $\det M_{\gamma^c}$. By (1.28) and (1.41) we have

$$(5) \qquad \mathcal{G}_{\beta,\varepsilon}(x,y) = < M^{-1}_{xy} > = \int e^{-\beta \mathcal{L}(t)} M^{-1}_{xy} \, [\det M]^{1/2} \prod_j \frac{dt_j}{\sqrt{2\pi}}.$$

Note the factors of e^{-t_j} appearing in (1.25) have been absorbed into the determinant. Now write

$$M^{-1}_{xy} [\det M]^{1/2} = \sqrt{M^{-1}_{xy}} \sqrt{M^{-1}_{xy} \det M}.$$

The first factor on the right hand side is bounded by $\epsilon_x^{-1/2} e^{t_x/2} + {\epsilon_y}^{-1/2} e^{t_y/2}$. For the second factor, we use the lemma. Let $\mathcal{L} = \mathcal{L}_\gamma + \mathcal{L}_{\gamma^c} + \mathcal{L}_{\gamma,\gamma^c}$ where $\mathcal{L}_\gamma$ denotes the restriction of $\mathcal{L}$ to γ. Then using the fact that

$$\int e^{-\beta \mathcal{L}_{\gamma^c}} [\det M_{\gamma^c}]^{1/2} \prod_j \frac{dt_j}{\sqrt{2\pi}} \equiv 1$$

we can bound

$$0 \le \mathcal{G}_{\beta,\varepsilon}(x,y) \le \sum_{\gamma_{xy}} \sqrt{\beta}^{|\gamma_{xy}|} \int e^{-\beta \mathcal{L}_\gamma + \beta \mathcal{L}_{\gamma,\gamma^c}} \, [\epsilon_x^{-1/2} e^{t_x/2} + {\epsilon_y}^{-1/2} e^{t_y/2}] \prod_j \frac{dt_j}{\sqrt{2\pi}}$$

where $|\gamma_{xy}|$ is the length of the self-avoiding path from x to y. The proof of Theorem 4 follows from the fact that the integral along γ is one dimensional and can be estimated as a product. See [DS] for further details.

Edge Reinforced Random Walk and Localization

Linearly edge reinforced random walk (ERRW) is a history-dependent walk which prefers to visit edges it has visited more frequently in the past. Consider a discrete time walk on $\mathbb{Z}^d$ starting at the origin and let $n(e,t)$ denote the number of times the walk has visited the edge e up to time t. Then the probability $P(v, v', t+1)$ that the walk at vertex v will visit a neighboring edge $e = (v, v')$ at time $t+1$ is given by

$$P(v, v', t+1) = (\beta + n(e,t))/S_\beta(v,t)$$

where S is the sum of $\beta + n(e', t)$ over all the edges e' touching v. The parameter β is analogous to β in the $\mathbb{H}^{2|2}$ model . Note that if β is large, the reinforcement is weak. This process was defined by Diaconis and is partially exchangeable which means that any two paths with the same stating point and same values of n(e,t) have the same probability. Thus the order in which the edges were visited is irrelevant. Such processes can be expressed as a superposition of Markov processes [Diac80]. In fact Coppersmith and Diaconis proved that this ERRW can be expressed as a random walk in a random environment. There is an explicit formula for the Gibbs weight of the local conductances across each edge, see [Diac88, Kean00, Merk06] which is quite close to that for $\mathbb{H}^{2|2}$ model with $\varepsilon_j = 0$ except at 0 where $\varepsilon_0 = 1$. It is nonlocal and also expressed in terms of a square root of a determinant. Moreover the partition function can be explicitly computed and there are identities similar to Ward identities (1.27). These presumably reflect conservation of probability.

