Equation of state of the unitary Fermi gas

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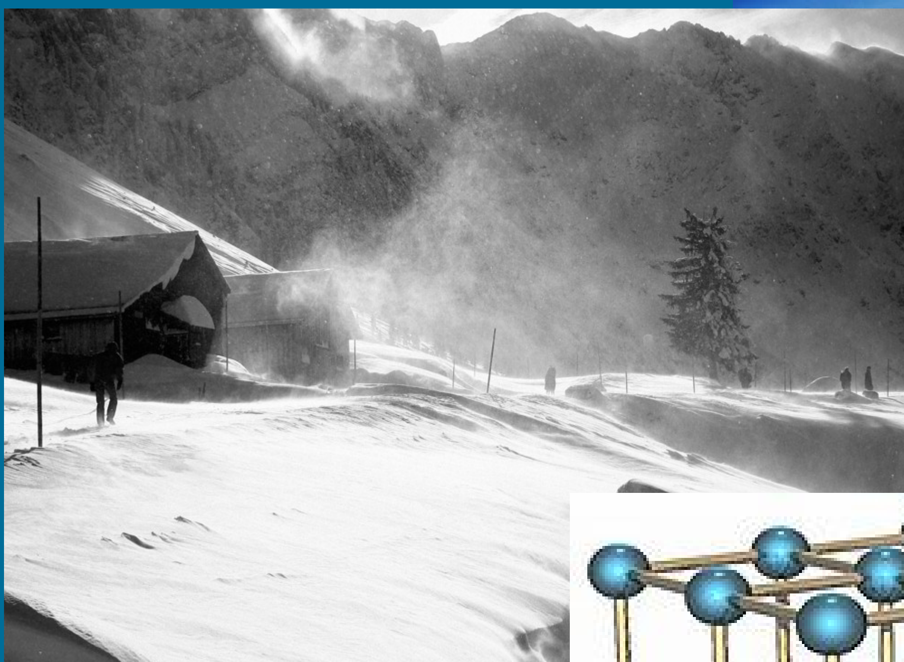
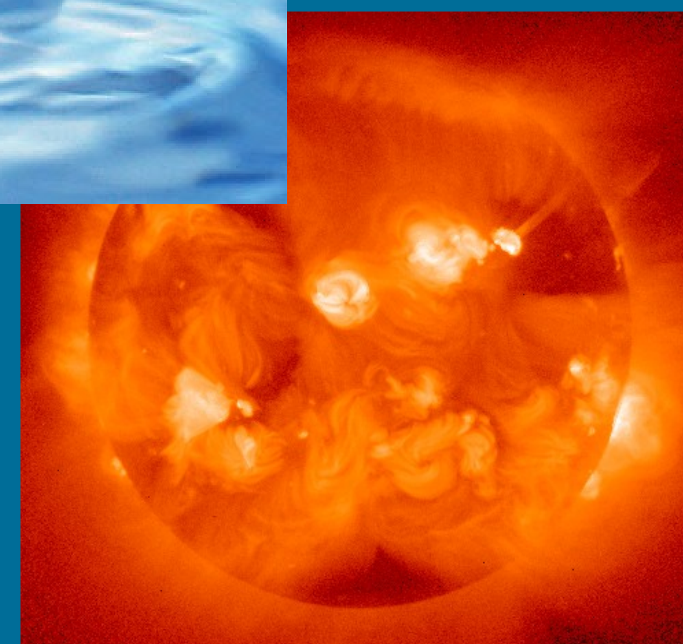
Δ13, 11. 1. 2013

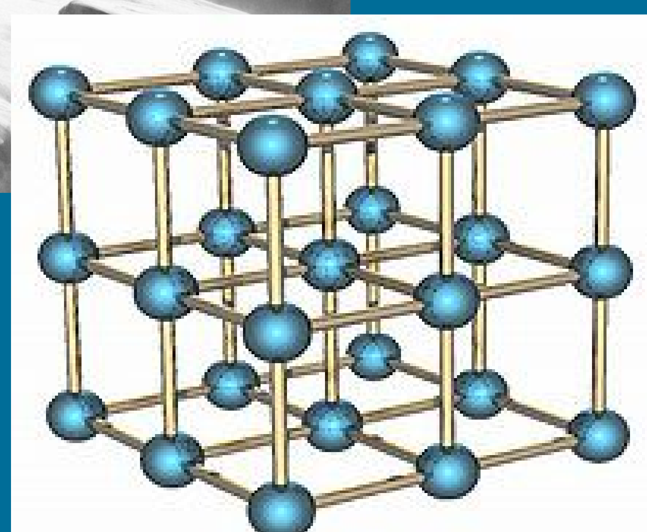
Cold atoms

Functional RG

The many-body problem



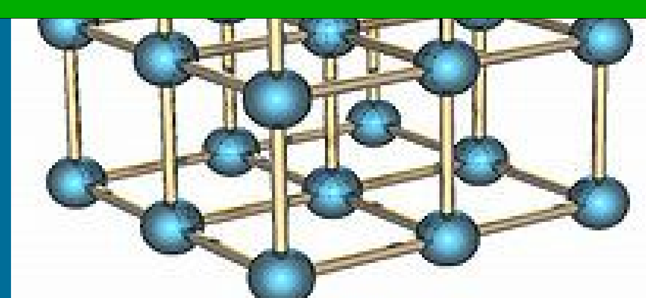




The many-body problem

possibility of a statistical description

collective degrees of freedom



The many-body problem

1st step: Find the right Hamiltonian H

2nd step: Determine the partition function Z

$$Z(\mu, T) = \mathrm{Tr}\left(e^{-\beta(H-\mu N)}\right)$$

The many-body problem

1st step: ~~Find the right Hamiltonian H~~

H is known for cold atoms and QCD!

2nd step: Determine the partition function Z

$$Z(\mu, T) = \mathrm{Tr}\left(e^{-\beta(H-\mu N)}\right)$$

The many-body problem

1st step: ~~Find the right Hamiltonian H~~

H is known for cold atoms and QCD!

2nd step: Determine the partition function Z

$$Z(\mu, T) = \mathrm{Tr}\left(e^{-\beta(H-\mu N)}\right) = \int \mathrm{D}\phi e^{-S[\phi]}$$

path integral

Euclidean quantum field theory

Shopping list

What are the generic features of
quantum many-body systems?

What are reliable theoretical methods
to describe such systems?

What observables reveal advancements
and short-comings of theory?

Shopping list

cold atoms

neutron stars

What are the generic features of
quantum many-body systems?

high-Tc superconductors

early universe

What are reliable theoretical methods
to describe such systems?

What observables reveal advancements
and short-comings of theory?

heavy ion collisions

nuclear matter

quark gluon plasma

Shopping list

Theory

Phase diagram and
Equation of state

$$P(\mu, T) = \frac{k_B T}{V} \log Z(\mu, T)$$

Momentum distribution

Transport coefficients

$$\eta(\mu, T)$$

Experiments with cold atoms

Density images

Collective mode frequencies and damping constants

Expansion after release from trap

Response functions

...

Shopping list

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Experiments with cold atoms

Density images

Collective mode frequencies and damping constants

Expansion after release from trap

Response functions

...

The equation of state

Classical ideal gas: $P(n, T) = nk_B T$

Virial expansion for interacting gas:

$$P(n, T) = nk_B T(1 + B_2(T)n + \dots)$$

Van-der-Waals equation of state:

$$P(n, T) = \frac{nk_B T}{1 - bn} - an^2 \simeq nk_B T \left(1 + (b - \frac{a}{k_B T})n + \dots\right)$$

Pressure P(μ,T)

Bose gas

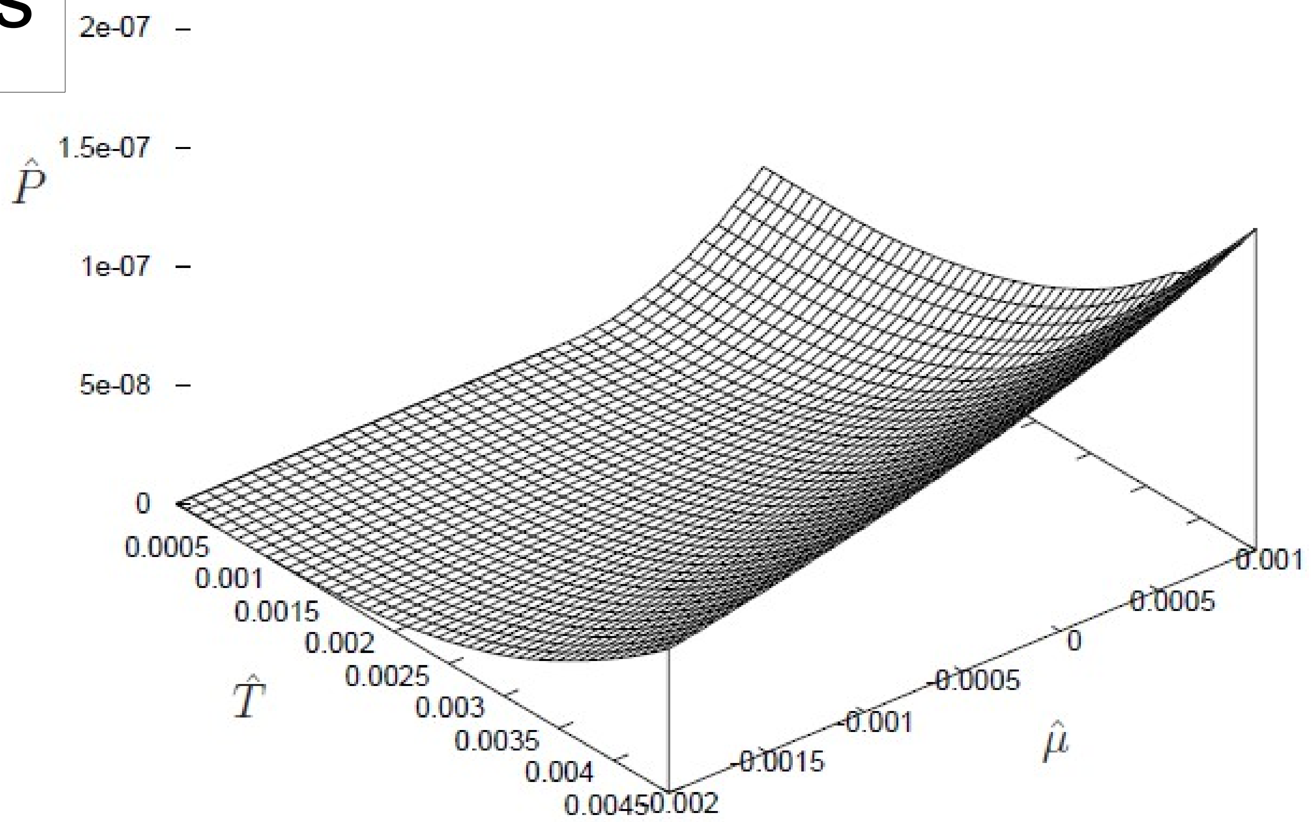


Figure 2.1: Pressure $\hat{P} = Pa^5 m/\hbar^2$ as a function of $\hat{T}$ and $\hat{\mu}$

$\hat{T} = Ta^2 m k_B/\hbar^2$

$\hat{\mu} = \mu a^2 m/\hbar^2$

Density $n=(\partial P/\partial \mu)_T$

Bose gas

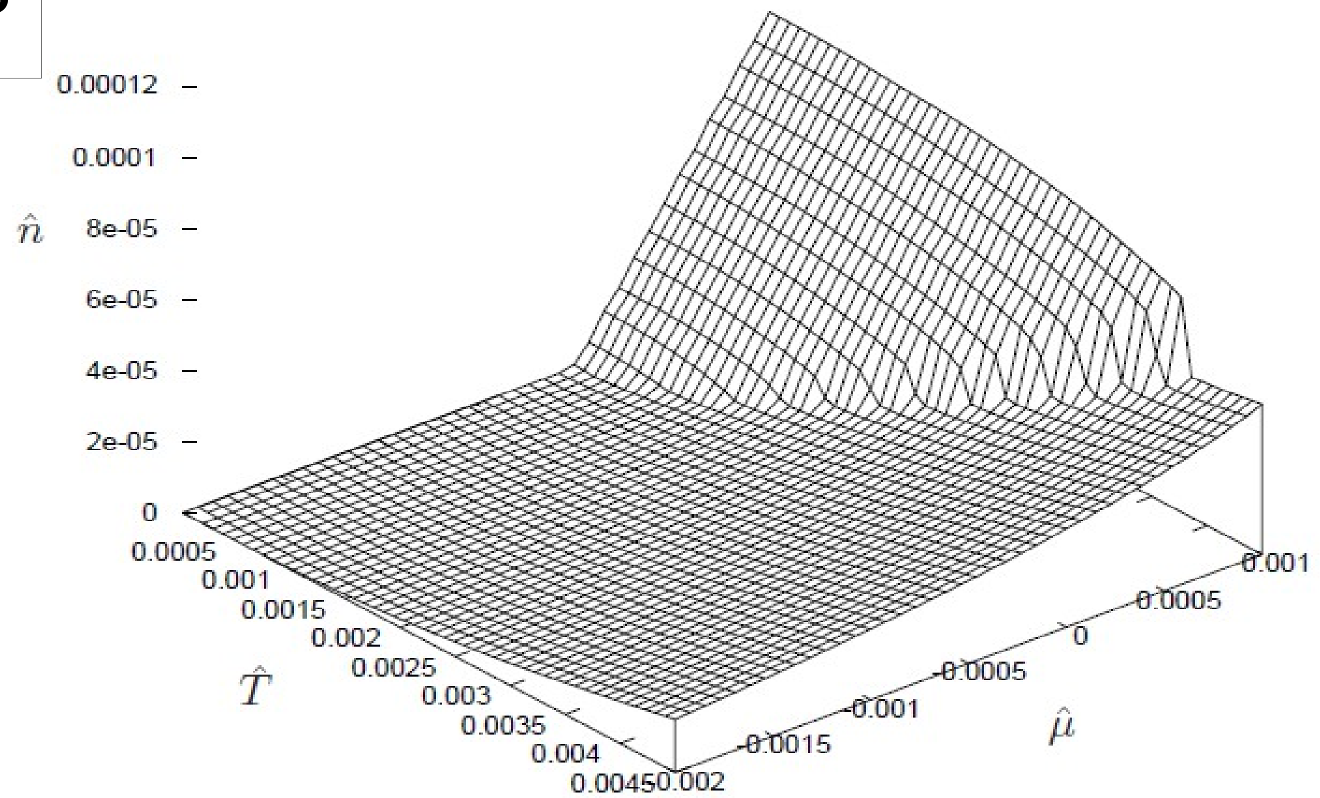


Figure 2.3: Density $\hat{n} = na^3$ as a function of $\hat{T}$ and $\hat{\mu}$

$\hat{T} = Ta^2 mk_B/\hbar^2$

$\hat{\mu} = \mu a^2 m/\hbar^2$

Isothermal compressibility $(\partial^2 P/\partial \mu^2)_T$

Bose gas

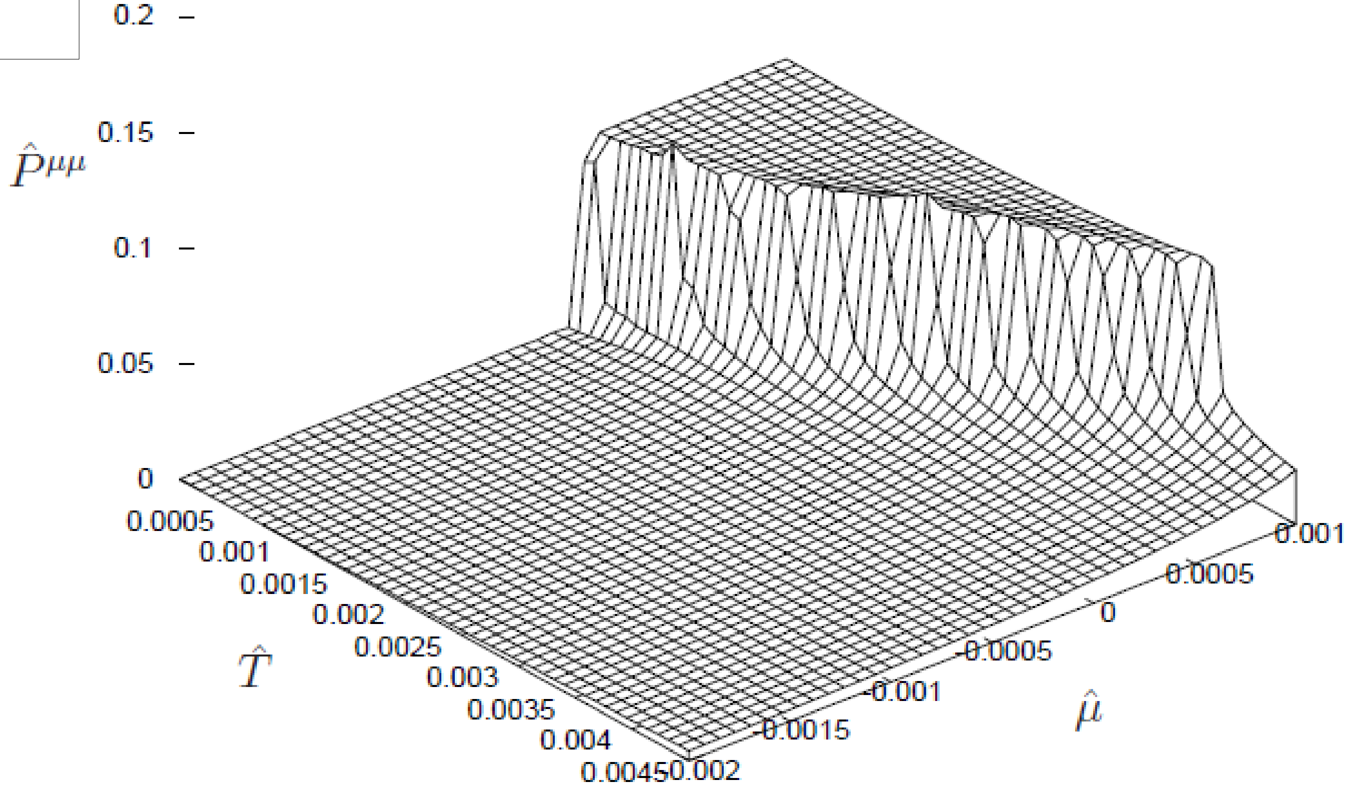


Figure 2.5: $\hat{P}^{\mu\mu} = P^{\mu\mu} a\hbar^2/m$ as a function of $\hat{T}$ and $\hat{\mu}$

