Universität Heidelberg WS 07/08

Exercise sheet 7, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in and class exercises discussed
in the tutorials of Week 9 (7/12/07)

Class exercise P7: Dirac notation, position and momentum space (2 points)

Consider a quantum particle moving in one dimension.

a) Let $|\, p\rangle$ be an eigenstate of the momentum operator with eigenvalue p. State the (unnormalised) representation in position space.

b) Let $|\, \psi\rangle$ be a state with wavefunction $\psi(x)$ in position space. Express the Fourier transform of $\psi(x)$ in Dirac notation.

c) Let A be a self-adjoint (and in general nonlocal) operator and $|\, \psi'\rangle = A\, |\, \psi\rangle$. Show that the Fourier transform of $\psi'(x)$ is given by the action of A (in its momentum space representation) on the Fourier transform of $\psi(x)$.

Ex. H13: Baker-Campbell-Hausdorff formula (5 points)

a) Verify the *Baker-Campbell-Hausdorff formula* for operators A and B,

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [^n A, B] \,.$$

where $[^n A, B]$ is recursively defined by $[^0 A, B] \equiv B$ and $[^{k+1} A, B] \equiv [A, [^k A, B]]$, thus $[^n A, B] = [A, \ldots [A, [A, B]] \ldots]$ with n brackets.

Hint: Consider $f(t) = e^{tA} B e^{-tA}$ with $t \in \mathbb{R}$.

b) Check the BCH-formula with $A = X$ and $B = P$, the position and momentum operators in one dimension, in the position space representation.

c) Suppose that $[A, B] = c\,\mathbb{1}$ with $c \in \mathbb{C}$. Show that

$$e^A e^B = e^B e^A e^c.$$

Ex. H14: Pauli matrices (5 points)

a) Show that a general Hermitian 2×2 matrix Σ can be written as $\Sigma = \mathbf{a} \cdot \boldsymbol{\sigma} + b\mathbb{1}$. Here $\mathbf{a} \in \mathbb{R}^3$, $b \in \mathbb{R}$, and $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) Show that:

$$e^{i\mathbf{a}\cdot\boldsymbol{\sigma}} = \cos|\mathbf{a}|\ \mathbb{1} + i\ \sin|\mathbf{a}|\ \frac{\mathbf{a}\cdot\boldsymbol{\sigma}}{|\mathbf{a}|}.$$

c) Find a unitary matrix that diagonalises σ^2, and describe it as $e^{i\mathbf{a}\cdot\boldsymbol{\sigma}}$.

Ex. H15: Ramsauer effect (5 points)

In 1921 Ramsauer investigated the penetrability of noble gases for low-energy electron beams. Classically one would expect that the collision probability falls monotonically with increasing energy of the electrons. But it transpired that, for particular values of the beam energy, the gas became practically completely transparent to the electrons. This is a quantum effect, which will be illustrated by means of a simple one-dimensional model.

In Ex. H2 you investigated tunnelling through a rectangular potential barrier. The result for the tunneling probability was

$$T(E) = \frac{4k^2q^2}{(k^2+q^2)^2\sinh^2(qa) + 4k^2q^2}.$$

where $k = \sqrt{2mE}/\hbar$ is the wavenumber of the incident plane wave arriving from the negative x-direction and $q = \sqrt{2m(V_0 - E)}/\hbar$, and where V_0 is the height and a the width of the barrier.

Now consider scattering states for the "inverted" potential

$$V = \begin{cases} -V_0, & 0 \le x \le a \\ 0, & \text{otherwise} \end{cases}$$

with $V_0 > 0$.

a) State the transmission probability $T(E)$, and determine the energies E_n for which the potential becomes "transparent".

b) Approximate $T(E)$ for energies close to E_n by resonances of the form

$$T(E) \approx \frac{(\Gamma_n/2)^2}{(E-E_n)^2 + (\Gamma_n/2)^2}$$

and determine the resonance widths Γ_n as functions of E_n, V_0, a and m.