Exercise sheet 3, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in and class exercises discussed
in the tutorials of Week 4 (9.11.07)

Class exercise P3: Hermite Polynomials (3 points)

The *Hermite Polynomials* H_n are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

a) Give the explicit forms of $H_n(x)$ for $0 \leq n \leq 3$.

b) Show that the Hermite polynomials fulfil

$$\left. \frac{\partial^n}{\partial t^n} e^{2xt-t^2} \right|_{t=0} = H_n(x),$$

or equivalently

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \qquad ☺$$

Ex. H4: Hermite Functions (5 points)

The *Hermite functions* Ψ_n are defined with the help of the Hermite polynomials from Ex. P3:

$$\Psi_n(x) = \left(n!\, 2^n \sqrt{\pi}\right)^{-1/2} \; e^{-x^2/2}\, H_n(x).$$

a) Show that the Hermite polynomials H_n solve the *Hermite differential equation*

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

Hint: Apply the operator $\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$ to both sides of Equation ☺.

b) Thus, show that the wavefunction $\psi_n(x) \equiv \Psi_n(\sqrt{m\omega/\hbar}\, x)$ fulfils the Schrödinger equation for the simple harmonic oscillator:

$$-\frac{\hbar^2}{2m}\psi_n''(x) + \frac{m\omega^2}{2}x^2\psi_n(x) = E_n\psi_n(x) \qquad \text{with } E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

Ex. H5: Coherent States (10 points)

Consider the one-dimensional simple harmonic oscillator of frequency ω from the lecture. We denote the n-th excited state by $|\, n\rangle$.

a) Show that for $\alpha \in \mathbb{C}$ the state

$$|\, \alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \,|\, n\rangle$$

is a normalized eigenstate of the lowering operator a with eigenvalue α. Such states are called *coherent* or *quasiclassical* states.

b) Calculate the expectation values $\langle X\rangle$, $\langle X^2\rangle$, $\langle P\rangle$, $\langle P^2\rangle$ und $\langle H\rangle$ in the state $|\, \alpha\rangle$. Calculate the squared fluctuations $(\Delta X)^2$ und $(\Delta P)^2$, and show that $(\Delta X)(\Delta P) = \hbar/2$.

Hint: Express the operators X, P and H in terms of the raising and lowering operators $a^\dagger$ and a.

c) Suppose that the oscillator is at time $t = 0$ in the state $|\, \alpha_0\rangle$ with $\alpha_0 = \rho e^{i\phi}$ (where $\phi \in \mathbb{R}$, $\rho \in \mathbb{R}_+$). Show that at any subsequent time t the state is also a coherent state, that can be written as $e^{-i\omega t/2}\,|\, \alpha(t)\rangle$. Determine the dependence of $\alpha(t)$ on ρ, ϕ, ω and t.

d) Calculate the time evolution of the expectation values $\langle X\rangle$ und $\langle P\rangle$. Thus, explain the description "quasiclassical state".