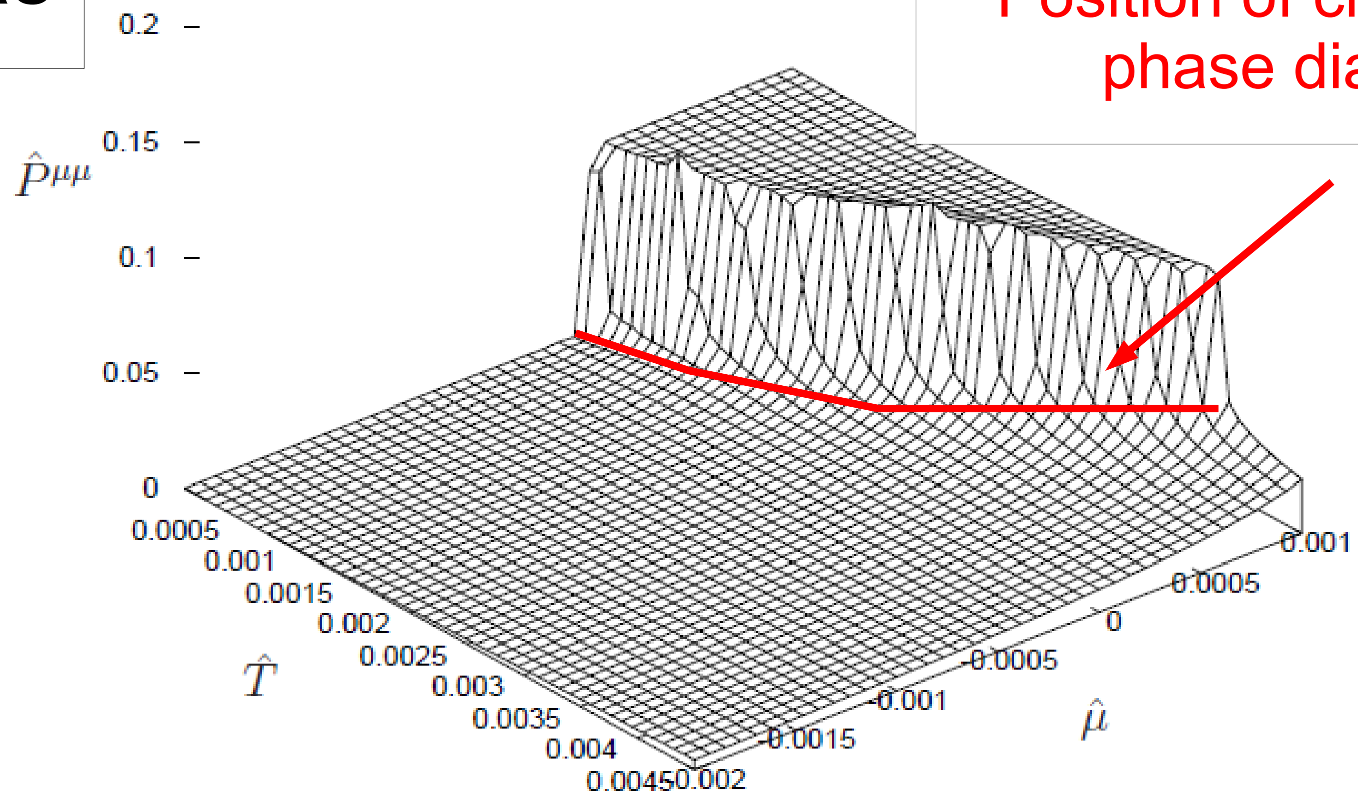
$\hat{T} = Ta^2 m k_B/\hbar^2$

$\hat{\mu} = \mu a^2 m/\hbar^2$

Isothermal compressibility $(\partial^2 P/\partial \mu^2)_T$

Bose gas

Position of critical line: phase diagram



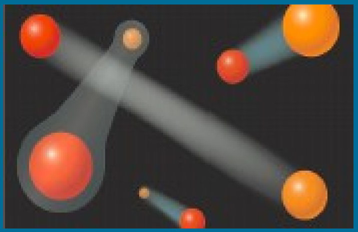
Superfluid phase transition

$\hat{T} = Ta^2 m k_B/\hbar^2$

$\hat{\mu} = \mu a^2 m/\hbar^2$

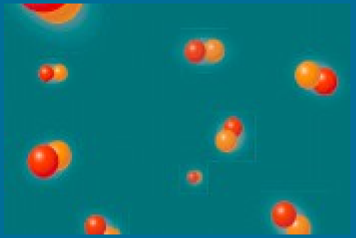
The BCS-BEC Crossover

Two cornerstones of quantum condensation:



BCS

Cooper pairing
of weakly attractive
fermions



BEC

Bose condensation
of weakly repulsive
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The BCS-BEC Crossover

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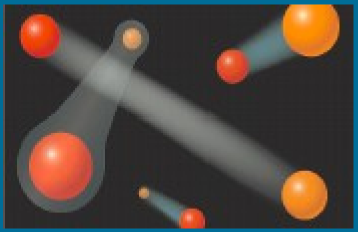
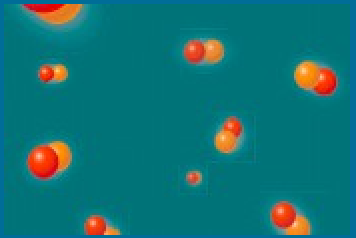
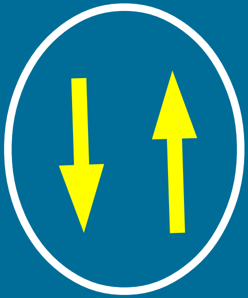


BCS



BEC

$(k_F a)^{-1}$



$\sigma = 4\pi a^2$

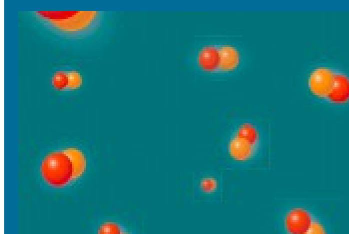
The BCS-BEC Crossover

Two cornerstones of quantum condensation:

BCS

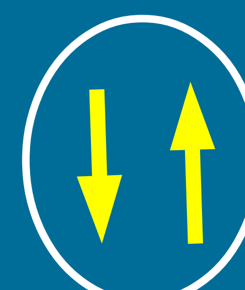
Unitary Fermi gas

$$(k_{\mathrm{F}}a)^{-1} = 0, \;\; \sigma = \frac{4\pi}{p^2}$$



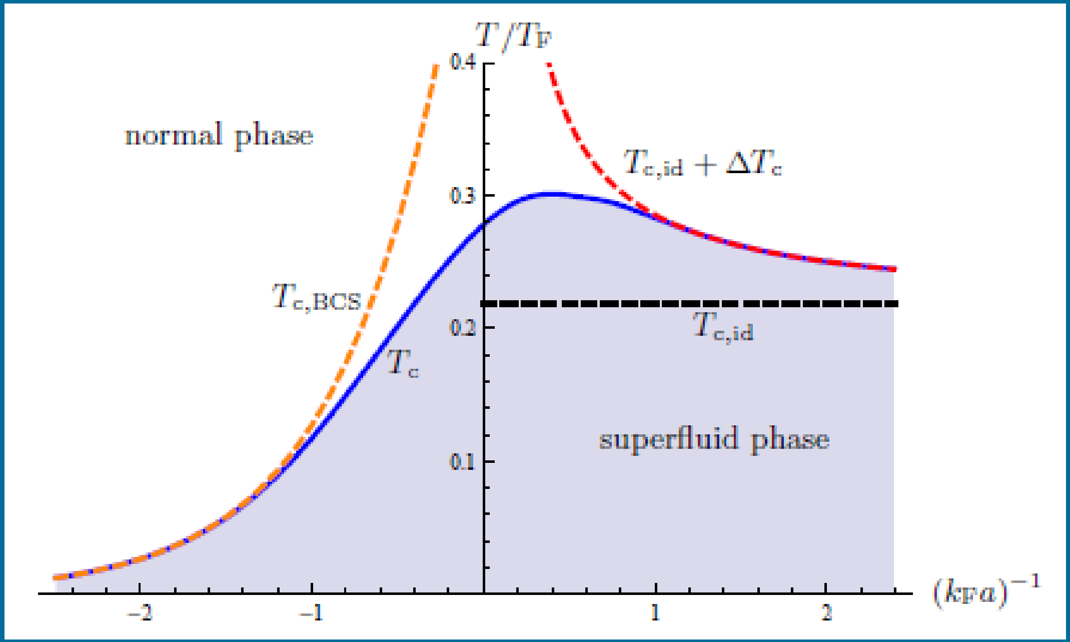
BEC

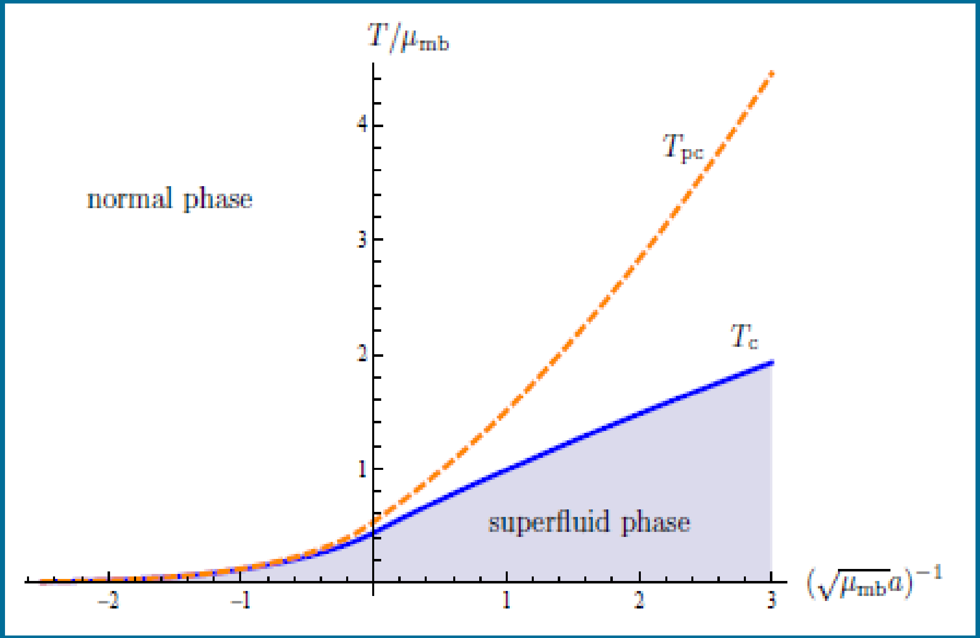
$$(k_{\mathrm{F}}a)^{-1}$$

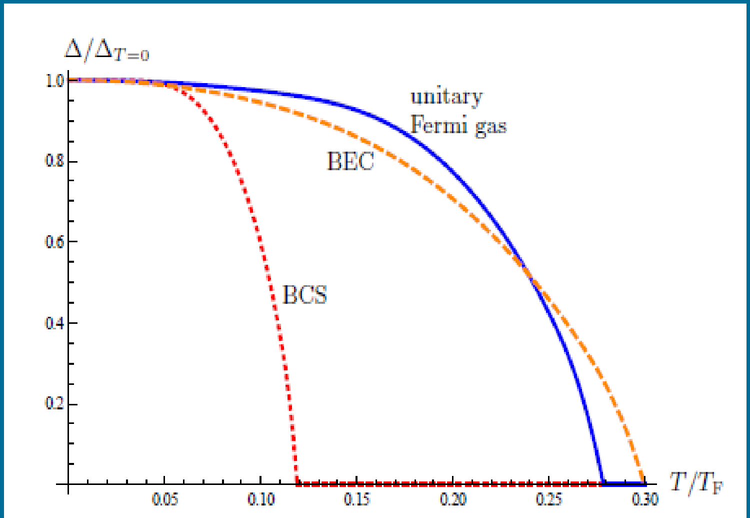


$$\sigma = 4\pi a^2$$

The BCS-BEC Crossover



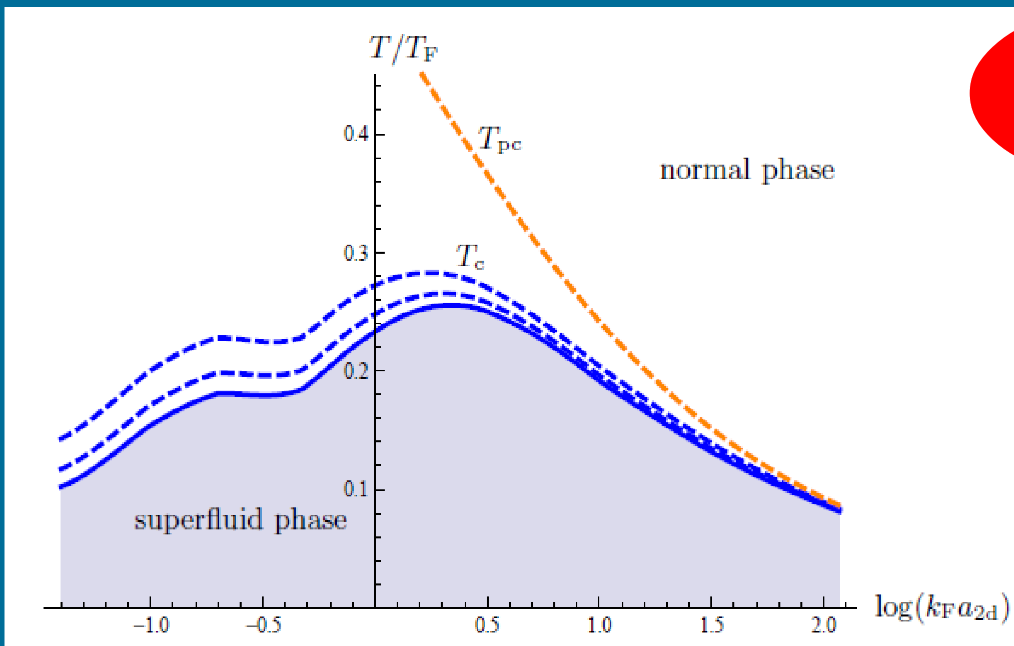


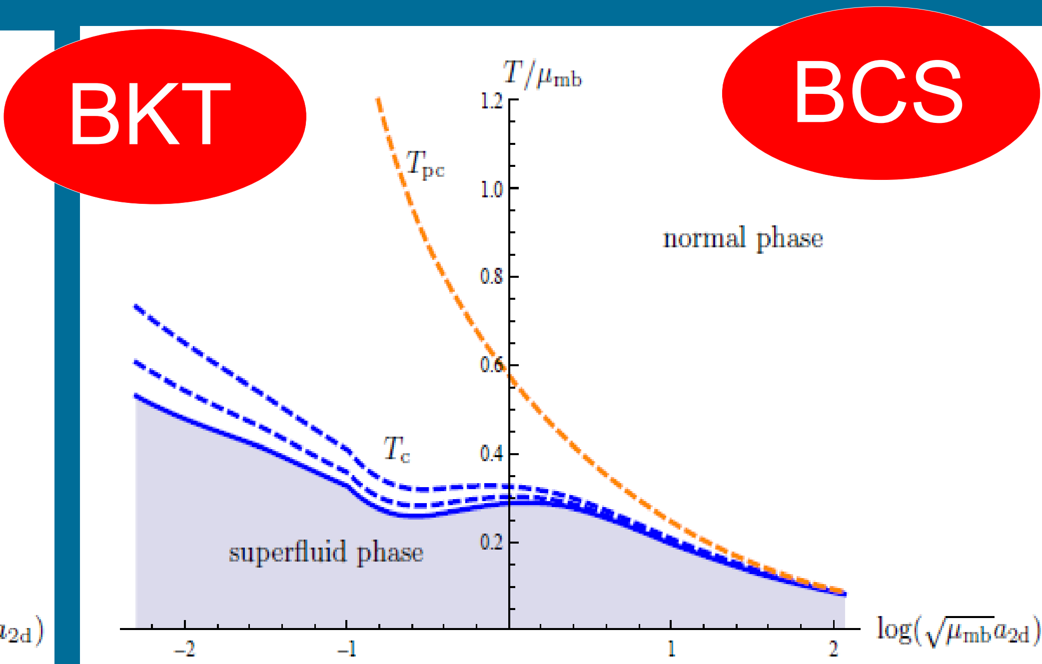


3D BCS-BEC crossover

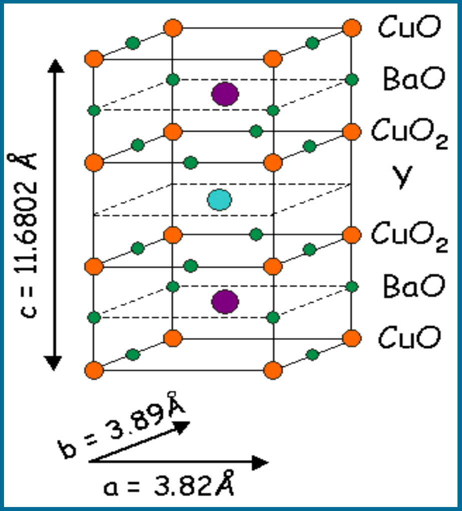
(results from Functional Renormalization Group)

The BCS-BEC Crossover





High Tc super-conductors!?



2D BCS-BEC crossover

(results from Functional Renormalization Group)

The BCS-BEC Crossover

Key observables

Tan contact

$$n_{\vec{p}\sigma} \simeq \frac{C}{p^4}$$

Equation of state

$$P(\mu, T, a)$$

Dimer-dimer scattering length

$$a_{\mathrm{dd}}/a = 0.6$$

Bertsch parameter

$\xi = \mu/\varepsilon_{\mathrm{F}}$ at $a^{-1} = 0$

Critical temperature

$$T_{\mathrm{c}}/T_{\mathrm{F}}$$

The BCS-BEC Crossover

Key observables

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The BCS-BEC Crossover

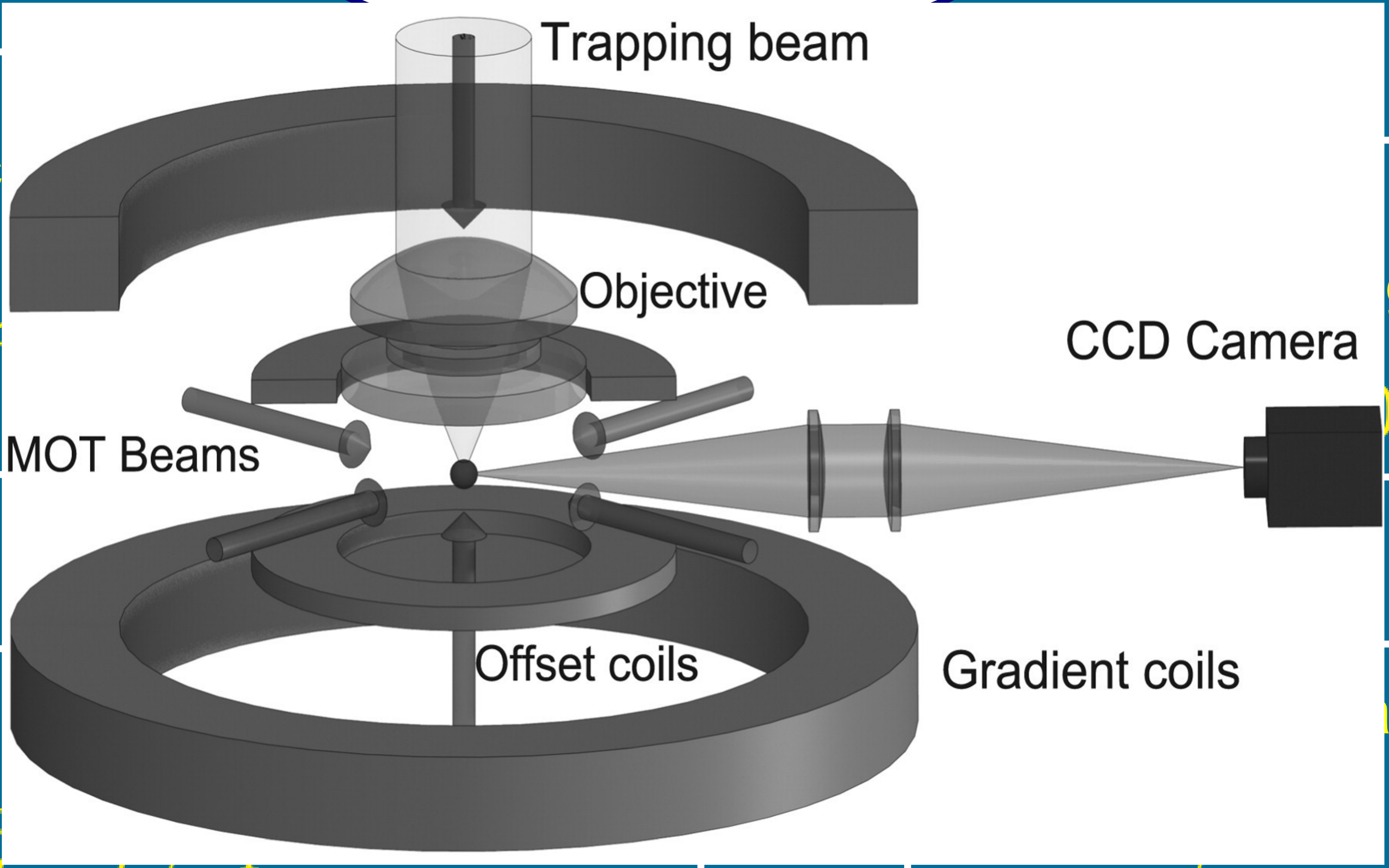
One problem:

Tan c

$n_{\vec{p}\sigma} \simeq$

gth

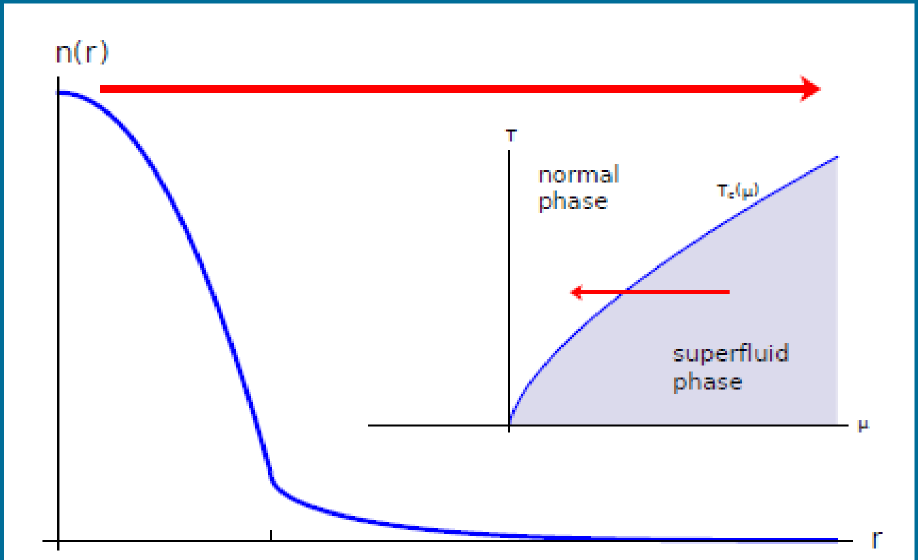
.6



ature

ξ

Thermodynamics from density profiles

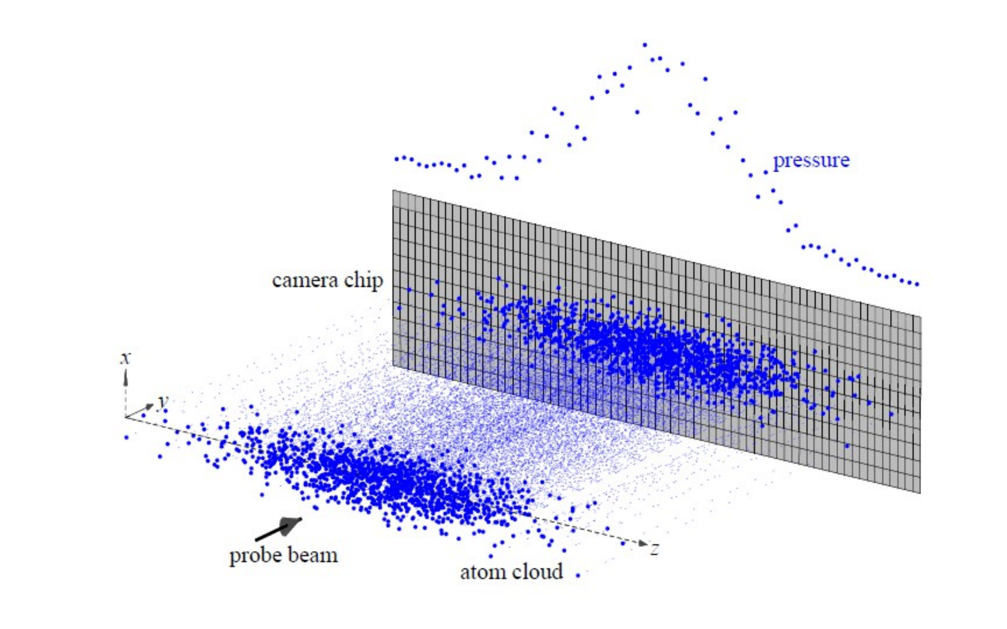


$$P(\mu, T) \to P(\mu - V_{\text{ext}}(\vec{x}), T)$$

local density approximation

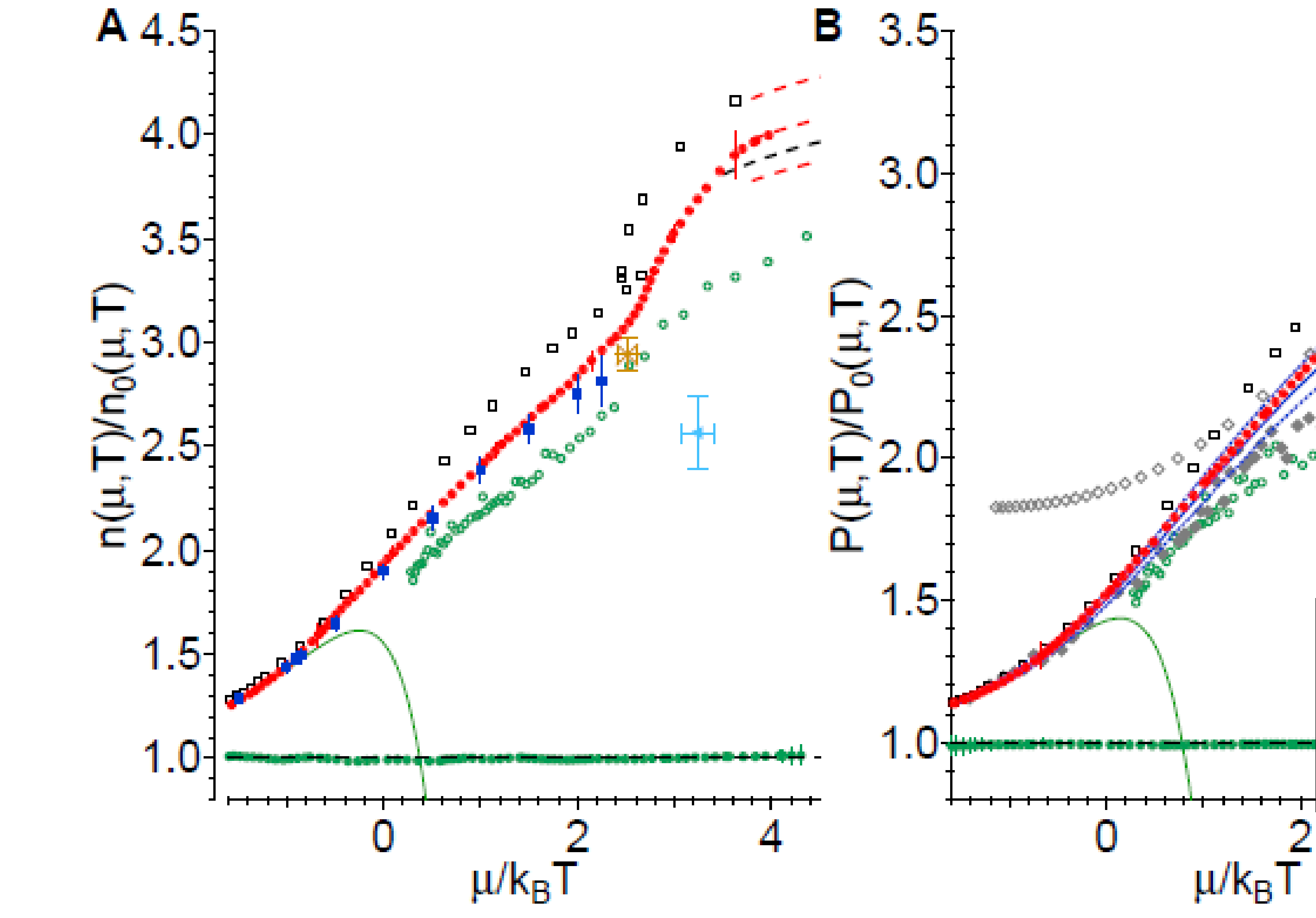
$$P(\mu_z, T) = \frac{m\omega_r^2}{2\pi}\overline{n}(z),$$

