# Quantum simulation of real-time dynamics in gauge theories

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# Outline

## When classical simulation fails for strongly coupled systems ···

··· there's usually a sign problem around the corner.
Various incarnations: Finite baryon density (QCD), fermions (repulsive Hubbard model, doping), non-zero $\theta$ angle, geometrically frustrated anti-ferromagnets, Real-time evolution

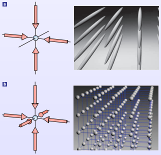
$$\langle \Phi_0 | \mathcal{O}(t) | \Phi_0 \rangle = \frac{1}{\mathcal{Z}} \sum_{m,n} \langle \Phi_0 | n \rangle \langle m | \Phi_0 \rangle \mathcal{O}_{mn} e^{-i(E_m - E_n)t}$$

MC evaluation is not possible

- In it's most general form sign problem is exponentially hard Troyer, Wiese (2005). General solution applicable to all problems unlikely.
- Use "quantum" degrees of freedom (atoms/molecules/ions) to represent the field variables and design the Hamiltonian dynamics to study the real time dynamics Feynman (1982).
- (here) Abelian and Non-Abelian Gauge theories with fermionic matter.

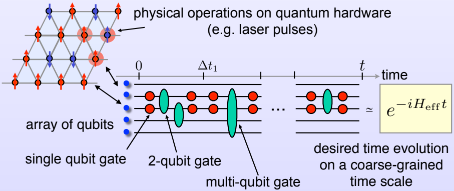
# Quantum Simulation: Analog vs Digital

Adjust parameters such that atoms in optical traps act as d.o.f



cold atoms in optical lattices realize Bosonic and Fermionic Hubbard type models.

Ions confined in ion-trap with interactions between individual ions can be controlled.

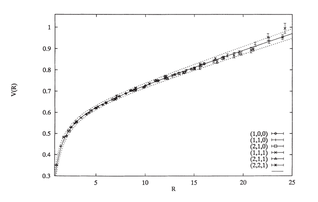


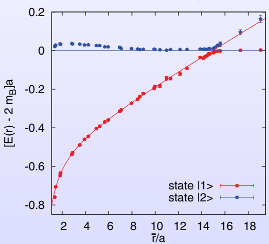

Advantage: Much more control over interactions; Challenge: Scalability.

Prepare the "quantum" system and let it evolve. Make measurements at times $t_i$ on identically prepared systems. Achievement: observation of Mott-insulator (disordered) to superfluid (ordered) phase. Greiner et. al. (2002)

# What to *see* in *real* time?

Confinement in QCD is phenomenologically described by a “string”
String breaking from a study of the spectrum:




G.S. Bali, K. Schilling (1992); Bali et. al. (2005).

Possibility of seeing string breaking in real time ...?
Chiral dynamics at finite density in U(N)/SU(N) gauge theories.

# Outline

# Abelian Quantum Link model

$$H = -t_F \sum_{\langle xy \rangle} \left( \psi_x^\dagger U_{xy} \psi_y + \text{h.c.} \right) + \frac{g^2}{2} \sum_{\langle xy \rangle} E_{xy}^2 - \frac{1}{4g^2} \sum_{\square} (U_\square + \text{h.c.})\,.$$

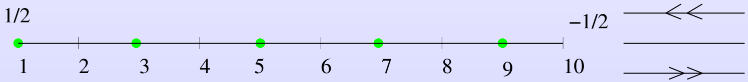
- Anti-commuting fermions; Gauge fields: $[E_{xy}, U_{xy}] = U_{xy}$
- $G_x|\Psi\rangle = 0$, $G_x = \psi_x^\dagger \psi_x - \sum_i \left( E_{x,x+\hat{i}} - E_{x-\hat{i},x} \right)$;
  $G_x$ generates Gauge transformations and $[H, G_x] = 0$
- Wilson: $U_{xy} = \exp(i\varphi_{xy}) \in U(1)$, infinite dimensional Hilbert space at each link $\rightarrow$ unsuitable for realization on optical lattices
- Generalized LGTs have discrete Hilbert spaces at each link, but generate continuous gauge transformations
  Horn (1981); Orland(1990); Chandrasekharan, Wiese (1996)
- Finite Hilbert space for each link field: $2S + 1$ states of an integer or half-integer quantum spin $\vec{S}_{xy}$ on each link
- Electric field: $E_{xy} = S_{xy}^3$ with eigenvalues $-S, \ldots, S$,

