Ëxercise sheet 9, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in in the tutorials of Week 10 (21st December '07)

Ex. H16: Translation operator (3 points)

The translation operator $T(\mathbf{a})$ is defined by

$$T(\mathbf{a}) = \exp\left(-\frac{i}{\hbar}\mathbf{a}\cdot\mathbf{P}\right),$$

where $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{P} = (P_1, P_2, P_3)$ is the momentum operator in three dimensions.

a) Show that $T(\mathbf{a})T(\mathbf{b}) = T(\mathbf{a}+\mathbf{b})$.

b) Show that, given the position operator $\mathbf{X} = (X_1, X_2, X_3)$, the following holds:

$$T(\mathbf{a})^\dagger\, \mathbf{X}\, T(\mathbf{a}) = \mathbf{X} + \mathbf{a}\mathbb{1}.$$

Ex. H17: Optical theorem (5 points)

Prove the *optical theorem* of scattering theory:

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\,\text{Im}\, f(\theta,\phi)\Big|_{\cos\theta=1}.$$

Here σ_{tot} is the total (integrated) cross-section, k is the wavenumber and f is the scattering amplitude.
Instructions: Integrate the continuity equation for the probability current over a sphere of radius r around the scattering centre, and consider the limit $r \to \infty$. Note furthermore that $f(\theta,\phi)$ is not dependent on ϕ for $\cos\theta = 1$.

Ex. H18: Partial wave expansion (8 points)

Consider the scattering of an incident plane wave travelling in the positive z-direction on a spherically symmetric, short-ranged potential. The aim of the exercise is to expand the scattering amplitude $f(\theta)$ in spherical harmonics (due to the spherical symmetry f is not dependent on ϕ) and to represent it in the following form:

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1) f_l P_l(\cos\theta).$$

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Here P_l is the lth Legendre polynomial given by $P_l = P_l^0$ with the associated Legendre function P_l^m already familiar from Ex. P4. The coefficient f_l (the so-called *partial wave amplitude*) may be calculated from the phase shift δ_l between the incident plane wave and the outgoing spherical wave.

a) Start by considering the expansion of a general ϕ-independent solution of the Schrödinger equation:

$$\psi(r,\theta) = \sum_{l=0}^{\infty} c_l \frac{u_l(r)}{r} Y_l^0(\theta) \equiv \sum_{l=0}^{\infty} (2l+1) a_l \frac{u_l(r)}{r} P_l(\cos\theta).$$

Show that, for $r \to \infty$, the radial part of the wavefunction $u_l(r)$ can be written in the following form:

$$u_l(r) = \frac{1}{k} \sin\left(kr - \frac{l\pi}{2} + \delta_l\right).$$

Here the value of δ_l depends on l, and we have chosen to split off the constant $\frac{l\pi}{2}$ from each of the terms (without loss of generality).

b) Show that $\delta_l = 0$ for an incident plane wave travelling in the positive z-direction with wavenumber k, und determine the corresponding coefficients a_l.

Hint: The following identity holds

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta)$$

with the *spherical Bessel functions* $j_l(x)$, that are related to the usual Bessel functions by $j_l(x) = \sqrt{\pi/2x}\, J_{l+1/2}(x)$. The asymptotic behaviour of J_α is

$$J_\alpha(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \qquad (x \to \infty).$$

c) Show that, for the complete scattering solution, the partial wave amplitudes in the decomposition ☺ are given by

$$f_l = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right).$$

d) Show that the total cross-section σ_{tot} can be described as follows:

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l.$$

Hint: Use the orthogonality of the Legendre polynomials:

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2\delta_{l,l'}}{2l+1}.$$