9th Exercise Sheet: Electrodynamics, Summer Term '06

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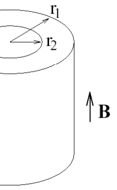
9. 1. **(Präsenzübung (taken from sheet 8): induction and mechanical angular momentum , 1+1 marks)** Consider a cylindrical capacitor with length l, outside radius r_1 with charge q, and inside radius r_2 with charge $-q$. There is vacuum between the two cylinders. Furthermore, there is a magnetic field B_0 parallel to the symmetry axes of the capacitor. Calculate the **mechanical torque** and the **mechanical angular momentum** of the system ...



(a) ... if the capacitor is discharged.

(b) ... if the magnetic field is turned off.

Hints: Calculate $\vec{S} = \frac{1}{4\pi c}\vec{E}\times\vec{B}$. Using $\vec{S}$, you can calculate an angular momentum in z-direction. What happens now, if the capacitor is discharged or the magnetic field is turned off?

9. 2. **(model of an inductor and self-inductance, 1.5+2.5 marks)** Consider a cylindrical inductor, formed by a helical winding of wire with radius R and height h, and with the current density:

$$\vec{j} = \frac{n}{h} I\delta(\rho - R)\left[\Theta\left(z+\frac{h}{2}\right) - \Theta\left(z-\frac{h}{2}\right)\right]\vec{e}_\phi\,.$$

The number of windings is given by n. Calculate the self-inductance L for large h, i. e. for $\frac{R}{h} \ll 1$ ...

(a) **(1.5 marks)** ... by calculation of the field strength $\vec{B}$ and the energy of the magnetic field for an infinitely long cylinder ($\frac{R}{h} \to 0$):

$$L = \frac{1}{4\pi I^2}\int d^3r\vec{B}\cdot\vec{B}\,.$$

(b) **(2.5 EXTRA-marks)** ... by means of the formula

$$L = \frac{1}{(cI)^2}\int d^3r\int d^3r'\frac{\vec{j}(\vec{r})\vec{j}(\vec{r'})}{|\vec{r}-\vec{r'}|}\,.$$

Give the first-order correction term for the self-inductance of an infinitely long cylinder!
Hints for exercise (b): Use

$$\int_{-\frac{y}{2}}^{\frac{y}{2}}\frac{dz'}{\sqrt{a^2+(z-z')^2}} = \ln\left(\sqrt{\left(\frac{\frac{y}{2}-z}{a}\right)^2+1}+\frac{\frac{y}{2}-z}{a}\right) - \ln\left(\sqrt{\left(\frac{\frac{y}{2}+z}{a}\right)^2+1}-\frac{\frac{y}{2}+z}{a}\right)$$

as well as

$$\int_0^{2\pi} d\psi\cos(\psi)\ln(1-\cos(\psi)) = -2\pi \quad\text{and}\quad \int_0^{2\pi} d\psi\cos(\psi)\sin\left(\frac{\psi}{2}\right) = -\frac{4}{3}\,.$$

9. 3. **(electromagnetic waves, 7 marks)** Consider an electromagnetic wave given by

$$\vec{E}(\vec{r},t) = a\vec{e}_x\cos(\vec{k}\vec{r}-\omega t) + b\vec{e}_y sin(\vec{k}\vec{r}-\omega t)\,,$$
$$\vec{B}(\vec{r},t) = c\vec{e}_x\sin(\vec{k}\vec{r}-\omega t) + d\vec{e}_y cos(\vec{k}\vec{r}-\omega t)\,,$$

where a, b, c, d are constants and $\vec{k} = k\vec{e}_z$ with $\omega = kc = |\vec{k}|c$.

(a) **(2 marks)** For given values of a and b, how does one have to choose c and d in order to fulfill the Maxwell-equations with $\vec{j} = 0$ and $\rho = 0$? Use these values also for exercise 9.3.(b) and 9.3.(c)!

