I. Simulating SU(2) gauge fields on the lattice

To implement a Metropolis Monte-Carlo update of a 2 dimensional Euclidean gauge field theory with gauge group SU(2) we need to consider an efficient way for the representation of the gauge field matrices, e.g. through Pauli matrices:

$$U \in SU(2), \quad U = u_0 1 + i\,\vec{u}\cdot\vec{\sigma} \quad \det[U] = u_0{}^2 + |\vec{u}|^2 = 1$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In the end each matrix is represented by four real numbers, the same number of degrees of freedom as in the conventional representation in terms of complex numbers.

a. One central part of the Wilson action is a sum over staples. Implement a separate function that carries out this operation, i.e. matrix multiplication and addition. To check the correctness of the simulation one needs to calculate the average of the plaquette as well.

b. We need a proposal matrix Y as starting point of each Metropolis step. If this candidate is chosen too far from the original link X it will be rejected with a high probability. Hence we wish to efficiently generate an SU(2) matrix close to unity which is subsequently multiplied with the original link.

   Generate four random real numbers r_i in the interval [-1,1] and choose a value $0 \le \varepsilon \le 1$ (controlling the departure from unity, and hence the acceptance rate) to construct:

$$\vec{v} = \varepsilon \frac{\vec{r}}{|\vec{r}|} \quad v_0 = \sqrt{1 - \varepsilon^2}$$

   The proposal matrix then reads: Y=VX

c. Implement a multihit Metropolis Monte Carlo update, i.e. at each link the algorithm attempts e.g. ten times to update the matrix before continuing to the next link. To keep rounding errors under control carry out a reunitarization of the SU(2) fields after each 10 sweeps (1 sweep = one full multihit update step over all links of the lattice).

d. Derive the formula for the matrix exponential of a SU(2) matrix in the Pauli representation.

For those of you who are interested in the project on heatbath simulations of SU(2) (or SU(3)) gauge fields, we recommend the chapter on simulating pure gauge theory in

C. Gattringer, C.B. Lang: Quantum Chromodynamics on the Lattice
Lecture Notes in Physics 788, Springer