Exercise sheet 6, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in and class exercises discussed
in the tutorials of Week 7 (30.11.07)

Class exercise P6: Linear Algebra (3 points)

Let $V \cong \mathbb{C}^n$ be a complex vector space and $\langle \cdot | \cdot \rangle$ be a scalar product, *i.e.* for $f, g, h \in V$ and $\alpha \in \mathbb{C}$ the following relations hold:

$$\langle f | g \rangle = \langle g | f \rangle^* \tag{1}$$
$$\langle f | \alpha g + h \rangle = \alpha \langle f | g \rangle + \langle f | h \rangle \tag{2}$$
$$\langle f | f \rangle \geq 0 \text{ and } \langle f | f \rangle = 0 \text{ if and only if } f = 0. \tag{3}$$

In the following we use Dirac notation, *i.e.* we write $| f \rangle$ instead of f, $\langle f |$ instead of $f^\dagger$ and $| f \rangle \langle g |$ instead of $f \otimes g^\dagger$. Let $\{| e_1 \rangle, \ldots, | e_n \rangle\}$ be an orthonormal basis of V.

a) What is $\langle e_i | e_j \rangle$? What are the matrix entries of $| e_i \rangle \langle e_j |$ and $\sum_{i=1}^n | e_i \rangle \langle e_i |$ in this basis?

b) Show that the components of an arbitrary vector $| x \rangle \in V$, $| x \rangle = \sum_{i=1}^n c_i | e_i \rangle$, are given by $c_i = \langle e_i | x \rangle$. Es gilt $\langle x | x \rangle = \sum_{i=1}^n c_i^* c_i$.

c) Let $A : V \to V$ be a linear transformation. Show that, if A acting on the basis $\{| e_1 \rangle, \ldots, | e_n \rangle\}$ has matrix elements (a_{ij}), then the Hermitian conjugate transformation $A^\dagger$ has matrix elements $(A^\dagger)_{ij} = a_{ji}^*$.

d) Let $| e_i' \rangle = U | e_i \rangle$ with a unitary transformation U. Show that $\{| e_1' \rangle, \ldots, | e_n' \rangle\}$ also form an orthonormal basis, and express U in terms of the $| e_i' \rangle$ and $\langle e_i |$.

e) Let $A = A^\dagger$ be a Hermitian transformation. Show that $\exp(iA)$ is unitary. (The exponential function for matrices is here defined by means of its power series.)

Ex. H10: Angular momentum and uncertainty relation (4 points)

Consider again the angular momentum operator $\mathbf{L} = \mathbf{X} \times \mathbf{P}$.

a) Show that $[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$.

b) State the uncertainty relation for simultaneous measurements of L_1 and L_2. How large is the minimal uncertainty for bound states ψ_{nlm} of the hydrogen atom?

Ex. H11: Two-body problem (6 points)

Consider once more the hydrogen atom, this time taking account of the motion of the proton. Let $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{P}_1, \mathbf{P}_2$ be the position and momentum operators for the proton and the electron respectively, and let m_1 (m_2) be the electron (proton) mass. The Hilbert space of the system is the product of the two single particle Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We can understand $\mathbf{X}_1$ as an operator on $\mathcal{H}$ by identifying $\mathbf{X}_1$ with $\mathbf{X}_1 \otimes \mathbb{1}$, and correspondingly for the other operators.

a) State the Hamiltonian.

b) Determine the position operators $\mathbf{X}_s$ and $\mathbf{X}_r$ for the centre-of-mass and relative coordinates respectively. Calculate the corresponding momentum operators $\mathbf{P}_s$ and $\mathbf{P}_r$.

c) Express the Hamiltonian in terms of the new position and momentum operators. Determine the general form of a solution of the Schrödinger equation in this basis, and state the corresponding energy.

   Hint: Use the known solutions for the Coulomb problem and the free particle.

Aufgabe H12: Dirac delta-function potential (6 points)

Consider a particle moving in one dimension under the potential

$$V(x) = -\frac{D\,\hbar^2}{m}\,\delta(x) \qquad (D > 0).$$

This potential has one bound state (with negative energy). Find its normalized wavefunction in position space and the energy eigenvalue.

Hints: The wavefunction should be continuous at $x = 0$. You can determine the energy by integrating the Schrödinger equation from $-\epsilon$ to ϵ and then considering the limit $\epsilon \to 0$.