In 1D and 1D strips, ERRW is *localized* for any value of $\beta > 0$. This means that the probability of finding an ERRW, $W(t)$, at a distance r from the origin at fixed time t is exponentially small in r, thus

$$Prob\,[|W(t)| \geq r] \leq Ce^{-mr}.$$

Merkl and Rolles [Merk09] established this result by proving that the conductance across an edge goes to zero exponentially fast with the distance of the edge to the origin. More precisely they show that the conductance c satisfies

$$< c_{jj'}^{1/4} > \leq Ce^{-m|j|}.$$

The local conductance $c_{jj'}$ corresponds to $e^{t_j+t_{j'}}$ hence the decay of $< c_{jj'}^{1/4} >$ should be closely related to that of $< e^{t_j/2} >$ in the $\mathbb{H}^{2|2}$ model. See the discussion just before Theorem 3. Note that the factor 1/2 is important, otherwise we have $< e^{t_j} > \equiv 1$. Their argument is based on a Mermin-Wagner like deformation of the Gibbs measure. It also shows that in 2D, $< c_{jj'}^{1/4} > \to 0$. In 3D, there are no rigorous theorems for ERRW. However, by analogy with Theorem 2, localization is expected to occur for strong reinforcement, i.e., for β small. It is natural to conjecture that in 2D ERRW is always exponentially localized for all values of reinforcement. On the Bethe lattice Pemantle [Pema88a] proved that ERRW has a phase transition. For $\beta \gg 1$ the walk is weakly reinforced and transient whereas for $0 < \beta \ll 1$ the walk is recurrent. It is an open question whether ERRW has the analog of the Anderson transition in 3D. See [Pema07, Merk06] for reviews of this subject.

Appendix on Integration

Complex Gaussian Integrals

Let $z = x + iy$ with $x, y \in \mathbb{R}$. Let dz $= dxdy/\pi$ and suppose $Re\, a > 0,\ a \in \mathbb{C}$. Then

$$\int e^{-a\bar{z}z} dz = \pi^{-1} \iint e^{-ar^2} r dr d\theta = a^{-1}. \qquad A.1$$

Also

$$\frac{1}{\sqrt{2\pi}} \int e^{-ax^2/2} dx = a^{-1/2}.$$

In the multi-dimensional case let $z = (z_1, z_2, ...z_n)$, $z^* = \bar{z}^t$. For an $n \times n$ matrix A with $ReA > 0$

$$\int e^{-z^* A z} Dz = (\det A)^{-1} \quad where \quad Dz = \prod_1^n dx_i dy_i/\pi \qquad A.2$$

We also use the notation $[z; Az] \equiv \sum \bar{z}_j A_{ij} z_j = z^* A z$. The pair correlation

$$< z_j \bar{z}_k > \equiv \det(A) \int e^{-z^* A z}\, z_j \bar{z}_k\, Dz = A^{-1}_{jk}. \qquad A.3$$

Note that $< z_j z_k > = < \bar{z}_j \bar{z}_k > = 0$. This is because the integral is invariant under the global transform $z \to e^{i\phi} z$, $\bar{z} \to e^{-i\phi} \bar{z}$. The generating function is given by

$$< e^{z^* v + w * z} > = e^{w^* A^{-1} v} = e^{[w; A^{-1} z]}$$

For real variables x $= (x_1, ...x_n)$ and A is symmetric

$$\int e^{-[x;Ax]/2}\, Dx = (detA)^{-1/2} \quad where \quad Dx = \prod_i^n dx_i/\sqrt{2\pi}. \qquad A.4$$

Its generating function is $< e^{[x;y]} > = e^{\,[y; A^{-1} y]/2}$.

There are similar formulas for integration over $N \times N$ matrices:

$$\int e^{-NTrH^2/2} e^{iTrMH}\, DH = e^{-TrM^2/2N} \int e^{-NTrH^2/2} DH \qquad A.5$$

For the case of Band matrices the generating function is

$$< e^{iTrHM} >= e^{-<(trHM)^2)/2>} = e^{-1/2\sum J_{ij}M_{ji}M_{ij}} = e^{-1/2\,Tr(M\sqrt{J})^2} \qquad A.6$$

Grassmann integration

Grassmann variables $\psi_i, \bar{\psi}_j$ are anticommunting variables $1 \le i, j \le N$ satifying $\psi_j^2 = \bar{\psi}_j^2 = 0$, *and* $\bar{\psi}_j\psi_i = -\psi_i\bar{\psi}_j$. Also

$$\psi_j\psi_i = -\psi_i\psi_j, \qquad \bar{\psi}_j\bar{\psi}_i = -\bar{\psi}_i\bar{\psi}_j. \qquad A.7$$

The $\bar{\psi}_j$ is simply convenient notation for another independent family of Grassmann variables. Even monomials in the Grassmann variables and complex numbers commute with Grassmann variables. The polynomials in the Grassann variables form a Z_2 graded algebra, with the even and odd polynomials belong to the even and odd gradings respectively.