(b) **(2 marks)** Calculate the electric and magnetic field as a function of time for fixed $z = 0$! Why is a wave with $ab = 0$ called linear polarised? Why is a wave with $a = b$ called circular polarised?

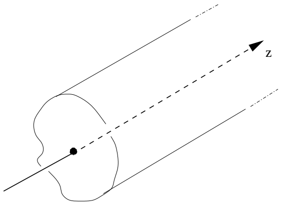
(c) **(3 marks)** Calculate the energy density and the Poynting-vector and its values averaged over one oscillation period.

See reverse!

9. 4. **(waveguides, 7 marks)**

(a) **(4 marks)** Consider an electromagnetic field in a waveguide (hollow conductor with no boundary in z-direction) with an arbitrary cross-section (see sketch).

i. **(1 mark)** The field equations for the electric and magnetic fields are given by (see lecture)



$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0 \qquad \text{and} \qquad \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0\,.$$

Use the ansätze

$$\vec{E} = \vec{E}_0(x,y)\mathrm{e}^{i(k_z z - \omega t)} \qquad \text{and} \qquad \vec{B} = \vec{B}_0(x,y)\mathrm{e}^{i(k_z z - \omega t)}$$

for the electric and magnetic field in the above-mentioned field equations in order to show that $\vec{E}_0(x,y)$ and $\vec{B}_0(x,y)$ satisfy the following equations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_\perp^2\right)\vec{E}_0(x,y) = 0 \qquad \text{and} \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_\perp^2\right)\vec{B}_0(x,y) = 0\,,$$

where $k_\perp^2 = \left(\frac{\omega}{c}\right)^2 - k_z^2$ with constant k_z and ω. Give the inequality between ω and $k_\perp$ that must be fulfilled for electromagnetic waves in z-direction?

ii. **(1.5 marks)** Which value does the electric field take on the boundary? Prove that the normal component $\vec{B}_\perp$ of the magnetic field vanishes on the boundary!

iii. **(1.5 marks)** Use the III. and IV. Maxwell equation (same nomenclature as in the lecture) as well as the ansatz for the $\vec{E}$- and $\vec{B}$-field from exercise 9.4.(a).i. to prove/derive the following relations:

$$k_\perp^2\,(\vec{e}_x E_{0,x} + \vec{e}_y E_{0,y}) = i\,k_z \vec{\nabla} E_{0,z} - i\,\frac{\omega}{c}\vec{e}_z \times \vec{\nabla} B_{0,z}\,,$$
$$k_\perp^2\,(\vec{e}_x B_{0,x} + \vec{e}_y B_{0,y}) = i\,k_z \vec{\nabla} B_{0,z} + i\,\frac{\omega}{c}\vec{e}_z \times \vec{\nabla} E_{0,z}\,.$$

(b) **(3 marks)** Consider a waveguide with rectangular cross-section (side length b in y-direction and side length a in x-direction).

i. **(1.5 marks)** transversal-magnetic (TM) modes: In this case, the z-component of the magnetic field is zero, i. e. $B_{0,z} \equiv 0$. Calculate the electric field in z-direction under consideration of the boundary conditions!

ii. **(1.5 marks)** transversal-electric (TE) modes: In this case, the z-component of the electric field is zero, i. e. $E_{0,z} \equiv 0$. Calculate the magnetic field in z-direction under consideration of the boundary conditions!

Remark: The labels 'TM' and 'TE' tells you, which field-vector ($\vec{B}$ or $\vec{E}$) is perpendicular to the wave-vector $\vec{k}$ of the electromagnetic wave in the hollow conductor.

Hints: Use the differential equation for $\vec{E}_0$ and $\vec{B}_0$ from exercise 9.4.(b).i. to calculate the electric and magnetic fields in z-direction. Make a separation ansatz for the electric and the magnetic field in z-direction, respectively, in order to solve the differential equation.