Ho, Zhou



S. Nascimbène et al.

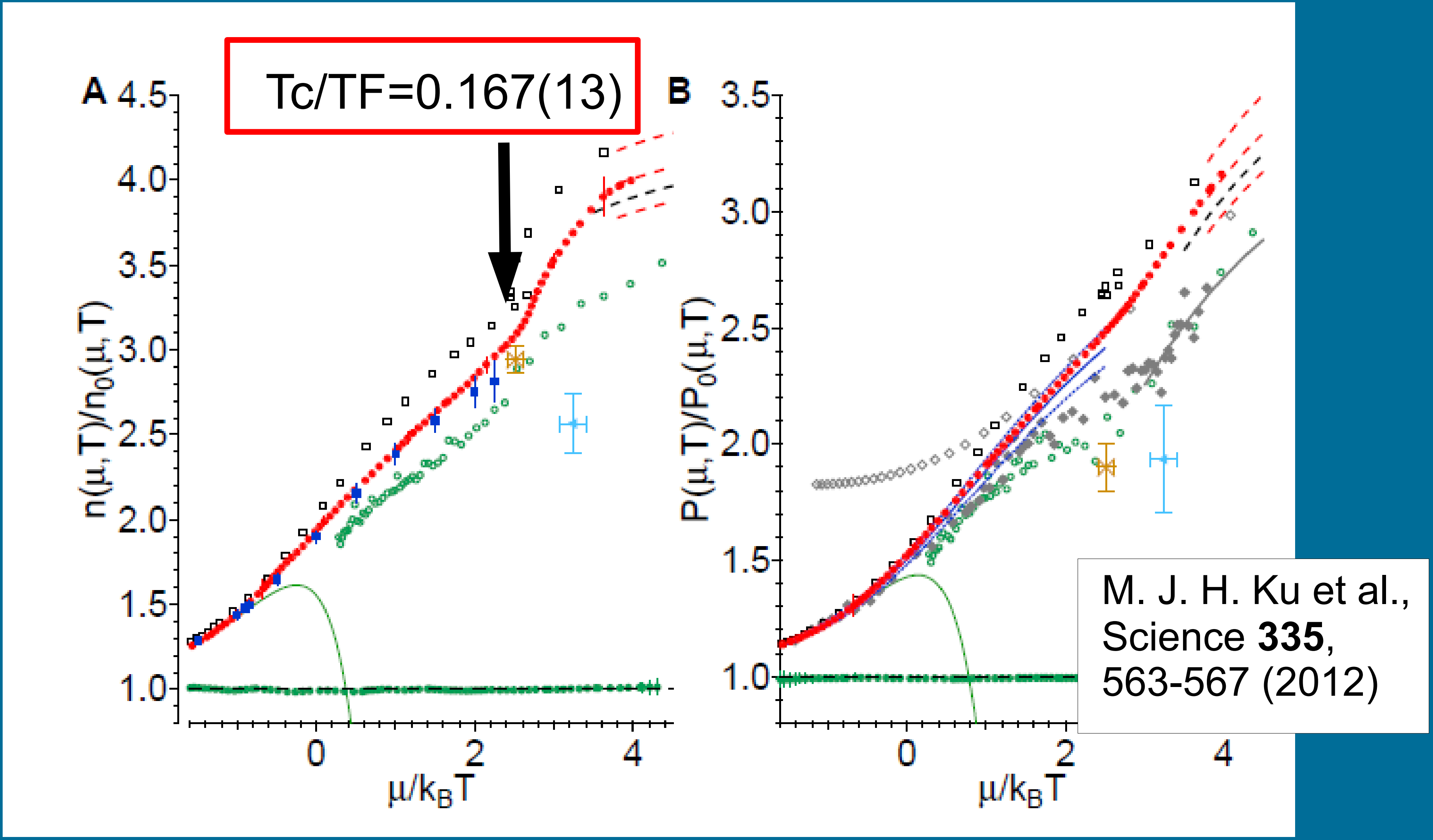
Thermodynamics from density profiles



M. J. H. Ku et al., Science **335**, 563-567 (2012)

Unitary Fermi gas at MIT by Zwierlein group

Thermodynamics from density profiles

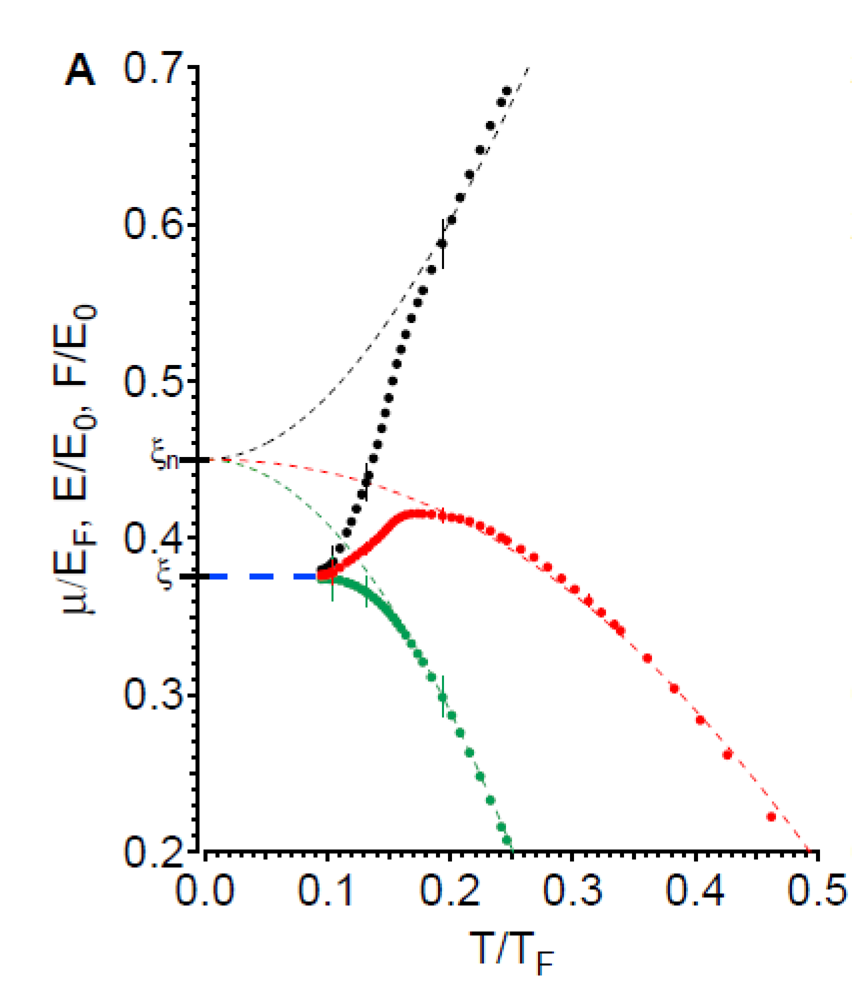


Unitary Fermi gas at MIT by Zwierlein group

Thermodynamics from density profiles

Bertsch parameter ξ:
EoS at T=0

$$E(0) = \xi \frac{3}{5} N \varepsilon_{\mathrm{F}}$$



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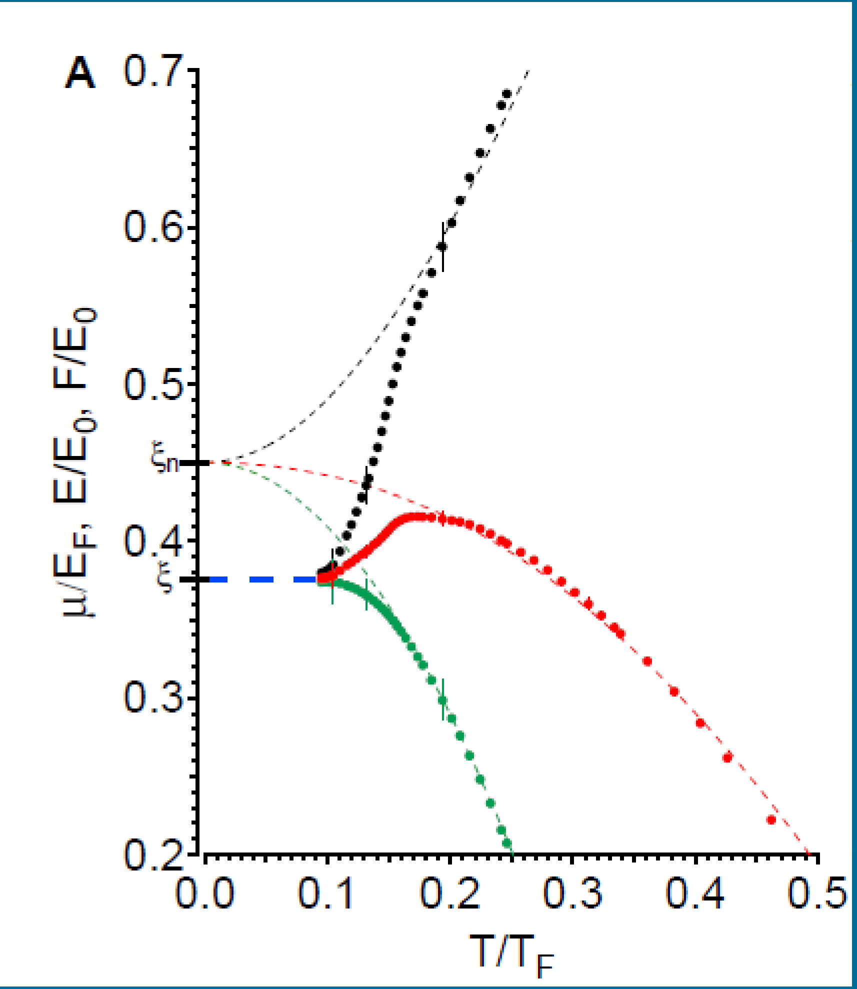
$$F(0) = E(0)$$



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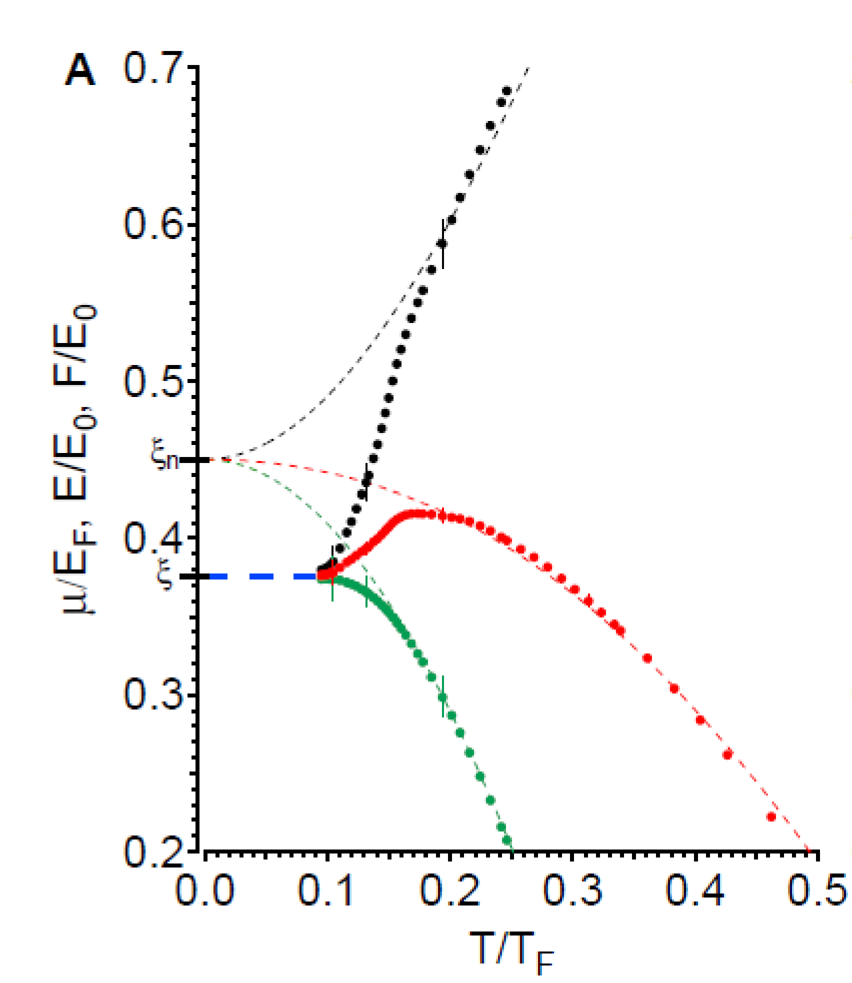

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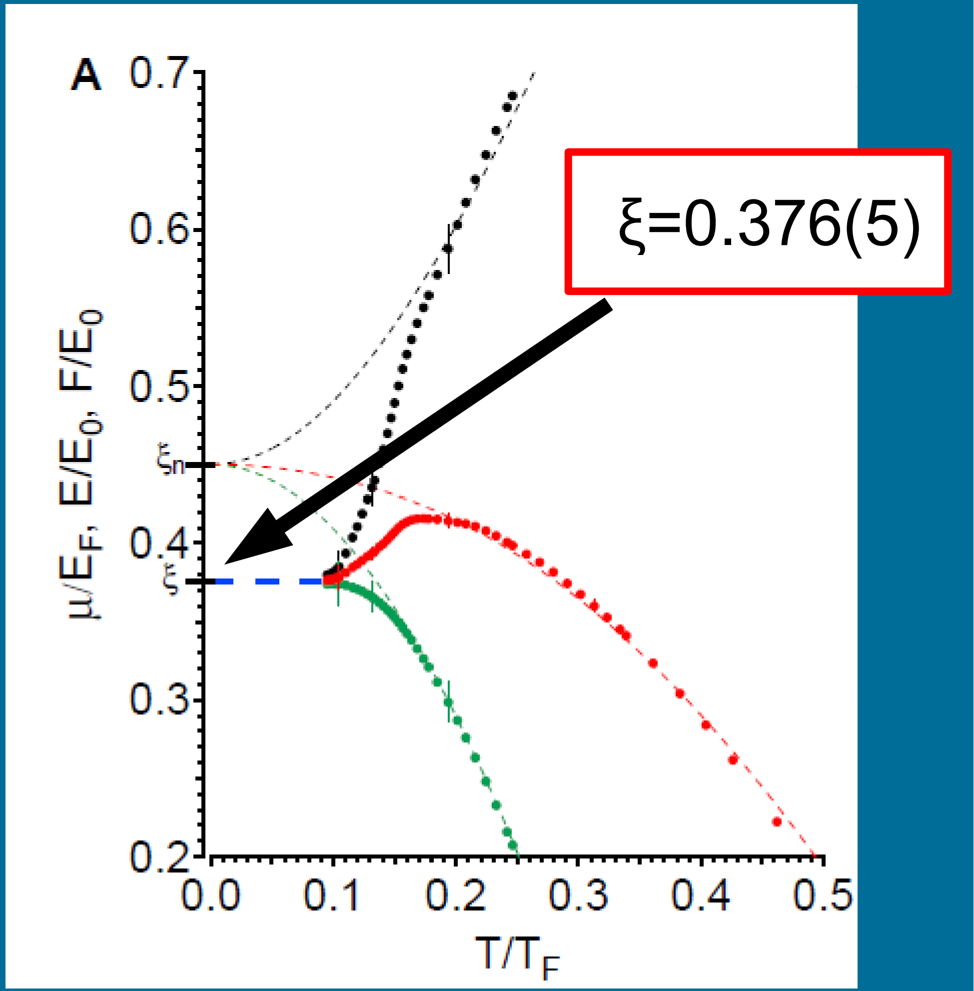

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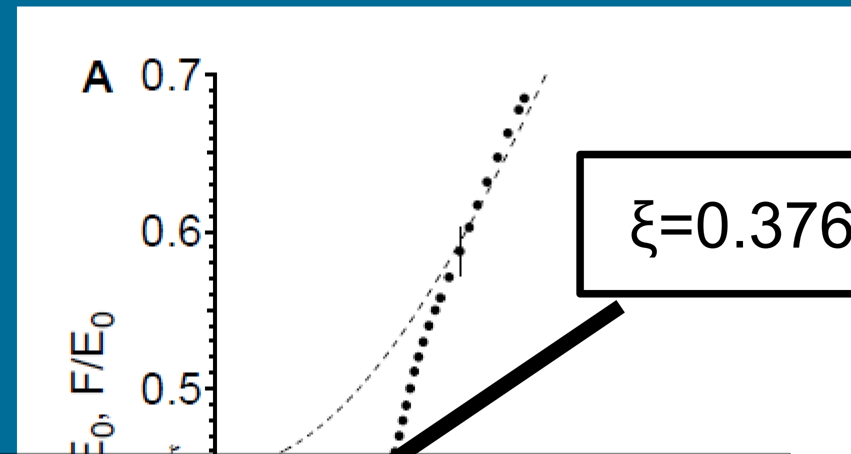


ξ=0.376(5)

Precise characterization of ^{6}Li Feshbach resonances
using trap-sideband resolved RF spectroscopy of weakly bound molecules

G. Zürn,[1,2] T. Lompe,[1,2,3,*] A. N. Wenz,[1,2] S. Jochim,[1,2,3] P. S. Julienne,[4] and J. M. Hutson[5,†]
[1] *Physikalisches Institut, Ruprecht-Karls-Universität Heidelberg, Germany*
[2] *Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany*
[3] *ExtreMe Matter Institute EMMI, GSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany*
[4] *Joint Quantum Institute, NIST and the University of Maryland, Gaithersburg, Maryland 20899-8423, USA*
[5] *Joint Quantum Centre (JQC) Durham/Newcastle, Department of Chemistry, Durham University, South Road, Durham, DH1 3LE, United Kingdom*
(Dated: November 8, 2012)

ξ=0.370(5)(8)

Unitary Fermi gas at MIT by Zwierlein group

Equation of state from Functional RG

Experiment:

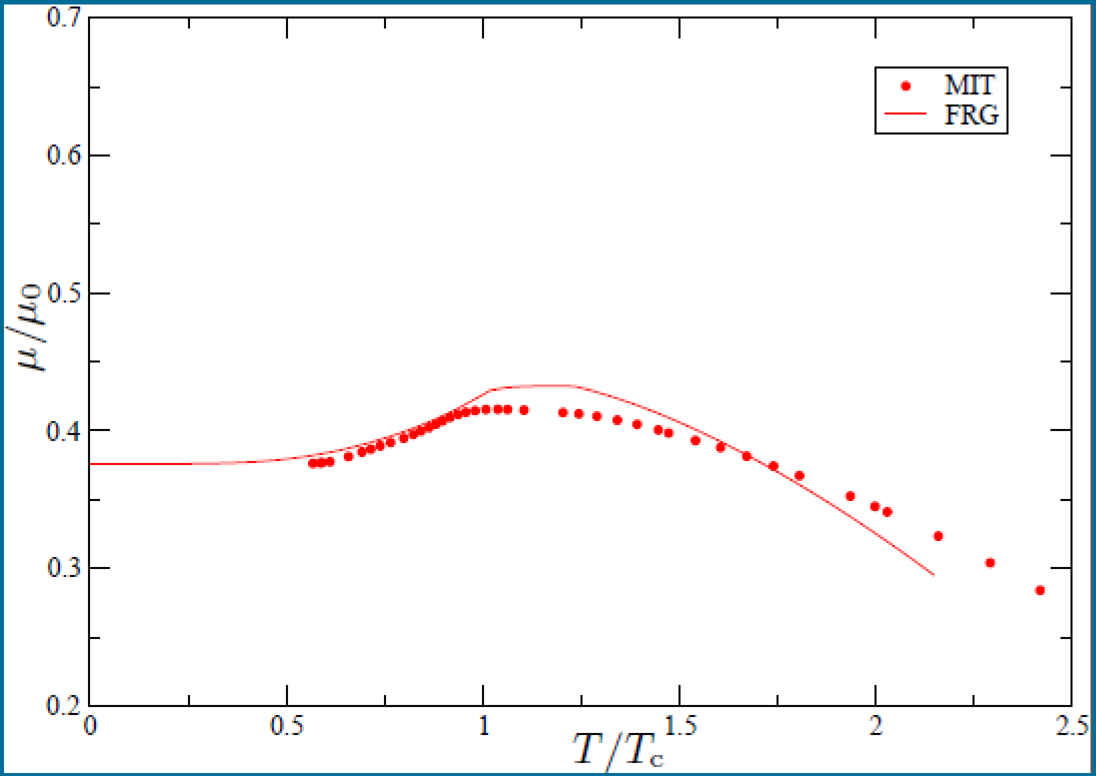
$$\xi_{\mathrm{exp}} = 0.370(5)(8)$$

$$(T_c/T_F)_{\mathrm{exp}} = 0.167(13)$$

Latest FRG:
(Floerchinger, Scherer, Wetterich)

$$\xi_{\mathrm{FRG}} = 0.51$$

$$(T_c/T_F)_{\mathrm{FRG}} = 0.248$$



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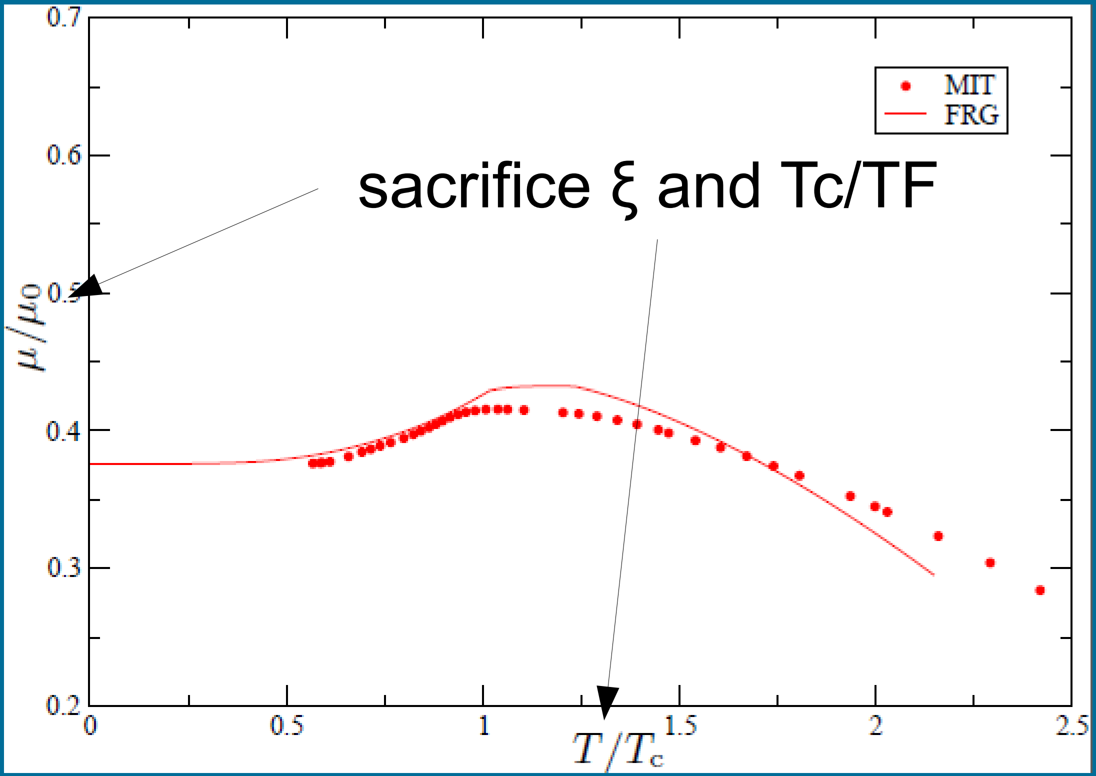

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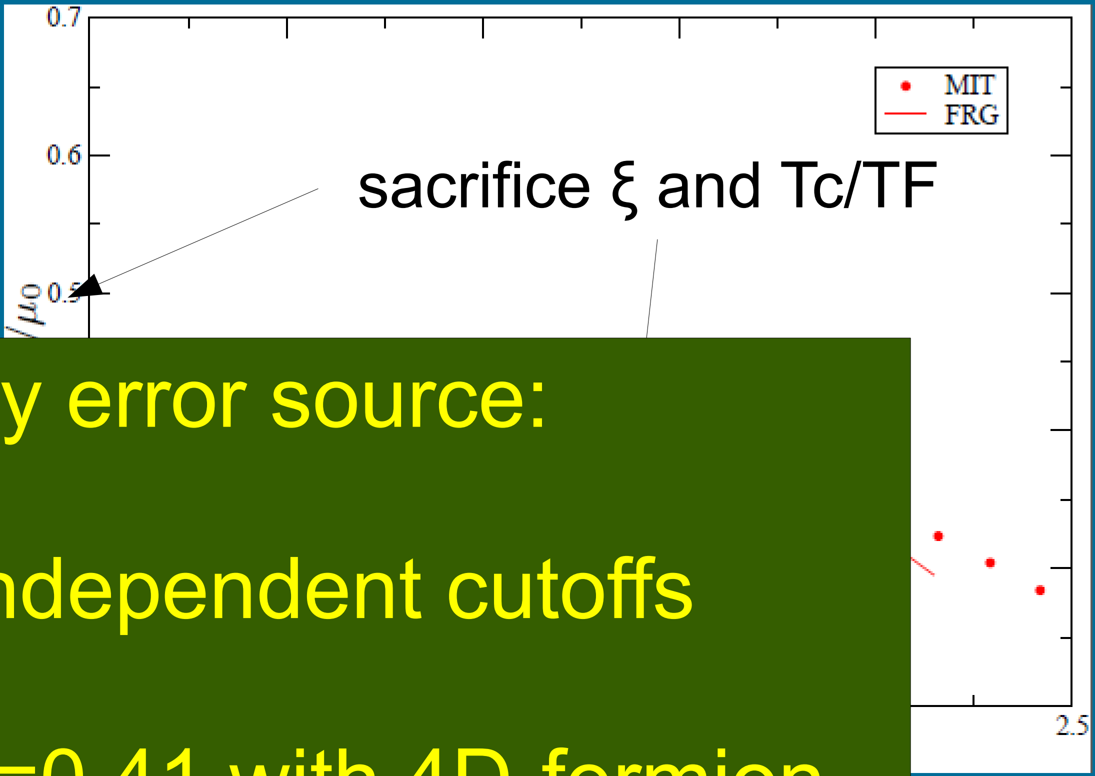
$(T_c/T$

Lates

(Floerchi

ξ

$(T_c/T$



Most likely error source:

Frequency-independent cutoffs

First estimate: ξ=0.41 with 4D-fermion cutoff in the simplest truncation

Equation of state from Functional RG

Experiment:

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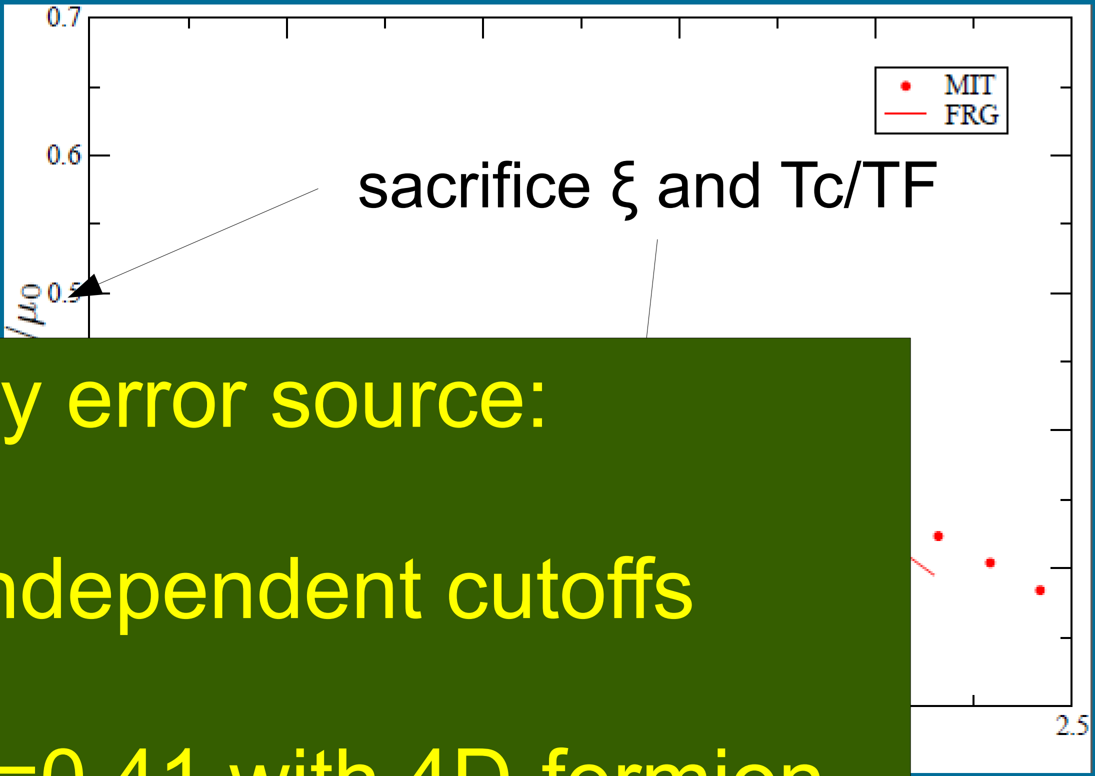

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ξ

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First estimate: ξ=0.41 with 4D-fermion cutoff in the simplest truncation

→ contact me during the workshop if you are interested in that

Tan contact

$$n_{\vec{p}\sigma} \simeq \frac{C}{p^4}$$

Momentum distribution

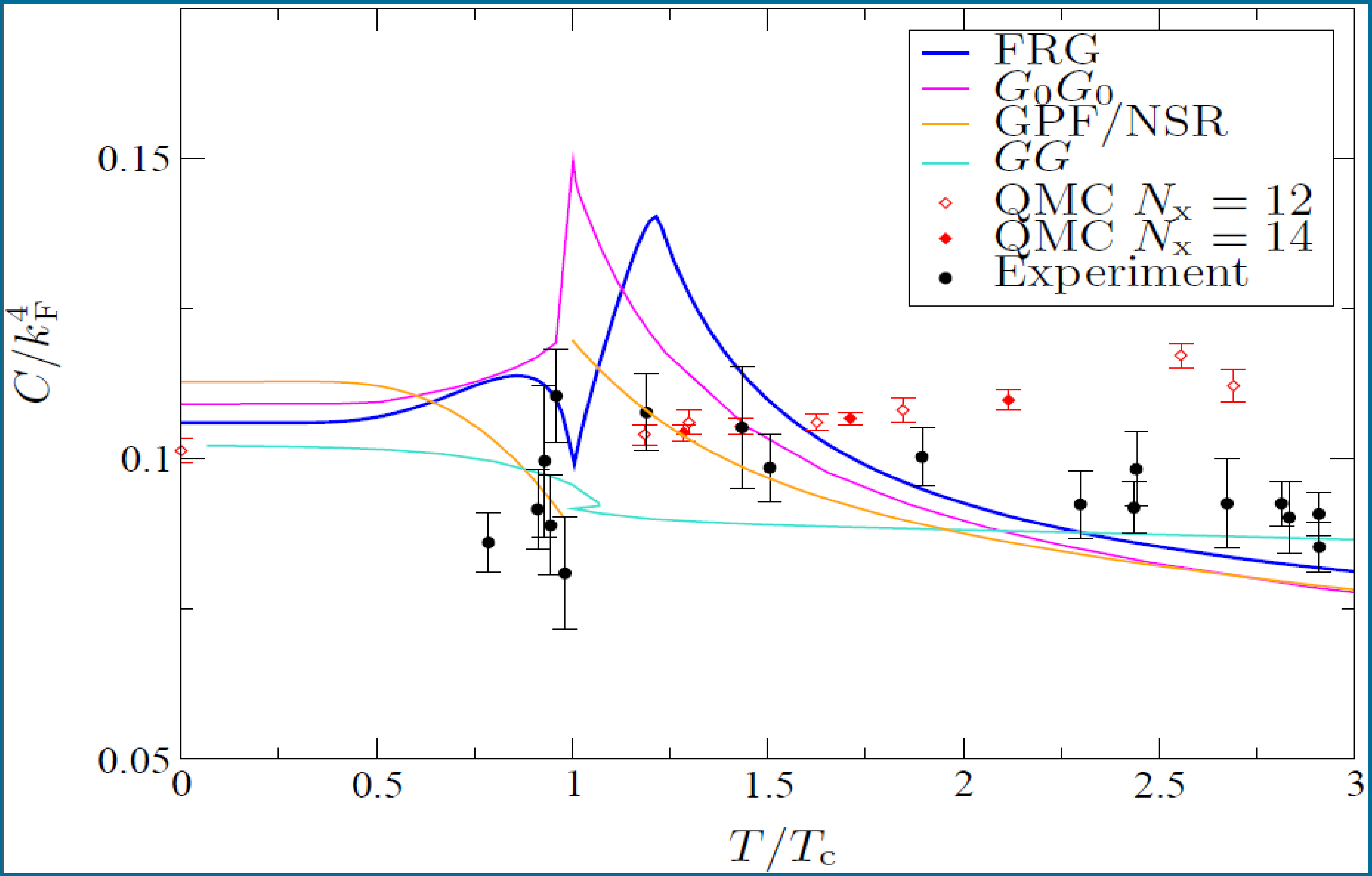
$$\left(\frac{\partial P}{\partial a^{-1}}\right)_{\mu,T} = \frac{C}{4\pi M}$$