$$U_{xy} = S_{xy}^+ = S_{xy}^1 + iS_{xy}^2, \quad U_{xy}^\dagger = S_{xy}^- = S_{xy}^1 - iS_{xy}^2,$$

# Schwinger model with staggered fermions

Stick to simple models in 1-d having the same qualitative features for **validating** the quantum simulator. Schwinger model with QL in spin-1 representation and staggered fermions:

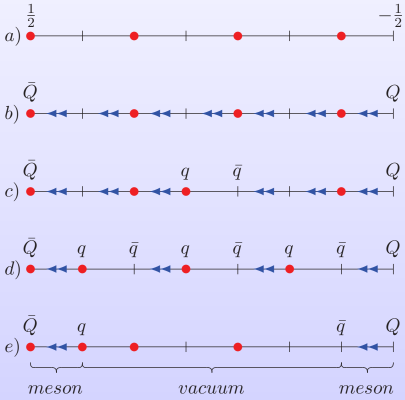
$$H_s = -\sum_x t_F \left[\psi_x^\dagger U_{x,x+1}\psi_{x+1} + h.c.\right] + m\sum_x (-1)^x \psi_x^\dagger \psi_x + \frac{g^2}{2}\sum_x E_{x,x+1}^2 + V\sum_x (\psi_x^\dagger \psi_x)^2$$




Symmetries: Translation symmetry by even number of lattice spacings;
**Gauss' Law:** $[H, G_x] = 0$; Parity; Charge Conjugation;
Discrete **Chiral Symmetry:** for $m = 0$

broken by Gauss law.

# The String and its breaking




Energetics in $t \to 0$ limit easy to analyze:

$E_0 = -m\frac{L}{2}$

$E_{\text{string}} - E_0 = \frac{g^2}{2}(L-1)$

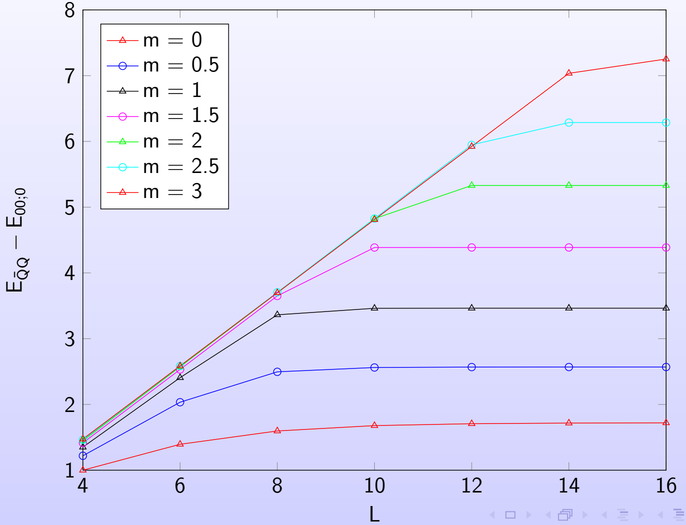
$E_{\text{mesons}} - E_0 = 2(\frac{g^2}{2} + m)$

