Universität Heidelberg WS 07/08

Quantum Mechanics, Problem Sheet 1

Homework solutions to be handed in and class exercises to be discussed
in the tutorials of the 2nd week (26.10.)

Class exercise P1: Fourier transformation (4 Punkte)

We define the Fourier transformed function $F[f]$ for a suitable function $f : \mathbb{R} \to \mathbb{C}$ by

$$F[f](k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}.$$

a) Show that

$$\begin{aligned} F[\lambda f + g] &= \lambda F[f] + F[g] \qquad (\lambda \in \mathbb{C}); \\ F[f'](k) &= ik F[f](k); \\ f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, F[f](k)\, e^{ikx}; \\ \int_{-\infty}^{\infty} dx\, f(x)\overline{g}(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, F[f](k)\overline{F[g]}(k). \end{aligned}$$

Hint: Use the following, extremely useful representation of the Dirac delta-function:

$$\int_{-\infty}^{\infty} dk\, e^{ik(x-y)} = 2\pi\, \delta(x-y). \qquad ☺$$

b) Calculate $F[f]$ for

$$f(x) = e^{-\alpha x^2 + \beta x} \quad (\mathrm{Re}\,\alpha > 0); \quad f(x) = \Theta(y-x)\Theta(y+x); \quad f(x) = \delta(x-y),$$

where $\Theta(x)$ is the Heaviside step function:

$$\Theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

c) Derive the equation ☺ by showing, that for all suitable test functions g the following equation holds:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk\, e^{ikx}\, g(x) = 2\pi\, g(0).$$

Instructions for a "Physicist's Proof": Assume that the ordering of integrations and limits can be freely interchanged. Use the result of Part b) for the integral over k, with the range of integration from $-K$ to K. Then substitute $t = Kx$ and finally calculate the integral over t in the limit $K \to \infty$, with the help of

$$\int_{-\infty}^{\infty} dt\, \frac{\sin t}{t} = \pi.$$

d) Use the results of a) and b) to find an alternative solution method for Ex. H1 d) via Fourier transformation.

Instructions: Write $\psi_0(x)$ as an integral over the Fourier modes $F[\psi_0](q)$, and consider their time evolution. In order to carry out the integration over q and to compare with the result of the homework, it is best to substitute $\kappa \equiv q - k$.

Homework exercise H1: Gaussian Wavepacket (10 Punkte)

Consider a Gaussian wavepacket in one dimension. The wavefunction at time $t = 0$ is

$$\psi_0(x) = N \exp\left(-\frac{x^2}{b^2} + ikx\right),$$

where N, b, k are positive real constants.

a) State the probability density in position space, and determine the normalisation factor N, so that the total probability is 1.

b) Calculate the expectation values $\langle X \rangle$, $\langle X^2 \rangle$, $\langle P \rangle$ und $\langle P^2 \rangle$. Here, P and X are the momentum and position operators, represented by

$$P = -i\hbar \frac{\partial}{\partial x}, \qquad X = x.$$

c) Calculate $(\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2$ and $(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2$. Show then that the uncertainty relation $\Delta P \Delta X \geq \hbar/2$ is saturated by a Gaussian wavepacket.

d) Calculate the wavefunction $\psi_t(x)$ at time t, given the Hamiltonian $H = P^2/2m$.

Hint: Make the ansatz

$$\psi_t(x) = N(t) \exp\left(-\frac{(x - a(t))^2}{b(t)^2} + ikx\right)$$

with (in general) complex functions $a(t), b(t), N(t)$, which you can determine from the Schrödinger equation..

e) Show that the resulting probability density at time t is also a Gaussian distribution. How does its width behave in the limit $t \to \infty$? What does this physically mean for a particle that is tightly localized at time $t = 0$ (i.e.ã wavepacket with small b)? Comment in view of the dispersion relation for the wave that you can obtain from part b).