Exercise Sheet 12, Theoretical Physics III (Quantum Mechanics)

Solutions to be handed in and class exercises discussed
in the tutorials of Week 13 (25/01/08)

Ex. H25: Aharonov-Bohm effect (5 points)

Electrons from a source at $\mathbf{x} = \mathbf{x}_0$ strike a wall with a double slit, behind which a screen is positioned. Between the two slits an infinitely long solenoid is mounted along the x_3-axis, that generates a magnetic field confined to the interior of the solenoid. Suppose that the radius of the solenoid is R and that it is shielded in such a way that the electrons cannot enter the magnetic field.

a) Let $r = \sqrt{x_1^2 + x_2^2}$ and $\mathbf{e}_3 = (0, 0, 1)$. Show that the vector potential $\mathbf{A}$, given by $\mathbf{A}_>$ for $r > R$ and by $\mathbf{A}_<$ for $r < R$ with

$$\mathbf{A}_>(\mathbf{x}) = -\frac{B}{2}\frac{R^2}{r^2}\mathbf{x} \times \mathbf{e}_3,$$
$$\mathbf{A}_<(\mathbf{x}) = -\frac{B}{2}\mathbf{x} \times \mathbf{e}_3,$$

describes the magnetic field for this apparatus.

b) Calculate

$$\Phi_C = \int_C \mathbf{A} \cdot d\mathbf{s}$$

for a closed circular path C in the (x_1, x_2)-plane. The centre of the circle is $(0, 0)$ and the radius r_0.

One can show that, for arbitrary closed curves C outside the solenoid, Φ_C depends only on the winding number of the curve around the x_3-axis.

c) Show that, within simply connected regions,

$$\psi_B(\mathbf{x}) = \exp\left(-\frac{ie}{\hbar c}\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{A} \cdot d\mathbf{s}\right)\psi_0(\mathbf{x})$$

is a solution of the Schrödinger equations with magnetic field, if $\psi_0(\mathbf{x})$ is a solution of the Schrödinger equation without field.

d) Deduce that the interference pattern on the screen changes if the magnetic field in the solenoid is switched on and off.

Ex. H26: Addition of angular momenta (5 points)

Consider the coupling of a spin-1 particle with a spin-1/2 particle. The respective spin operators are $\mathbf{\Sigma}$ und $\mathbf{S}$ and the corresponding normalized eigenstates are $\{|\pm1\rangle, |0\rangle\}$ resp. $\{|\pm\frac{1}{2}\rangle\}$. Determine the (correctly normalized) common eigenstates of $\mathbf{J}^2$ und J_3 and the corresponding eigenvalues. Here

$$\mathbf{J} = \mathbf{S} + \mathbf{\Sigma} \equiv \mathbf{S} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{\Sigma}$$

is the total angular momentum.

Instructions: First determine, as demonstrated in the lecture, the state with the highest eigenvalue of J_3. Construct from this the remaining eigenstates of J_3 with the same $\mathbf{J}^2$ with the help of the lowering operator $\mathbf{J}_-$. Eigenstates with different $\mathbf{J}^2$-eigenvalue can be obtained from the orthogonality condition.

Ex. H27: Perturbed simple harmonic oscillator (4 points)

A charged particle moving in a simple harmonic oscillator potential in one dimension is additionally exposed to a homogeneous electric field. The potential energy is

$$V = \frac{m\omega^2}{2}x^2 - eEx.$$

a) Determine the exact energy spectrum.

b) Calculate the energy spectrum by perturbation theory to the leading non-vanishing order, by considering the eEx term as a small perturbation. Compare the result with a).