Tan relation

$$\Sigma_\psi(P) \simeq \frac{4C}{-\mathrm{i}p_0 + p^2 - \mu} - \delta\mu$$

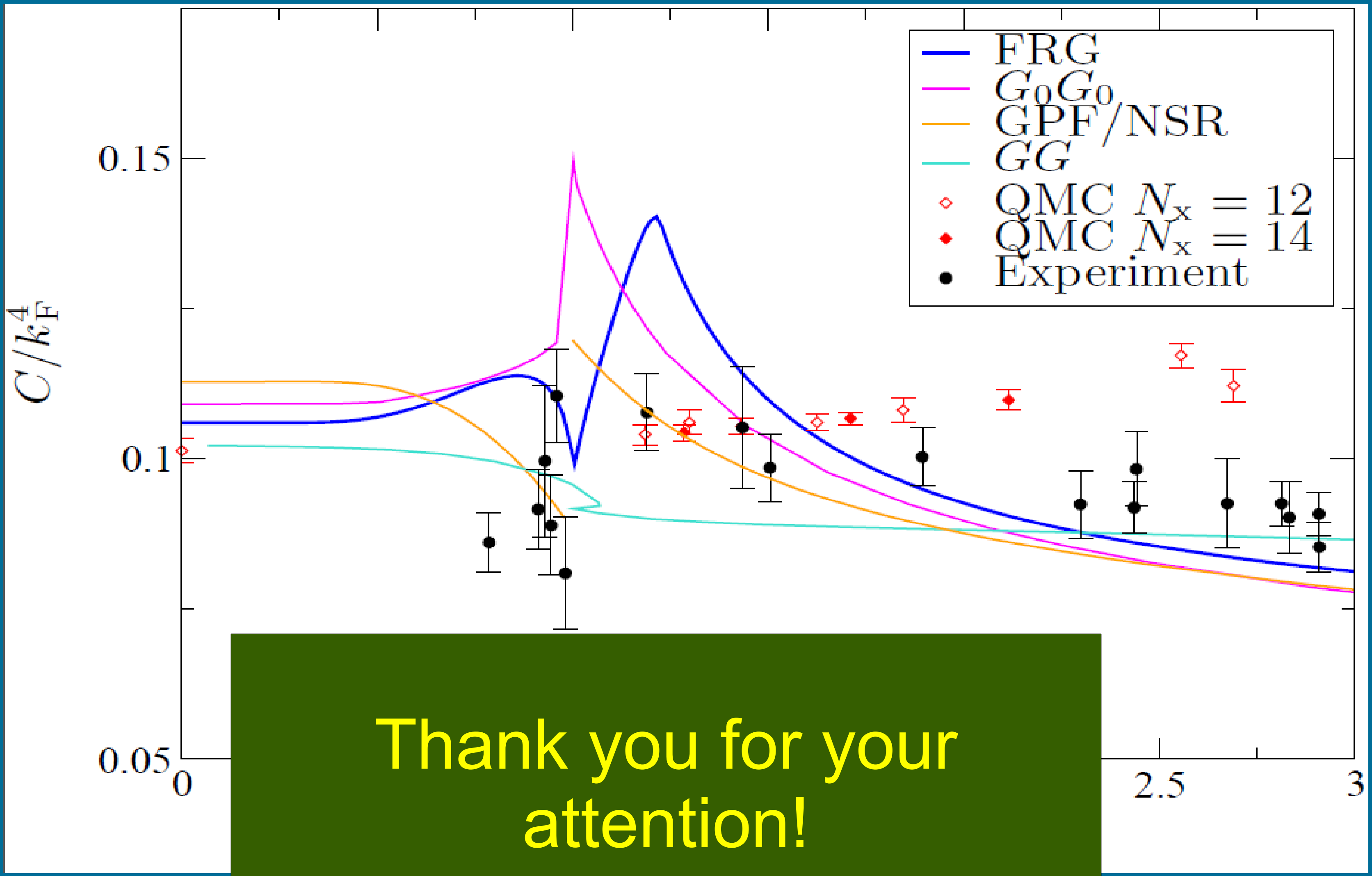
Asymptotic fermion self-energy

Tan contact



FRG: IB, S. Diehl, J. M. Pawlowski, C. Wetterich

Tan contact



Thank you for your attention!

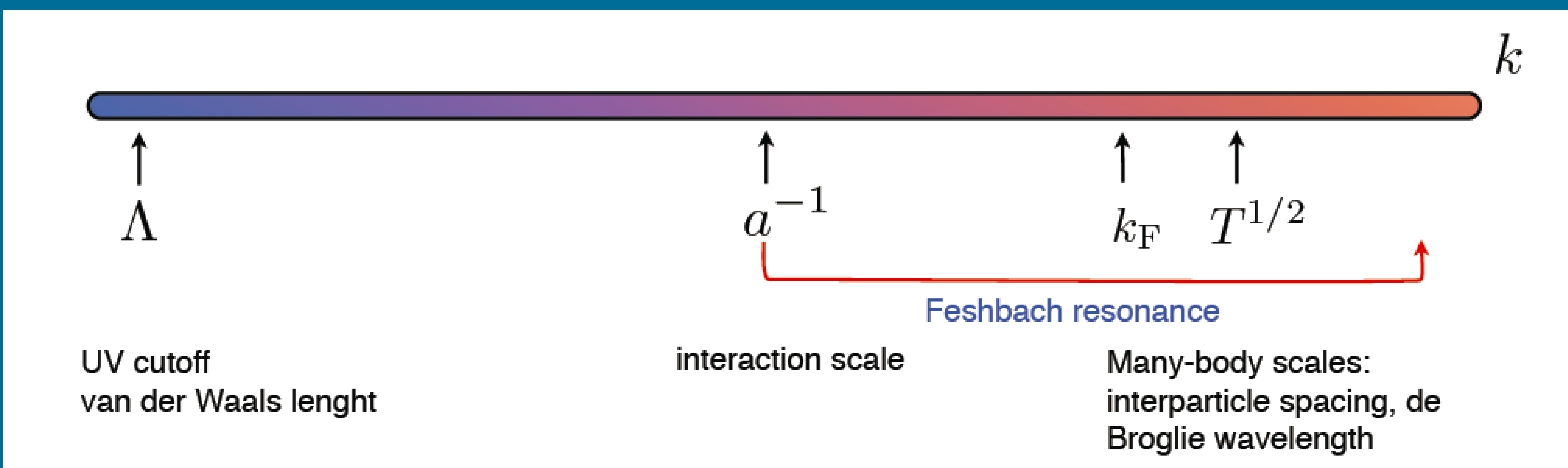
FRG: IB, S. Diehl, J. M. Pawlowski, C. Wetterich

Additional slides

Microscopic Model

Many-body Hamiltonian

$$\hat{H} = \int d^3x \left(\sum_{\sigma=1,2} \hat{\psi}^\dagger_\sigma (-\nabla^2) \hat{\psi}_\sigma + \lambda_{\psi,\Lambda} \hat{\psi}^\dagger_1 \hat{\psi}^\dagger_2 \hat{\psi}_2 \hat{\psi}_1 \right)$$



Microscopic Model

Many-body Hamiltonian

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Microscopic action

$$S[\varphi, \psi] = \int_X \left(\sum_{\sigma=1,2} \psi_\sigma^* (\partial_\tau - \nabla^2 - \mu) \psi_\sigma + m_{\varphi,\Lambda}^2 \varphi^* \varphi \right.$$

$$\left. - h_\varphi (\varphi^* \psi_1 \psi_2 - \varphi \psi_1^* \psi_2^*) \right)$$

Macroscopic physics

How to compute the partition function?

$$Z(\mu, T) = \int \mathrm{D}\varphi \mathrm{D}\psi \, \mathrm{e}^{-S[\varphi,\psi]}$$

Integration

Macroscopic physics

How to compute the partition function?

$$Z_k(\mu, T) = \int D\varphi D\psi e^{-S[\varphi,\psi]+\Delta S_k}$$

scale dependent partition function

$$\partial_k Z_k(\mu, T) = \ldots$$

Solve flow equation

Wetterich equation

$\Gamma[\Phi] = J \cdot \Phi - \log Z[J]$ effective action

$$\partial_k \Gamma_k = \frac{1}{2} \mathrm{STr} \left(\frac{1}{\Gamma_k^{(2)} + R_k} \partial_k R_k \right)$$

fluctuations

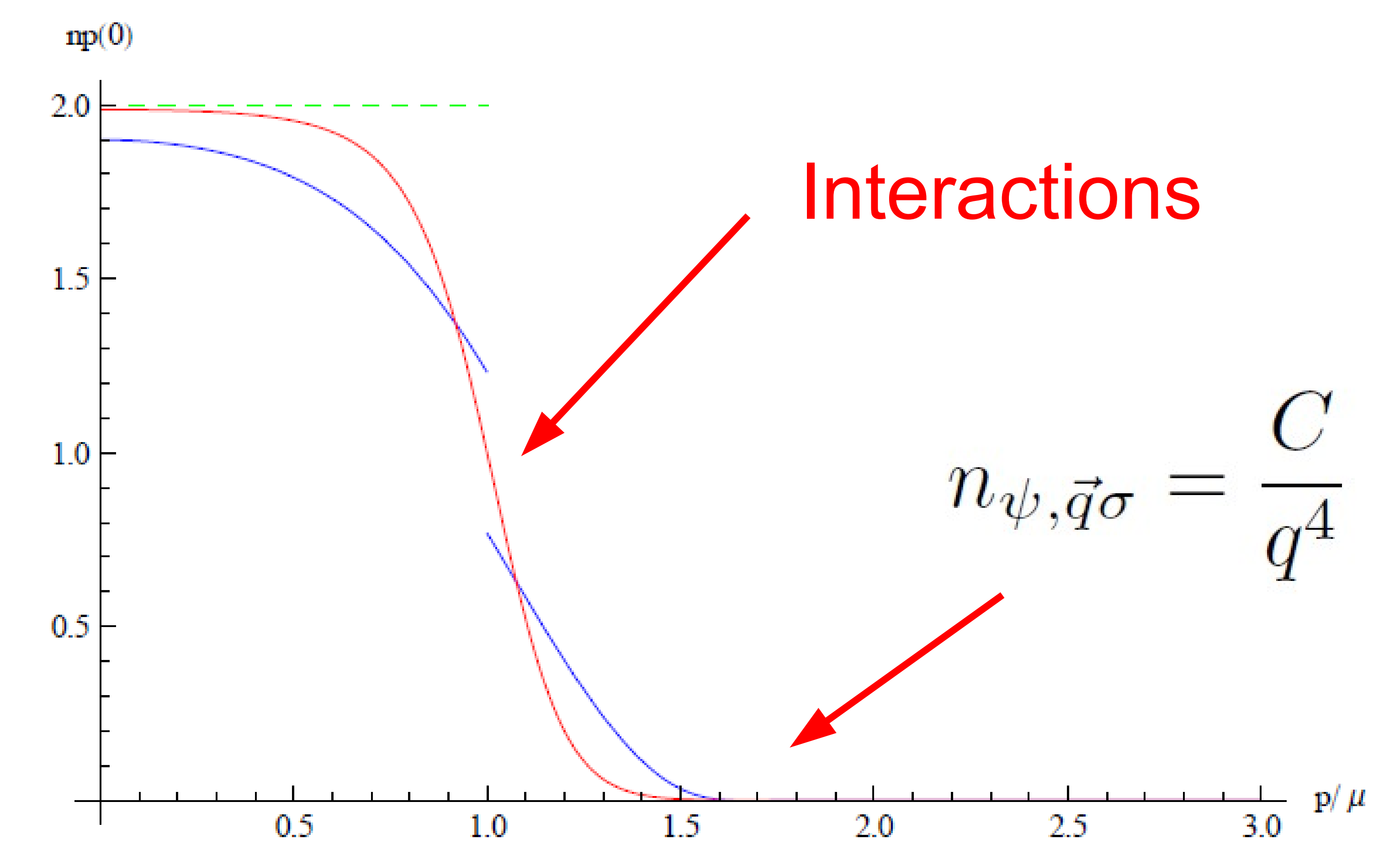
$\Gamma_{k=\Lambda} = S$ → $\Gamma_{k=0} = \Gamma$

Microphysics Macrophysics

Contact in the BCS-BEC Crossover

Momentum distribution

Ideal Fermi gas: Fermi-Dirac distribution



$$n_{\psi,\vec{q}\sigma} = \frac{C}{q^4}$$

Momentum distribution

$$n_{\vec{p}\sigma} \simeq \frac{C}{p^4}$$

Tan contact C

Several exact relations, e.g.:

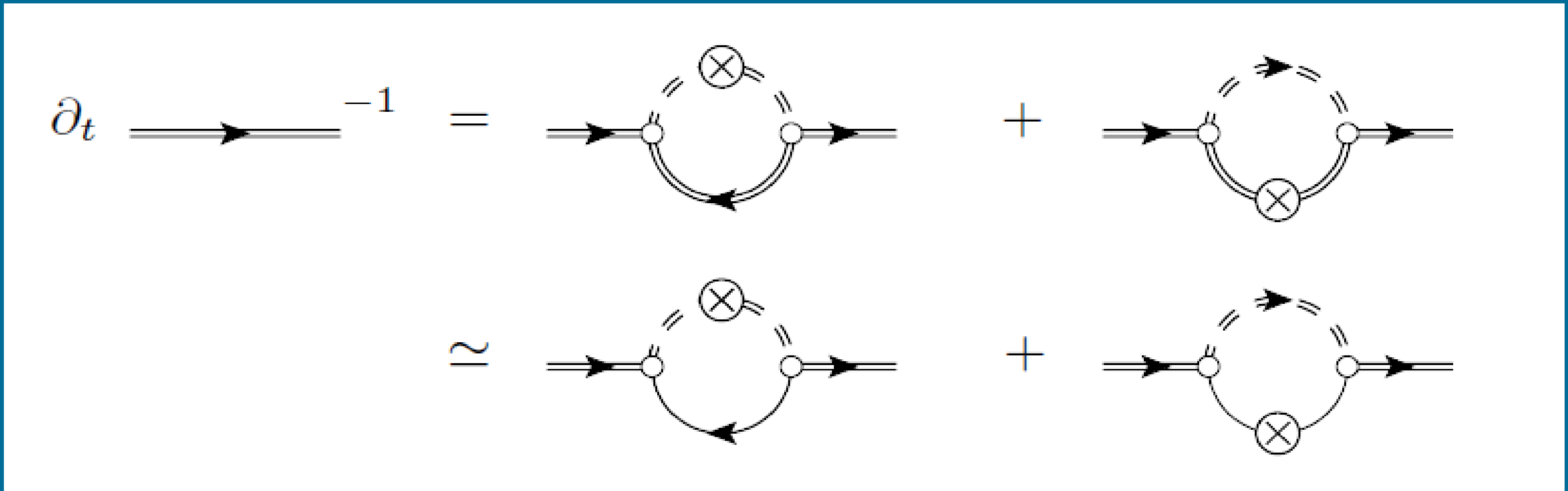
$$\frac{1}{V}\frac{\mathrm{d}E}{\mathrm{d}(-1/a)} = \frac{C}{4\pi M}$$

$$E = \frac{C}{4\pi M a} + \sum_{\sigma=1,2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{p^2}{2M} \left(n_{\vec{p}\sigma} - \frac{C}{p^4} \right)$$

Contact from the FRG

$$n_{\vec{p}\sigma} = -\int_{p_0} G_{\psi\sigma}(p_0, \vec{p})$$

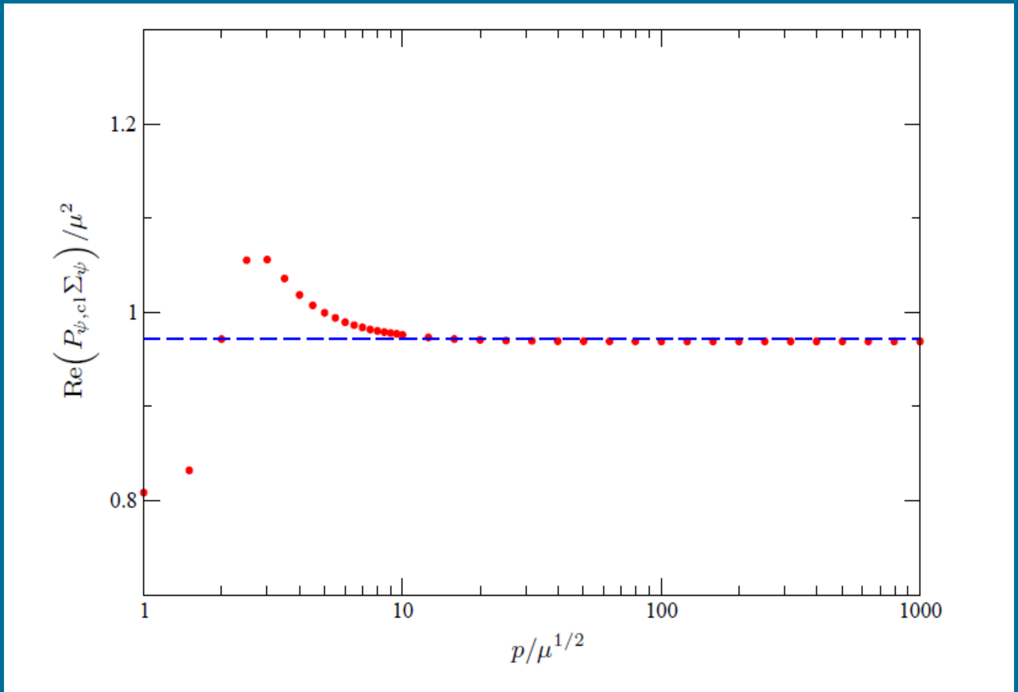
full macroscopic propagator



Contact from the FRG

Factorization of the RG flow for large p:

$$\partial_k G_{\psi,k}^{-1}(P) \simeq \frac{4}{-\mathrm{i}p_0 + p^2 - \mu} \partial_k C_k$$



