Exercise Sheet 10, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in and class exercises discussed
in the tutorials of Week 11 (11th Jan. '08)

Class exercise P10: Free path integral (3 points)

Consider a free particle in one dimension. The action between the times t_a and t_b is

$$S = \int_{t_a}^{t_b} dt\, \frac{1}{2} m \dot{x}^2.$$

The amplitude for propagation between x_a (at $t = t_a$) and x_b (at $t = t_b$) is thus given by the path integral

$$\begin{aligned}\langle x_b, t_b | x_a, t_a \rangle &= \int_{x_a}^{x_b} Dx \; \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \frac{m}{2} \dot{x}^2 \right) \\ &= \lim_{n\to\infty} \sqrt{\frac{m}{2\pi i \hbar \epsilon}} \int \prod_{i=1}^{n} \left\{ \frac{dx_i}{\sqrt{2\pi i \hbar \epsilon / m}} \right\} \exp\left(\frac{i}{\hbar} \epsilon \sum_{i=1}^{n+1} \frac{m}{2} \frac{(x_i - x_{i-1})^2}{\epsilon^2} \right)\end{aligned}$$

where $\epsilon = (t_b - t_a)/n$.

a) Show by using the result of Ex. H19 b) (for the special case of one dimension) that

$$\int dx_1 \exp\left(\frac{im}{2\hbar\epsilon} \left[(x_2 - x_1)^2 + (x_1 - x_0)^2 \right] \right) = \left(\frac{\pi \hbar \epsilon}{im} \right)^{1/2} \exp\left(\frac{im}{2 \cdot 2\hbar\epsilon} (x_2 - x_0)^2 \right).$$

b) Show also that

$$\int dx_2 \exp\left(\frac{im}{2\hbar\epsilon} \left[(x_3 - x_2)^2 + \frac{(x_2 - x_0)^2}{2} \right] \right) = \left(\frac{4\pi \hbar \epsilon}{3im} \right)^{1/2} \exp\left(\frac{im}{2 \cdot 3\hbar\epsilon} (x_3 - x_0)^2 \right).$$

c) Deduce that the amplitude is given by

$$\begin{aligned}&\langle x_b, t_b | x_a, t_a \rangle \\ &= \lim_{n\to\infty} \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{(n+1)/2} \prod_{j=1}^{n} \left(\frac{j}{j+1} \frac{2\pi\epsilon\hbar}{im} \right)^{1/2} \exp\left(\frac{im}{2(n+1)\epsilon\hbar} (x_b - x_a)^2 \right) \\ &= \sqrt{\frac{m}{2\pi i \hbar (t_b - t_a)}} \exp\left(\frac{im}{2\hbar} \frac{(x_b - x_a)^2}{t_b - t_a} \right).\end{aligned}$$

Ex. H19: Gaussian and Fresnelic integrals (6 points)

Let M be a real symmetric positive definite $n \times n$ matrix. Prove that:

a)

$$\int d^n x \exp\left(-\frac{1}{2} \mathbf{x}^T M \mathbf{x} \right) = \sqrt{\frac{(2\pi)^n}{\det M}}.$$

b)

$$\int d^n x \exp\left(\frac{i}{2}\mathbf{x}^T M \mathbf{x}\right) = e^{in\pi/4}\sqrt{\frac{(2\pi)^n}{\det M}}.$$

Ex. H20: Spins in a magnetic field (6 points)

An electron beam enters a magnetic field at time $t = 0$. The magnetic field is homogeneous with field strength B pointing in the z-direction. The Hamiltonian is

$$H = \mu_B B \sigma^3,$$

where $\mu_B = \frac{\hbar e}{2m_e c}$ denotes the Bohr magneton and σ^3 is, as usual, the third Pauli matrix. The electron spins are initially, at time $t = 0$, polarized in the positive x-direction (*i.e.* all the electrons are in an eigenstate of S_x with eigenvalue $+\hbar/2$).

a) Calculate the expectation value of the spin in the x-direction at time t ($t > 0$).

Now suppose that half the electron spins at time $t = 0$ are polarized in the positive x-direction and the other half in the positive y-direction.

b) State the density matrix at time $t = 0$.

c) Calculate the density matrix in the Schrödinger picture at arbitrary time t ($t > 0$).

d) Calculate once more the time evolution of $\langle S_x \rangle$.

Ex. H21: Preparation measurements (2 points)

Suppose that we are given a quantum statistical system with a k-fold degenerate energy level E_n. We denote the pure states of energy E_n with $|\, n, \alpha\rangle$ ($\alpha = 1, \dots, k$). The projection operator onto states with energy E_n is thus

$$P_{E_n} = \sum_{\alpha=1}^{k} |\, n, \alpha\rangle \langle n, \alpha \,|,$$

and the density matrix in the micro-canonical ensemble is $\rho = P_{E_n}/k$.
Now a further observable is measured that commutes with the Hamiltonian. As a result of the measurement the equipartition passes over onto the subspace of states with the measured eigenvalue; suppose that these are only k'-fold degenerate (where $k' < k$), and the projection operator onto this space is

$$P'_{E_n} = \sum_{\beta=1}^{k'} |\, n, \beta\rangle \langle n, \beta \,|.$$

Show that after the measurement the density matrix is given by

$$\rho' = \frac{P'_{E_n}}{k'}.$$