Exercise Sheet 4, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in and class exercises discussed
in the tutorials of Week 5 (16.11.07)

Class exercise P4: Spherical harmonics (3 points)

Reminder: The *spherical harmonics* Y_l^m are an orthonormal system of eigenfunctions of the Laplace operator on the 2-sphere. They are given in polar coordinates by

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, e^{im\phi}\, P_l^m(\cos\theta),$$

where m and l are integers with $l \geq 0$ und $-l \leq m \leq l$, and the *associated Legendre functions* P_l^m are given by

$$P_l^m(x) = \frac{(-1)^m}{2^l\, l!}\left(1-x^2\right)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

a) Give explicit forms of Y_0^0, Y_1^0, Y_1^1, Y_1^{-1} and Y_2^0.

b) Make sure that your results are mutually orthogonal.

c) Expand $f(\theta,\phi) = \sin\theta\sin\phi$ in spherical harmonics.

Exercise H6: Spectrum of the H atom (8+2 points)

In this exercise you determine the spectrum of the hydrogen atom with the help of the raising and loweing operators. Let $\psi_{Elm}(r,\theta,\phi) = f_{El}(r)\, Y_l^m(\theta,\phi)$ be the wavefunction of a bound state with energy $E < 0$, $\mathbf{L}^2$-eigenvalue $\hbar^2 l(l+1)$ and L_3-eigenvalue $\hbar m$. We define $\epsilon \equiv -2m_e E/\hbar^2$, $\beta \equiv m_e e^2/\hbar^2$ and $g_{\epsilon l}(r) \equiv r\, f_{El}(r)$.

a) Beginning with the Schrödinger equation for the hydrogen atom

$$\left(-\frac{\hbar^2}{2m_e}\left(\frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) + \frac{\mathbf{L}^2}{2m_e r^2} - \frac{e^2}{r}\right)\psi_{Elm} = E\,\psi_{Elm}$$

show that $h_l\, g_{\epsilon l}(r) = \epsilon\, g_{\epsilon l}(r)$ with $h_l = \left(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} + \frac{2\beta}{r}\right)$.

b) Show that the operators $a_l \equiv \frac{\partial}{\partial r} + \frac{l}{r} - \frac{\beta}{l}$ and $a_l^\dagger \equiv -\frac{\partial}{\partial r} + \frac{l}{r} - \frac{\beta}{l}$ are Hermitian conjugate with respect to the scalar product

$$\langle f|g\rangle \equiv \int_0^\infty dr\,\overline{f(r)}g(r) \quad \text{(for functions } f\,, g \text{ with } f(0) = g(0) = 0).$$

c) Show that: $h_l = -a_l^\dagger a_l + \frac{\beta^2}{l^2} = -a_{l+1}a_{l+1}^\dagger + \frac{\beta^2}{(l+1)^2}$.

d) Show that: $a_{l+1}^\dagger g_{\epsilon l}$ is either 0 or an eigenfunction of h_{l+1} with eigenvalue ϵ.

e) From the positivity of the norm $\left\langle a^\dagger_{l+1}\, g_{\epsilon l} \middle| a^\dagger_{l+1}\, g_{\epsilon l} \right\rangle \geq 0$, derive that there exists a maximum value of l for every possible ϵ (consider $\epsilon > 0$). Call this value $\bar{l}$, so that $a^\dagger_{\bar{l}+1}\, g_{\epsilon \bar{l}} = 0$. Thus show that the possible binding energies are of the form $E = -m_e e^4/(2\hbar^2 n^2)$ with positive integer n.

f (2 Extrapoints) Solve the differential equation $a^\dagger_{\bar{l}+1}\, g_{\epsilon \bar{l}}(r) = 0$ for arbitrary $\bar{l} \in \mathbb{N}_0$, and deduce from the normalizability of the solution that every $n \in \mathbb{N}^*$ yields a possible binding energy according to the formula above.

Exercise H7: Neutron in a gravitational field (8 points)

A team in the Physical Institute in Heidelberg in the gravitational field of the Earth drops cold neutrons onto a horizontal mirror, and then measures the resulting energy levels (see e.g. `http://www.physi.uni-heidelberg.de/~abele/nature.pdf`). It is assumed that the mirror reflects the neutrons perfectly and is at $x = 0$. The potential is then $V(x) = mgx$ for $x \geq 0$ and ∞ for $x < 0$, such that the Schrödinger equation in position space for positive x is

$$-\frac{\hbar^2}{2m}\psi''(x) + (mgx - E)\psi(x) = 0.$$

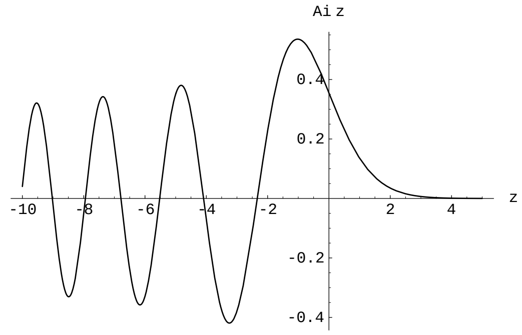
The general solution of this equation (a variant of the *Airy equation*) cannot be expressed via elementary functions. However, it is possible to obtain an integral expression.

a) State the Schrödinger equation in momentum space.

b) Find the general solution of the Schrödinger equation in momentum space. Write the wavefunction in position space in the region $x \geq 0$ as an integral over the Fourier modes thus determined. You can initially leave the normalization and energy indefinite. The possible energies are then determined by the boundary condition at $x = 0$, see below.

c) What boundary condition must the wavefunction satisfy at $x = 0$? Thus, relate the energies E_n of the bound states to the zeros of the *Airy function* Ai. This is defined by

$$\mathrm{Ai}\,(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \cos\left(\frac{t^3}{3} + tz\right).$$



The zeros of the Airy function are negative real numbers. There is no known closed expression for them, however they can easily be calculated with arbitrary accuracy. The first zero is at $z_0 = -2.338\ldots$

d) Calculate the ground state energy in eV.