$E_{\text{string}} - E_{\text{mesons}} = \frac{g^2}{2}(L-3) - 2m$

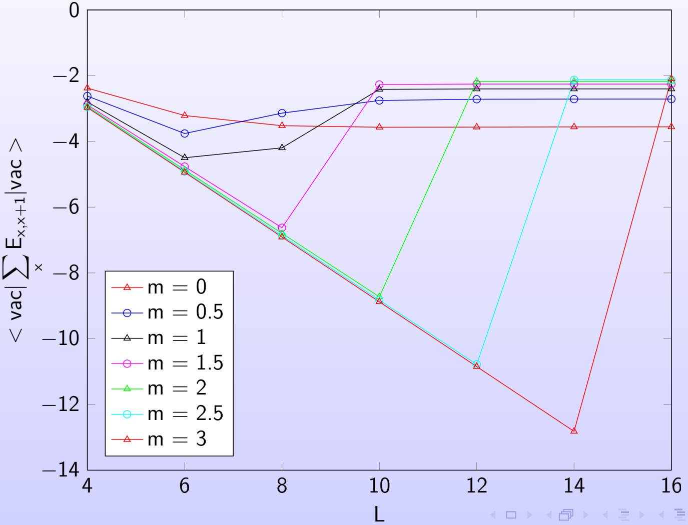
$E_{\text{string}} - E_{\text{mesons}} = 0$

$\Longrightarrow L = \frac{4m}{g^2} + 3$

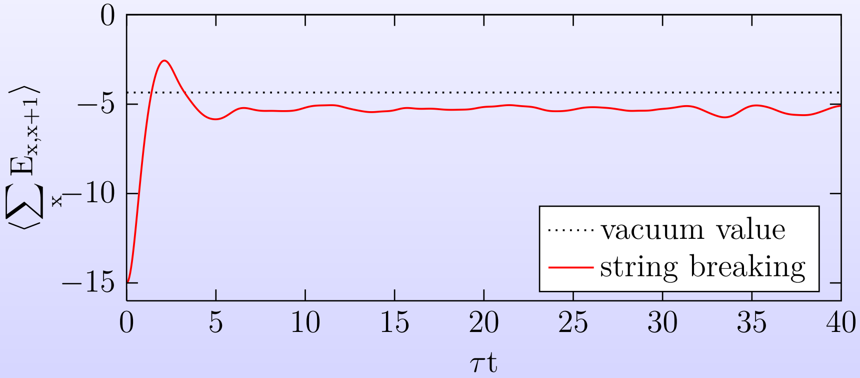
# Static Properties



# Static Properties



# String breaking



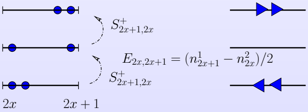
# Outline

# Microscopic Model and *effective* gauge invariance

Schwinger model acts as an **effective** theory $\mathcal{H}_{ph}$, induced at low-energies by a microscopic **Hubbard** type model $H$.

Express dynamical gauge fields in terms of **rishons** ("Schwinger" bosons for U(1)):

$$U_{2x,2x+1} = S^{+}_{2x+1,2x} = b^{\sigma\dagger}_{2x+1} b^{\sigma}_{2x}; \quad E_{2x,2x+1} = S^{z}_{2x+1,2x} = (n^{\sigma}_{2x+1} - n^{\sigma}_{2x})/2; \quad n^{\sigma}_{2x} + n^{\sigma}_{2x+1} = 2S = \mathcal{N}$$




Rishon for spin-1;
$S = 1; \mathcal{N} = 2$

Impose Gauss law by the Hamiltonian:

$$H_{\mathcal{U}} = \mathcal{U} \sum_{x} (G_x)^2 = \mathcal{U} \{ \sum_{x,\sigma=1,2} \left[ (n^{\sigma}_{x})^2 + 2 n^{F}_{x} n^{\sigma}_{x} + (-1)^{x} (n^{F}_{x} + n^{\sigma}_{x}) \right] + 2 \sum_{x} n^{1}_{x} n^{2}_{x} \}$$

$\mathcal{U} \gg$ all other scales in the system $\Longrightarrow$ ground states have $G_x |\psi\rangle = 0$
Violating gauge invariance costs energy $\mathcal{O}(\mathcal{U})$; limit $\mathcal{U} \to \infty$ Gauss law exact!

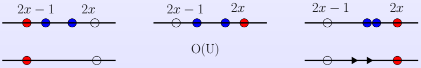
Gauss Law satisfied in the low-energy sector.

