Exercise sheet 11, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in and class exercises discussed
in the tutorials of Week 12 (18/01/08)

Ex. H22: Kronig-Penney model (5 points)

A one-dimensional periodic potential is given with period l, thus $V(x) = V(x+l)$.

a) Prove *Bloch's Theorem*: there exists a basis of energy eigenstates, whose wavefunctions assume the form $\psi_\kappa(x) = e^{-i\kappa x} u_\kappa(x)$ with periodic functions $u_\kappa(x) = u_\kappa(x+l)$.

   Hint: The law of simultaneous diagonalisability of commuting operators holds not only for Hermitian operators. Apply it here to the unitary translation operator $T(l)$ (see Ex. H16) and the Hamiltonian.

Consider now a particle with energy E in the range $0 < E < V_0$, moving in the following potential with period $l = a + b$:

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a \\ V_0, & a < x < a+b \end{cases}$$

(periodically continued for all $x \in \mathbb{R}$). This is a simple one-dimensional model for an electron in a solid. The positively charged ions are located on a crystal lattice at a separation of $a + b$, separated by potential barriers of height V_0 and width b. The energy spectrum shows a band structure, in which certain ranges of energy are allowed and others are forbidden.

b) Show that the following condition holds for the energy:

$$-1 \leq \cos(ka)\cosh(qb) + \frac{q^2 - k^2}{2qk}\sin(ka)\sinh(qb) \leq 1,$$

   where

$$k = \frac{\sqrt{2mE}}{\hbar}, \qquad q = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

c) Consider the limiting case

$$b \to 0, \qquad V_0 \to \infty, \qquad V_0 b = \text{const.}$$

   and energies $E \ll V_0$. Show that

$$-1 \leq \cos(ka) + \gamma\frac{\sin(ka)}{ka} \leq 1,$$

   where $\gamma = mabV_0/\hbar^2$. Determine the allowed ranges for ka graphically for $-4\pi \leq ka \leq 4\pi$ and $\gamma = 5$.

Ex. H23: Three-dimensional harmonic oscillator (5 points)

The potential of the three-dimensional harmonic oscillator is

$$V = \frac{m\omega^2}{2} r^2, \qquad \text{mit } r = |\mathbf{x}|.$$

a) The corresponding Schrödinger equation separates in Cartesian coordinaten. State the possible energy eigenvalues, and determine the degeneracy of the nth energy eigenstate.

b) Since we are dealing with a central potential, the Schrödinger equation also separates in spherical polar coordinates, for which the solutions of the angular part are already known to be the spherical harmonic functions $Y_l^m(\theta, \phi)$:

$$\psi(r, \theta, \phi) = R(r)\, Y_l^m(\theta, \phi).$$

It now remains to determine the radial part $R(r)$. Set up the radial equation, and discuss the limits $r \to 0$ and $r \to \infty$ with the ansatz

$$u(r) = r\, R(r) = r^\delta\, e^{-\gamma r^2}\, g(r)$$

(where $g(0) \neq 0$ and $|g(r)|$ does not either grow or fall off too quickly for $r \to \infty$). Determine δ and γ.

c) Set up an equation to determine $g(r)$.

d) Give a reason why, with the Ansatz

$$g(r) = \sum_n a_n r^n,$$

the power series must terminate at a finite value of n. Determine thereby the energy spectrum and the degeneracies of energies, and compare with a).

Ex. H24: Landau levels (5 points)

Consider an electron (whose spin is to be neglected) moving in a constant magnetic field pointing in the z-direction, $\mathbf{B} = (0, 0, B)$.

a) Show that the vector potential can be chosen as $\mathbf{A} = (0, Bx, 0)$.

b) Show that the Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{P} + \frac{e}{c}\mathbf{A}\right)^2$$

commutes with both the y- and z-components of the momentum operators. The eigenstates of H can thus be determined with the separation of variables $\psi(x, y, z) = f(x)g(y)h(z)$. State g and h. We wish to choose $h(z)$ such that eigenvalue of P_z is precisely 0.

c) Derive an eigenvalue equation for $f(x)$, and trace this back to the eigenvalue equation of the simple harmonic oscillator.

d) Read off the energy spectrum. The energy levels resulting from the equation for f are called *Landau levels*.