The Grassman integral, defined below, plays an important role in many aspects of physics. It is an extremely efficient and useful notation analysis of interacting Fermi systems, Ising models (Free Fermions), and SUSY. Athough most of the time we shall eliminate the Grassmann variables by integrating them out, they are nevertheless an essential tool for obtaining the identities we shall analyze. See [Ab1, FKT, Mir, Sal] for more details about Grassmann integration.

We define integration over

$$D\psi \equiv \prod^{N} d\bar{\psi}_j d\psi_j \qquad A.8$$

as follows. For N=1

$$\int (a\psi_1\bar{\psi}_1 + b\psi_1 + c\bar{\psi}_1 + d)\, D\psi = a.$$

The general rule is that the integral of a polynomial in 2N variables with respect to $D\psi$ is by coefficent of the top monomial of degree 2N ordered as $\prod^N \psi_j\bar{\psi}_j$. Note that since the factors in the product are even, their order does not matter. Any element of the Grassmann alagebra can be expressed as a polynomial and the top monomial can always be rearranged using the anticommutation rules so that it coincides with $\prod^N \psi_j\bar{\psi}_j$.

To differentiate a Grassmann monomial, use the rule $\frac{\partial}{\partial\psi_j}\psi_k = \delta_{jk}$ and that the derivative anticommutes with other Grassmann variables. We have

$$\frac{\partial}{\partial\psi_j}\psi_j \prod \psi_k = \prod \psi_k.$$

To differentiate a general monomial in ψ_j, use the anticommutation relations so that it is of the above form. If ψ_j is not a factor the derivative is 0.

For any $N \times N$ matrix A we have the following analog of Gaussian integration

$$\int e^{-[\psi;A\psi]}\, D\psi = detA \quad where \quad [\psi;A\psi] = \sum \bar{\psi}_i A_{ij} \psi_j. \qquad A.9$$

Moreover,

$$< \psi_i \bar{\psi}_j > \equiv detA^{-1} \int \psi_i \bar{\psi}_j \, e^{-[\psi;A\psi]} D\psi = A^{-1}_{ij}. \qquad A.10$$

This formula can be proved by integration by parts and using the fact that

$$\int \frac{\partial}{\partial\psi_k} F(\psi,\bar{\psi}) D\psi = 0 \quad with \quad F = \psi_i e^{-[\psi;A\psi]}$$

Let us work out a very simple example:

$$\int e^{-a\bar{\psi}_1\psi_1} D\psi = \int (1 - a\bar{\psi}_1\psi_1) D\psi = a \int \psi_1\bar{\psi}_1 D\psi = a.$$

Exercise: Show that if A is a 2×2 matrix, A.9 holds.

To prove the general case note that the exponential can be written as a product $\prod_i (1 - \sum_j A_{ij}\bar{\psi}_i\psi_j)$ and we look at the terms:

$$\sum A_{1j_1} A_{2j_2} ... A_{N,j_N} \int \bar{\psi}_1 \psi_{j_1} \, \bar{\psi}_2 \psi_{j_2} ... \bar{\psi}_N \psi_{j_N} \, D\psi.$$

The j_i are distinct and hence are a permutation of $1...N$. The integral then is the sign of the permutation and thus we obtain the determinant. The generating function is given by

$$< e^{\bar{\rho}^t\psi + \bar{\psi}^t\rho} > = detA^{-1} \int e^{-[\psi;A\psi]} \, e^{\bar{\rho}^t\psi + \bar{\psi}^t\rho} D\psi \; = e^{\bar{\rho}^t A^{-1} \rho} \qquad A.11$$

where $\rho, \bar{\rho}$ are independent Grassmann variables.

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