# Low energy physics

Low energy physics induced by $H_{\text{pert}}$:

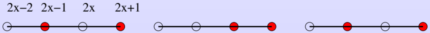
$$H_{\text{pert}} = -t_F \sum_x (\psi_x^\dagger \psi_{x+1} + \psi_{x+1}^\dagger \psi_x) + m \sum_x (-1)^x n_x + \frac{g^2}{4} \sum_{x,\sigma} (n_x^\sigma)^2 - \frac{t_B}{2} \sum_{x,\sigma} (b_x^{\sigma\dagger} b_{x+1}^\sigma + b_{x+1}^{\sigma\dagger} b_x^\sigma)$$

Fermion-gauge coupling generated in 2^nd-order perturbation theory




$$-\frac{t_F t_B}{2U} \psi_{2x}^\dagger U_{2x-1,2x} \psi_{2x-1}$$

Four-fermi interaction is also generated (suppressed by $\mathcal{O}(t_F/t_B)$)!

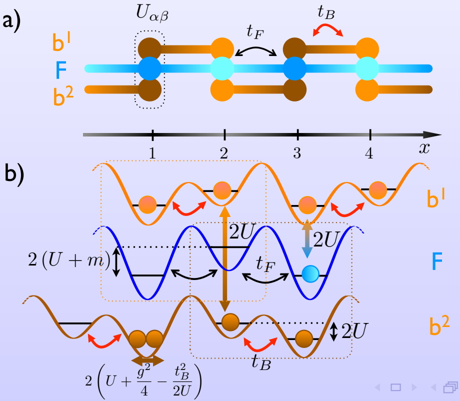



$$\sim \frac{t_F^2}{U}(n_{2x-1}^F - n_{2x-1}^F n_{2x}^F)$$

one-to-one correspondence between states in $\mathcal{H}_{ph}$ and the physical Hilbert space of $H_s$.

# Schematic representation

Each term of the Schwinger model can be implemented via a Bose-Fermi Hubbard model and using superlattices (optical potential created by superposition of different harmonics)



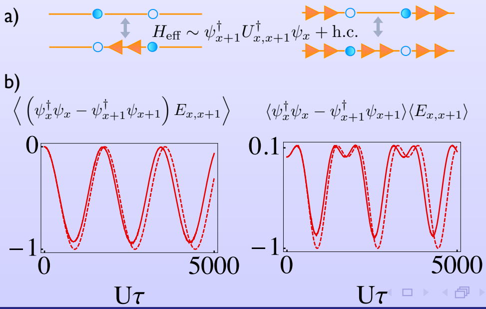
# How good is the approximation?

**Static properties**: Agreement of the ground state spectra to per-cent level for $U \simeq 10t_F$

**Gauss law**: Probability of remaining in the gauge invariant subspace about 90% for $U = 10t_F$ and about 98% for $U = 20t_F$ even for $\tau \simeq 5000t^{-1}$

Note: perturbative violation of gauge invariance is unimportant when taking the continuum limit Foerster, Nielsen, Ninomiya (1980)

**Time evolution:**



# Outline

# Non-Abelian Quantum Link Models

The Hamiltonian with staggered fermions are given by:

$$H = -t \sum_{\langle xy \rangle} \left( s_{xy} \psi_x^{i\dagger} U_{xy}^{ij} \psi_y^j + \text{h.c.} \right) + m \sum_x s_x \psi_x^{i\dagger} \psi_x^i + V \sum_x (\psi_x^{i\dagger} \psi_x^i)^2$$

where $s_x = (-1)^{x_1 + \cdots + x_d}$ and $s_{xy} = (-1)^{x_1 + \cdots + x_{k-1}}$, with $y = x + \hat{k}$.
The non-Abelian Gauss law:

$$G_x^a = \psi_x^{i\dagger} \lambda_{ij}^a \psi_x^j + \sum_k \left( L_{x,x+\hat{k}}^a + R_{x-\hat{k},x}^a \right), \quad G_x = \psi_x^{i\dagger} \psi_x^i - \sum_k \left( E_{x,x+\hat{k}} - E_{x-\hat{k},x} \right),$$

$\lambda^a$: Gell-Mann matrices; $L_{xy}^a, R_{xy}^a$ : $SU(N)$ electric fluxes, $E_{xy}$: Abelian $U(1)$ flux.

