Exercise Sheet 5, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in and class exercises discussed
in the tutorials of Week 6 (23/11/07)

Class exercise P5: Cross products and operators (3 points)

a) Show that the angular momentum operators $L_i = (\mathbf{X} \times \mathbf{P})_i$ $(i = 1, 2, 3)$ are Hermitian.

b) Let $\mathbf{A}$ and $\mathbf{B}$ be vectors whose components A_i und B_i are Hermitian operators. Which condition must the A_i und B_i fulfil for the components of the cross product $(\mathbf{A} \times \mathbf{B})_i$ to be Hermitian?

c) The classical Runge-Lenz vector of the Coulomb potential is

$$\mathbf{a} = \mathbf{p} \times \mathbf{l} - me^2 \frac{\mathbf{x}}{|\mathbf{x}|}$$

where the angular momentum is $\mathbf{l} = \mathbf{x} \times \mathbf{p}$. Show that a non-Hermitian operator arises from $\mathbf{a}$ under the substitution $\mathbf{x} \to \mathbf{X}, \mathbf{p} \to \mathbf{P}$. Find a Hermitian operator $\mathbf{A}$ that represents a sensible quantum mechanical substitute for $\mathbf{a}$ (*i.e.* that goes to $\mathbf{a}$ under the substitution $\mathbf{X} \to \mathbf{x}$, $\mathbf{P} \to \mathbf{p}$).

d) Express $\mathbf{L} \cdot \mathbf{L}$ and $\mathbf{l} \cdot \mathbf{l}$ by means of scalar products of $\mathbf{X}$ and $\mathbf{P}$ and $\mathbf{x}$ und $\mathbf{p}$ respectively.

Ex. H8: Commutators (6 points)

Let A, B, C be operators acting on a Hilbert space. Show that the following identities hold:

a)

$$[AB, C] = A[B, C] + [A, C]B,$$

b)

$$[[A, B], C] + [[C, A], B] + [[B, C], A] = 0 \qquad (\text{"Jacobi identity"}).$$

Suppose that $[A, B] = c\mathbb{1}$, where $\mathbb{1}$ is the unity operator and $c \in \mathbb{C}$. Let $f(A)$ be defined by a power series:

$$f(A) = \sum_{n=0}^{\infty} c_n A^n, \qquad c_n \in \mathbb{C}.$$

c) Show that

$$[f(A), B] = cf'(A).$$

d) Let $\mathbf{P} = (P_i)$ and $\mathbf{X} = (X_i)$ be the momentum and position operators in three dimensions (with $i = 1, 2, 3$). Calculated the commutator $[X_i, \mathbf{P}^2]$. Calculate in addition $\{x_i, \mathbf{p}^2\}$, where the Poisson bracket $\{\cdot, \cdot\}$ of two functions of position and momentum is defined by

$$\{f, g\} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}.$$

Ex. H9: Two-state system (8 points)

Consider a NH_3 molecule. The measured position of the N atom may be above or below the plane defined by the three H atoms. We represent the measured quantity "Position of the N atom" by the operator Σ acting on the Hilbert space $\mathbb{C}^2$. In the Schrödinger picture this operator is

$$\Sigma = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The measured quantity can take the values 1 (N atom upstairs) or -1 (N atom downstairs). The corresponding eigenstates are

$$\psi_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \psi_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since the Hamiltonian should be symmetrical under exchange of ψ_u und ψ_d, it has the form

$$H = \begin{pmatrix} E & W \\ W & E \end{pmatrix}$$

with real E and W.

a) Determine the (normalized) energy eigenstates and the corresponding energies.

b) Suppose that the molecule is in the state ψ_u at time $t = 0$. Calculate the time evolution of the system and the expectation value of H at arbitrary $t > 0$.

c) Determine the respective probabilities for the N atom to be found upstairs or downstairs at a measurement at time t. State the time evolution of the expectation value of Σ.

Now consider the same system in the Heisenberg picture.

d) Determine the operator Σ in the Heisenberg picture, and calculate the time evolution of its expectation value in the state ψ_u.

Hint: Write $H = E\,\mathbb{1} + W\sigma^1$ with

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and then consider the matrix entries of $e^{-iHt/\hbar}$, exploiting the fact that $(\sigma^1)^2 = \mathbb{1}$.