Contact from the FRG

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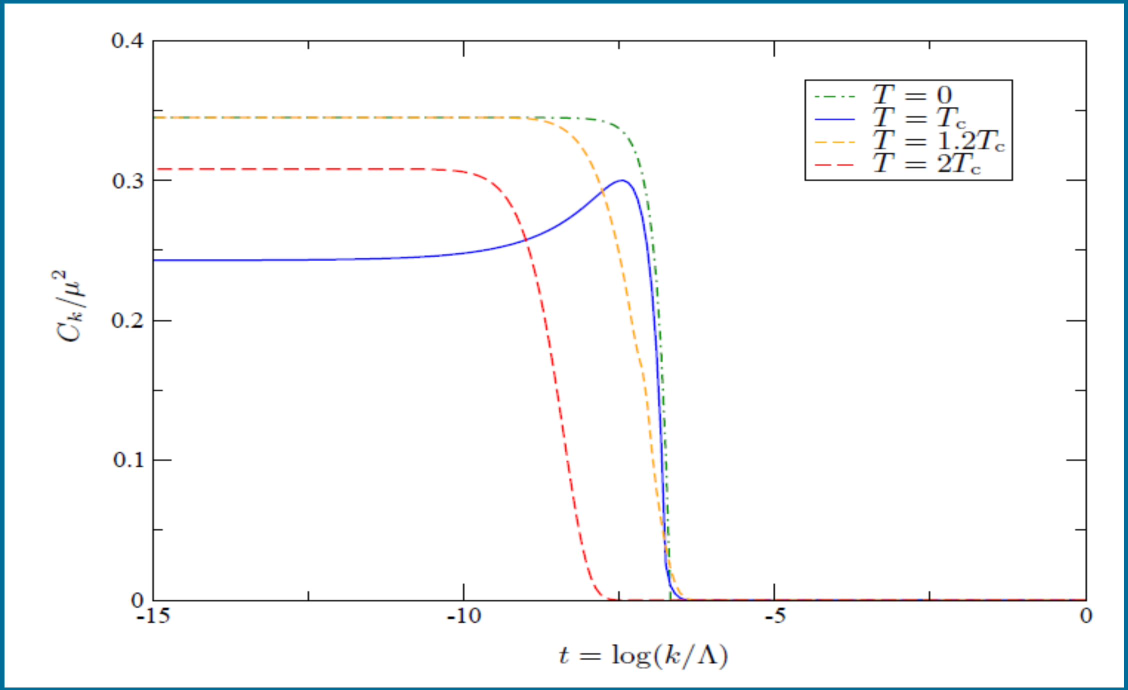
$$\partial_k G_{\psi,k}^{-1}(P) \simeq \frac{4}{-\mathrm{i}p_0 + p^2 - \mu} \partial_k C_k$$

Flowing contact

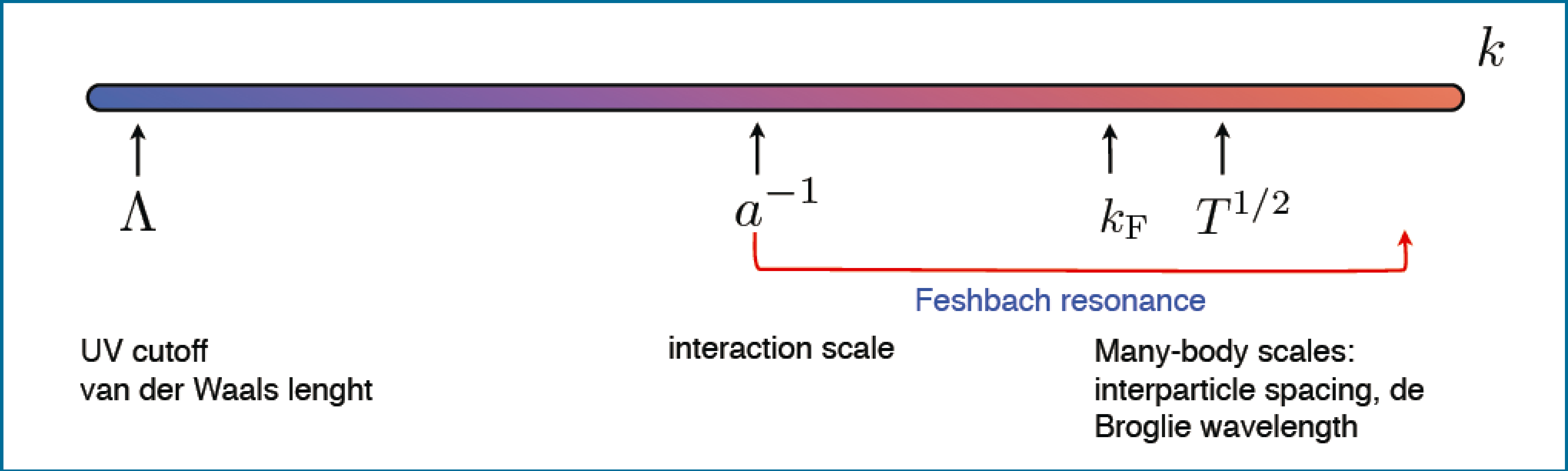
$$\partial_k C_k = \dots$$



Contact from the FRG

Universal regime is enhanced for the Unitary Fermi gas

$$\Sigma_{\psi}(P) \simeq \frac{4C}{-\mathrm{i}p_0 + p^2 - \mu} - \delta\mu$$



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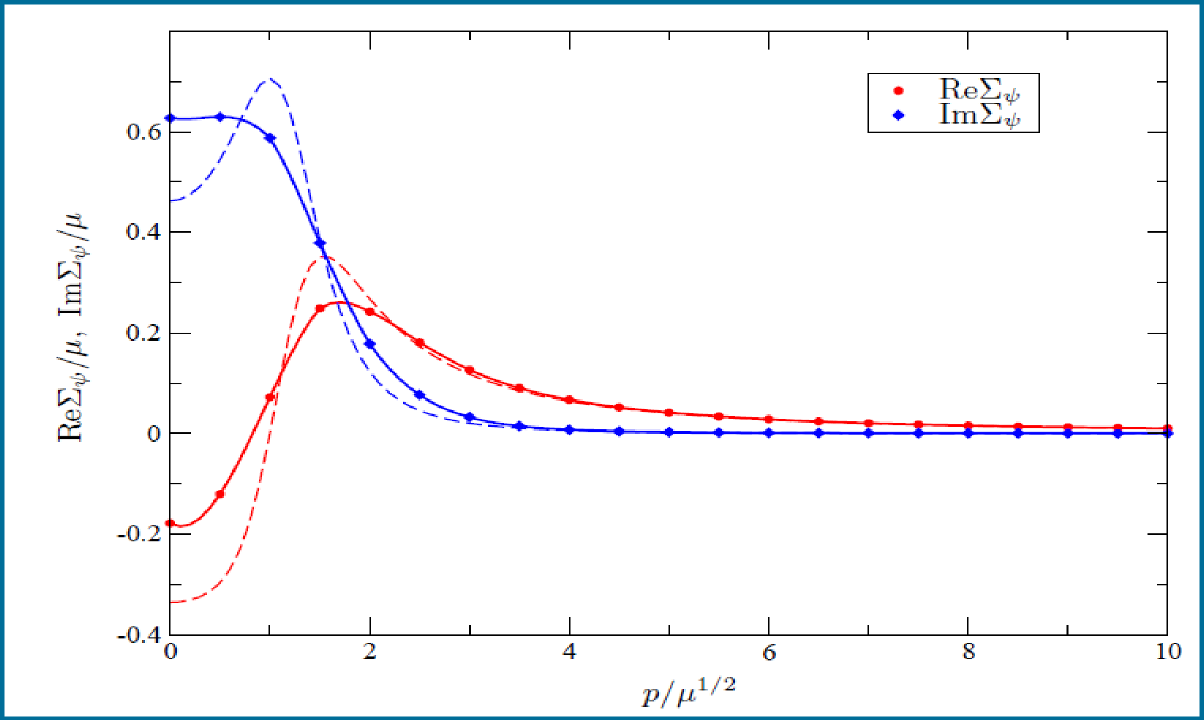
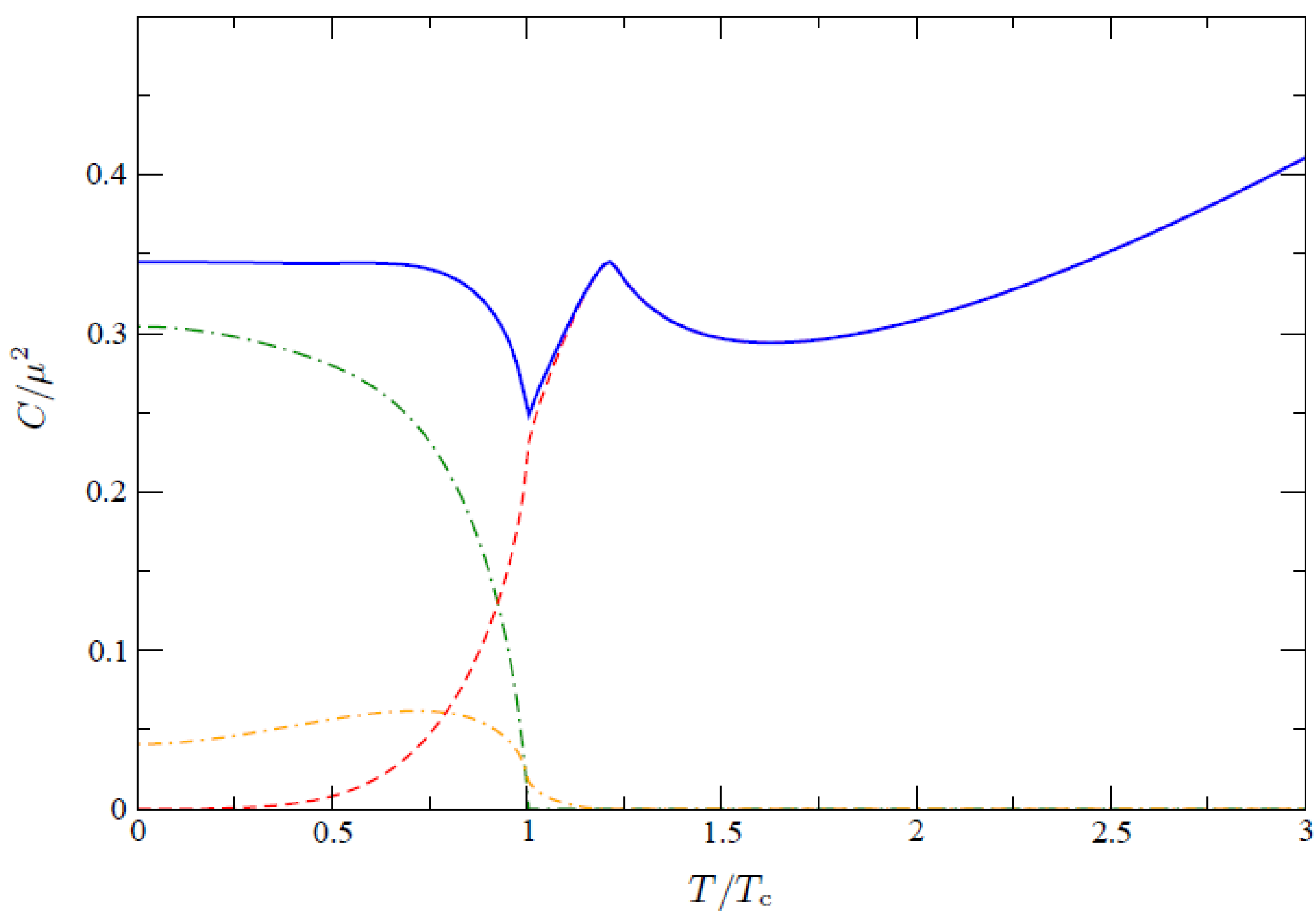