Other possible terms in the Hamiltonian: $\frac{g^2}{2} \sum_{\langle xy \rangle} \left( L_{xy}^a L_{xy}^a + R_{xy}^a R_{xy}^a \right)$, $\frac{g'^2}{2} \sum_{\langle xy \rangle} E_{xy}^2$, $\frac{1}{4g^2} \sum_\Box \left( U_\Box + \text{h.c.} \right)$. Not included in our study.
$U(N)$ gauge invariance requires:

$$[L^a, L^b] = 2i f_{abc} L^c, [R^a, R^b] = 2i f_{abc} R^c, \quad [L^a, R^b] = [E, L^a] = [E, R^a] = 0,$$
$$[L^a, U] = -\lambda^a U, \ [R^a, U] = U \lambda^a, \ [E, U] = U$$

To study SU(N) theories, we must include the term $\gamma \sum_{\langle xy \rangle} (\det U_{xy} + \text{h.c.})$

# Rishons: the magic of the QLMs

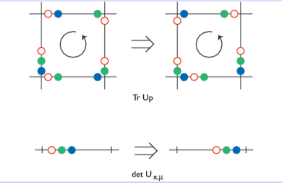
Non-Abelian link fields can be represented by a finite-dimensional fermionic representation:

$$L^a_{xy} = c^{i\dagger}_{x,+}\lambda^a_{ij}c^j_{x,+},\ R^a_{xy} = c^{i\dagger}_{y,-}\lambda^a_{ij}c^j_{y,-},\ E_{xy} = \frac{1}{2}(c^{i\dagger}_{y,-}c^i_{y,-} - c^{i\dagger}_{x,+}c^i_{x,+}),\ U^{ij}_{x,y} = c^i_{x,+}c^{j\dagger}_{y,-}.$$

$c^i_{x,\pm k}$, $c^{i\dagger}_{x,\pm k}$ with color index $i \in \{1, 2, ..., N\}$ are rishon operators. They anti-commute:

$$\{c^i_{x,\pm k}, c^{j\dagger}_{y,\pm l}\} = \delta_{xy}\delta_{\pm k,\pm l}\delta_{ij},\ \{c^i_{x,\pm k}, c^j_{y,\pm l}\} = \{c^{i\dagger}_{x,\pm k}, c^{j\dagger}_{y,\pm l}\} = 0.$$

By fixing the no of rishons on a link, the Hilbert space can be truncated in a completely gauge-invariant manner: $\mathcal{N}_{xy} = c^{i\dagger}_{y,-}c^i_{y,-} + c^{i\dagger}_{x,+}c^i_{x,+}$.

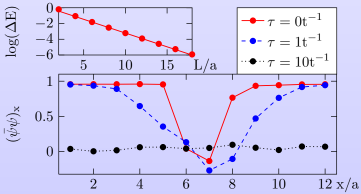



Action of the plaquette and the determinant on a SU(3) theory with $\mathcal{N}_{xy} = 3$ rishons per link.

# Chiral Dynamics

| dimension | group | $\mathcal{N}$ | $C$ | flavor | baryon | phenomena |
|---|---|---|---|---|---|---|
| $(1+1)$D | $U(2)$ | 1 | no | no | no | $\chi$SB, $T_c = 0$ |
| $(2+1)$D | $U(2)$ | 2 | yes | $\mathbb{Z}(2)$ | no | $\chi$SB, $T_c > 0$ |
| $(2+1)$D | $SU(2)$ | 2 | yes | $\mathbb{Z}(2)$ | $U(1)$ boson | $\chi$SB, $T_c > 0$ $\chi$SR, $n_B > 0$ |
| $(3+1)$D | $SU(3)$ | 3 | yes | $\mathbb{Z}(2)^2$ | $U(1)$ fermion | $\chi$SB, $T_c > 0$ $\chi$SR, $n_B > 0$ |

**Table:** Symmetries and phenomena in some QLMs.




Top: Chiral symmetry breaking in a $U(2)$ QLM with $m = 0$ and $V = -6t$.

Bottom: Real-time evolution of the order parameter profile $(\overline{\psi}\psi)_x(\tau) = s_x \langle \psi_x^{i\dagger} \psi_x^i - \frac{N}{2} \rangle$ for $L = 12$, mimicking the expansion of a hot quark-gluon plasma.

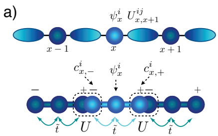
# Outline

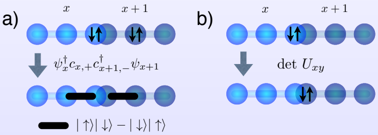
# Outlook

- Although quantum simulating QCD is still far away, many of the simpler models have similar physical phenomena. Very useful for insight into the physics of QCD.
- Necessiates analytical and numerical developments understanding real-time dynamics in quantum systems to complete the picture with data from quantum simulation
- It is very important to validate quantum simulators. At present, mostly exact-diagonalization and DMRG techniques are used in 1-d for this purpose. Interesting to consider developing classical simulation algorithms to validate quantum simulators.
- The way to QCD involves adding extra flavor degrees of freedom, and multi-component Dirac fermions with appropriate symmetries.

## Backup: Implementation of the non-Abelian models

Lattice with quark and rishon sites as a physical optical lattice for fermionic atoms.




- Force the rishon number constraint per link by the term: $U \sum_{\langle xy \rangle} (\mathcal{N}_{xy} - n)^2$.
- Hopping is induced perturbatively with a Hubbard-type Hamiltonian.
- Color d.o.f are encoded in the internal states ( the $2I+1$ Zeeman levels of the electronic ground state ${}^1S_0$) of fermionic alkaline-earth atoms.
- Remarkable property: scattering is almost exactly $2I+1$ symmetric.
- Since the hopping process between quarks and rishon sites is gauge invariant, the induced effective theory is also gauge invariant.

# Backup: An example of real-time evolution

Use the Trotter-Suzuki decomposition

$$e^{-i\mathcal{H}t} \simeq e^{-i\mathcal{H}_1 t} e^{-i\mathcal{H}_2 t} e^{[\mathcal{H}_1,\mathcal{H}_2]t^2/2}$$

to study the real time evolution of

2-quantum spins

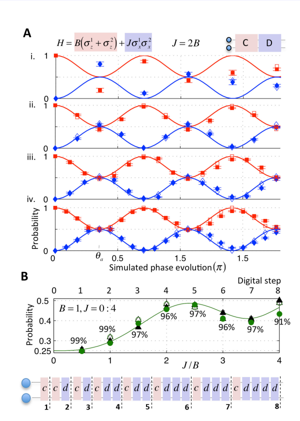
Time-dependent variation of parameters possible

Trotter errors known and bounded;

gate errors under control;

Implementation with upto 6

ions/spins Lanyon et. al. 2011



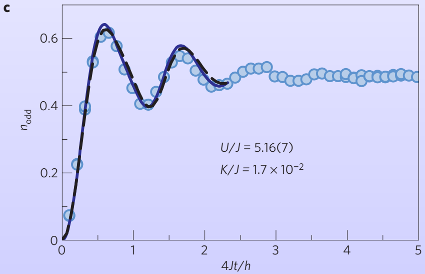
## Backup: Classical vs Quantum Simulation

Example of a quantum quench in a strongly correlated Bose gas.

S. Trotzky et. al., Nature Physics (2012).

$$H = \sum_j \left[ -J(a_j^\dagger a_{j+1} + \text{h.c.}) + \frac{U}{2} n_j(n_j - 1) + \frac{K}{2} n_j j^2 \right]$$

Start the system in the state $|\psi(t=0)\rangle = |\cdots, 1, 0, 1, 0, 1, \cdots\rangle$ and then study the evolution by the Hamiltonian




Measured: no of bosons on odd lattices. Solid curves are from DMRG results.