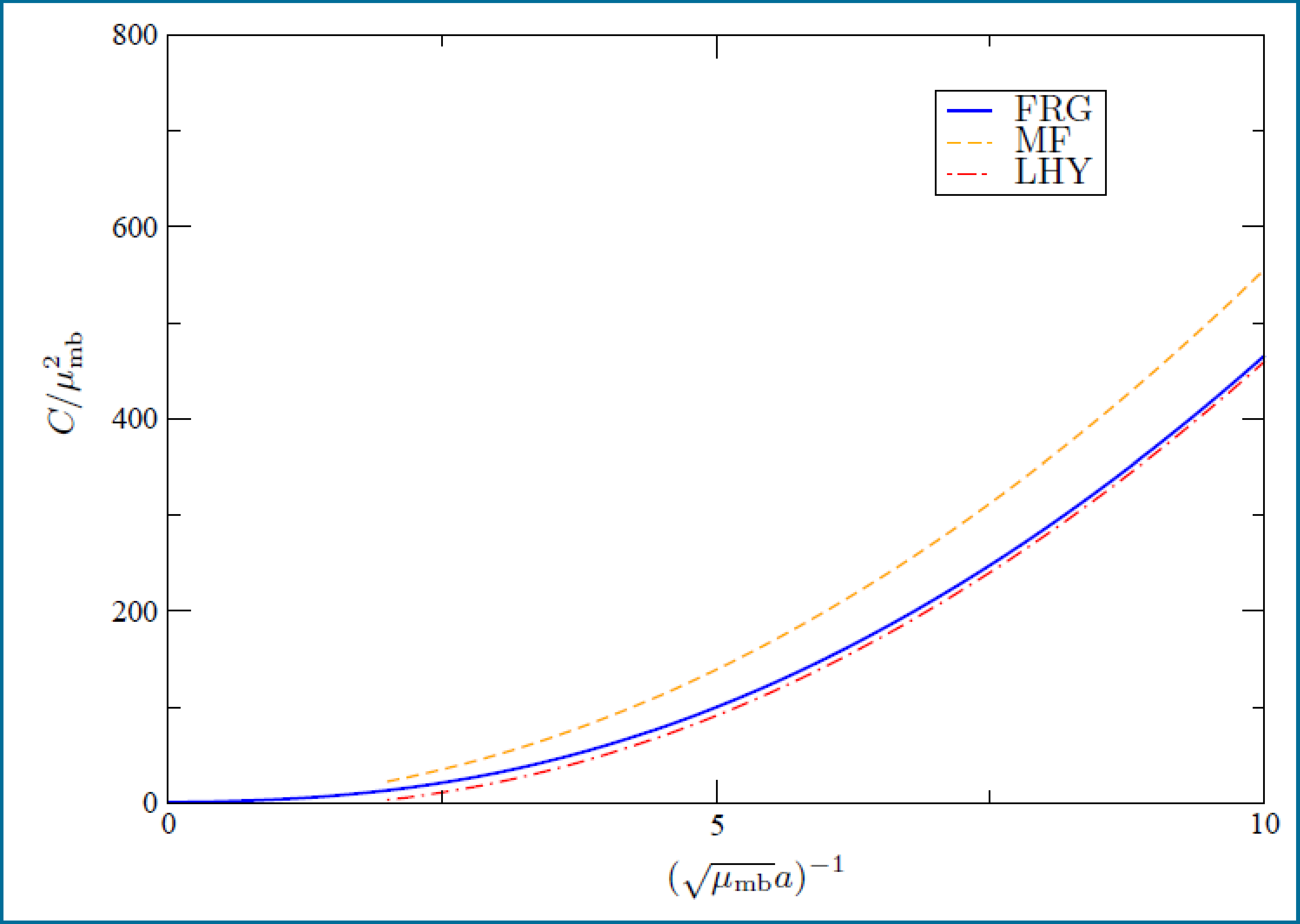
Contact from the FRG

Temperature dependent contact of the Unitary Fermi gas



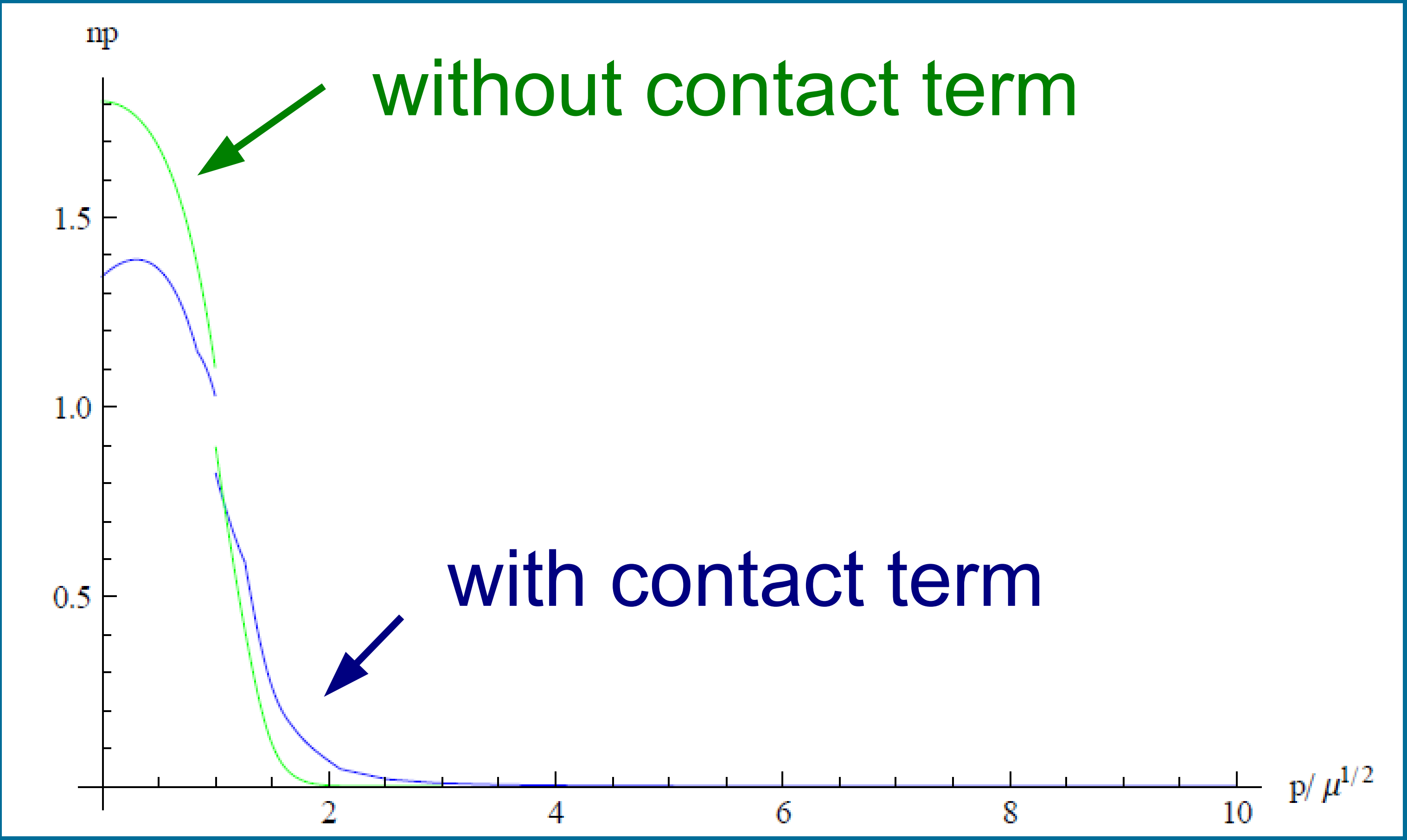
Contact from the FRG

Contact at T=0 in the BCS-BEC crossover



Contact from the FRG

Momentum distribution of the Unitary Fermi Gas at the critical temperature



Increase of density

Contribution from high energetic particles to the density

$$n = 2 \int \frac{d^3 p}{(2\pi)^3} n_{\vec{p}\sigma}$$

⇒ $\frac{\delta n^{(C)}}{n} = 27.5\%$ at Tc

⇒ Substantial effect on $\frac{T_c}{T_F} \propto \frac{T_c}{n^{2/3}}$

Two-dimensional BCS-BEC Crossover

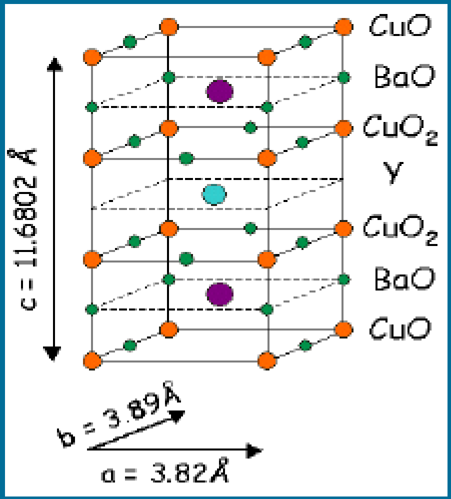
Two-dimensional BCS-BEC Crossover

Why two dimensions?

- Enhanced effects of quantum fluctuations
  → test and improve elaborate methods
- Understand pairing in two dimensions
  → high temperature superconductors

How?

Highly anisotropic traps!



What is different?

Scattering physics in two dimensions

$$f_{2\mathrm{d}}(q) \sim \frac{1}{\log(1/q^2 a_{2\mathrm{d}}^2) + \mathrm{i}\pi + \ldots}$$

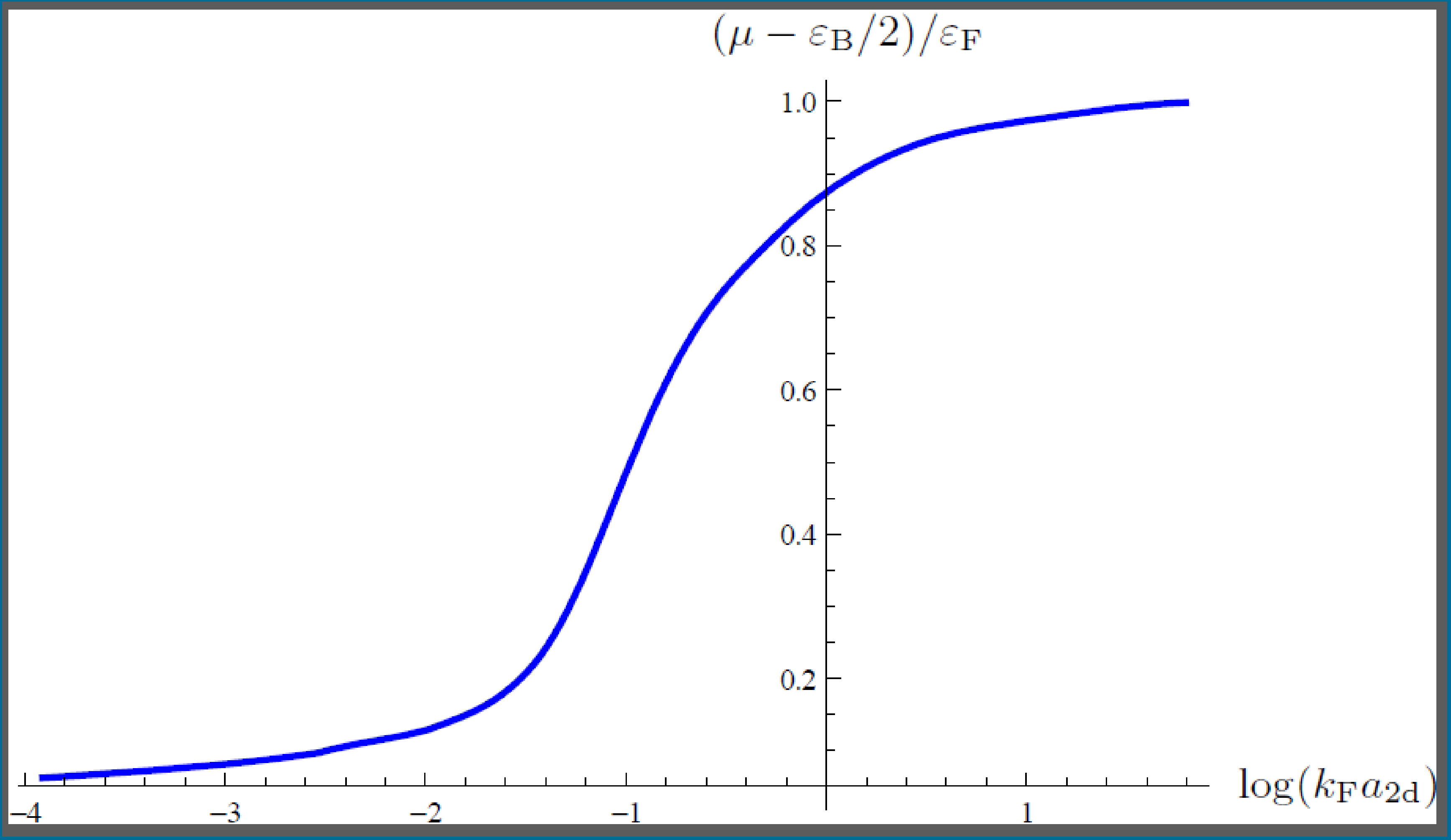
$$f_{3\mathrm{d}}(q) \sim \frac{1}{-\frac{1}{a} + \frac{1}{2} r_{\mathrm{e}} q^2 - \mathrm{i}q + \ldots}$$

Scattering amplitude

→ Crossover parameter $\log(k_{\mathrm{F}} a_{2\mathrm{d}})$

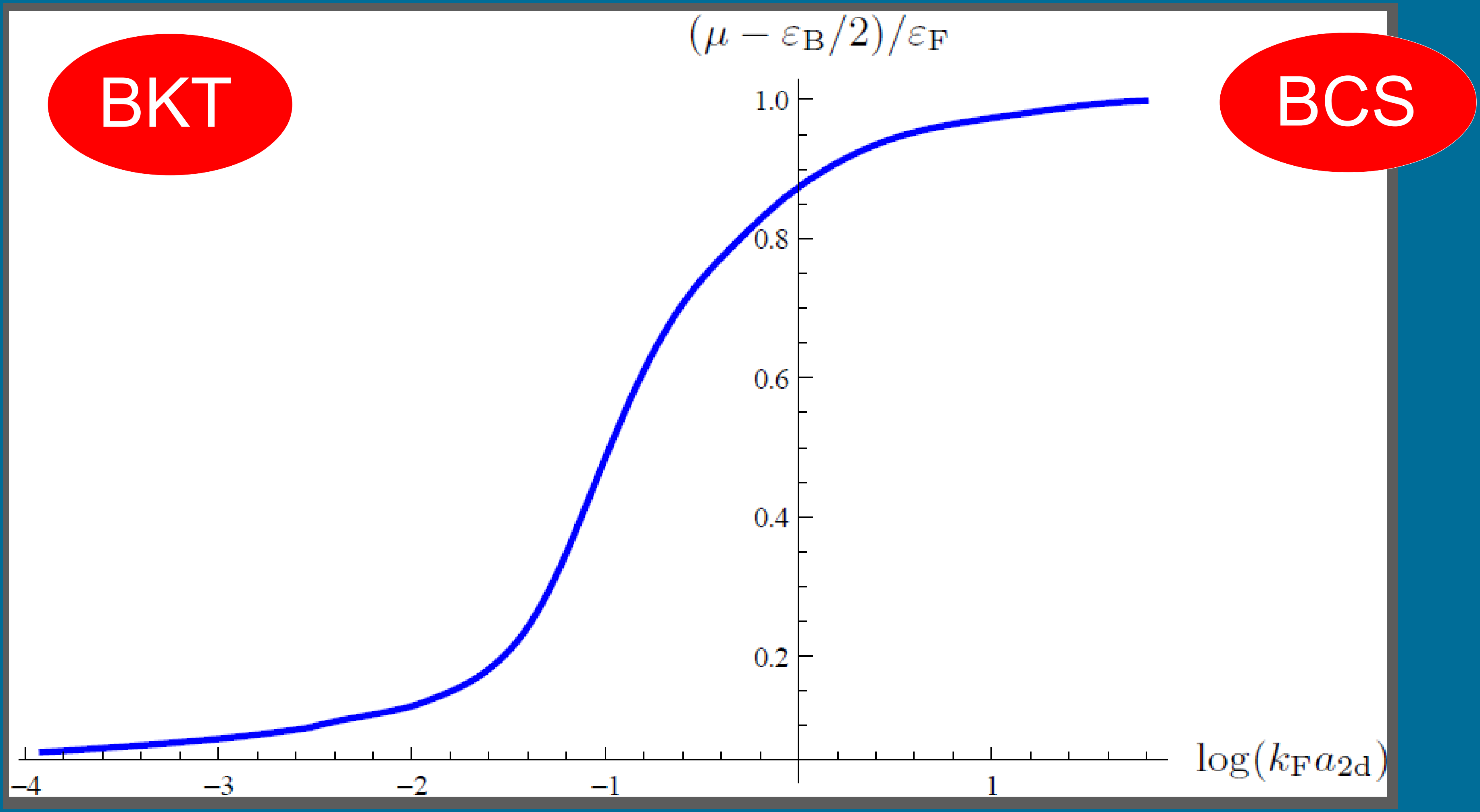
→ No scale invariance, but strong correlations for $k_{\mathrm{F}} \sim \frac{1}{a_{2\mathrm{d}}}$

Equation of state at T=0



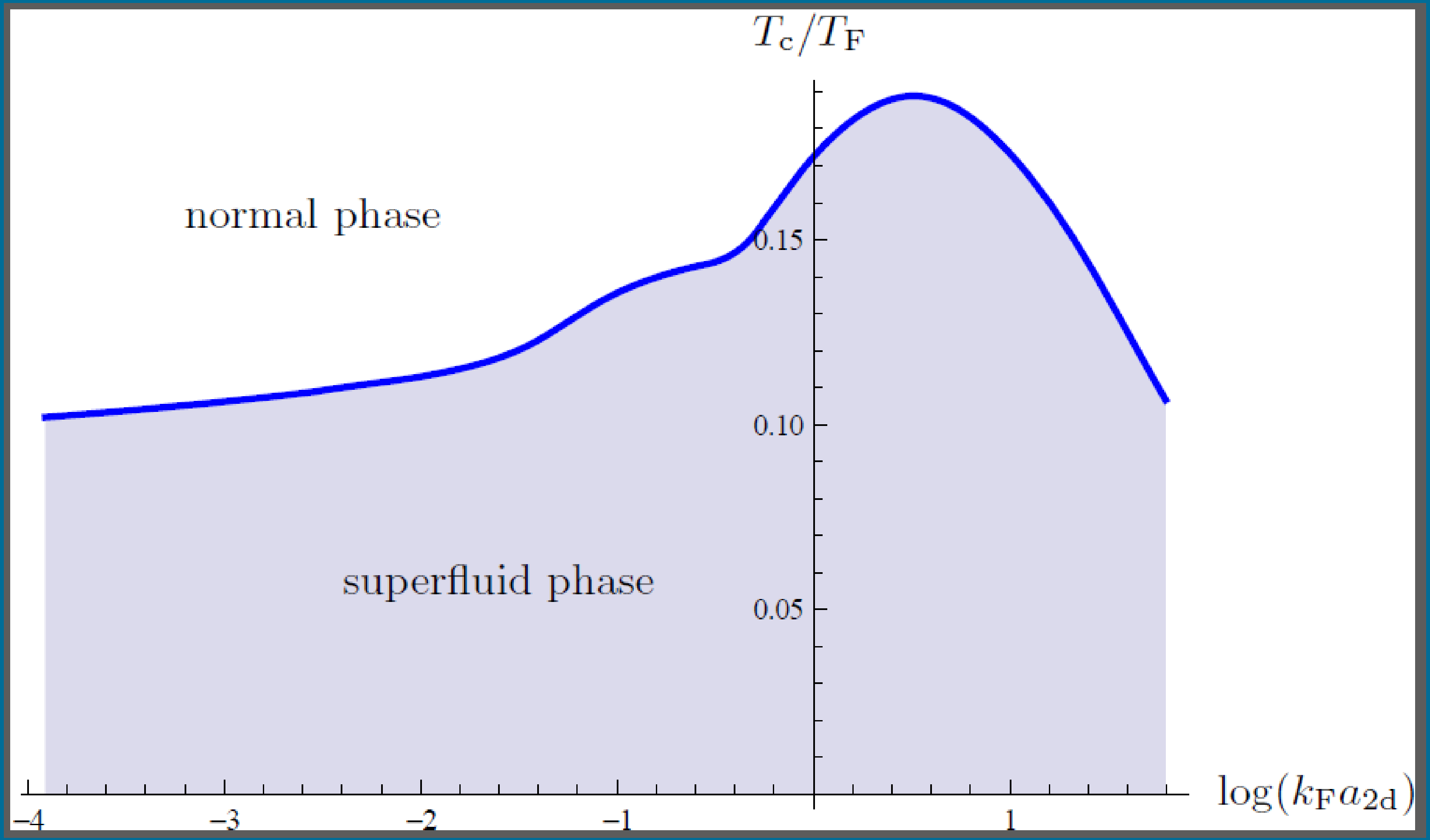
$$(\mu - \varepsilon_B/2)/\varepsilon_F = 0.874 \quad \text{for} \quad \log(k_F a_{2d}) = 0$$

Equation of state at T=0



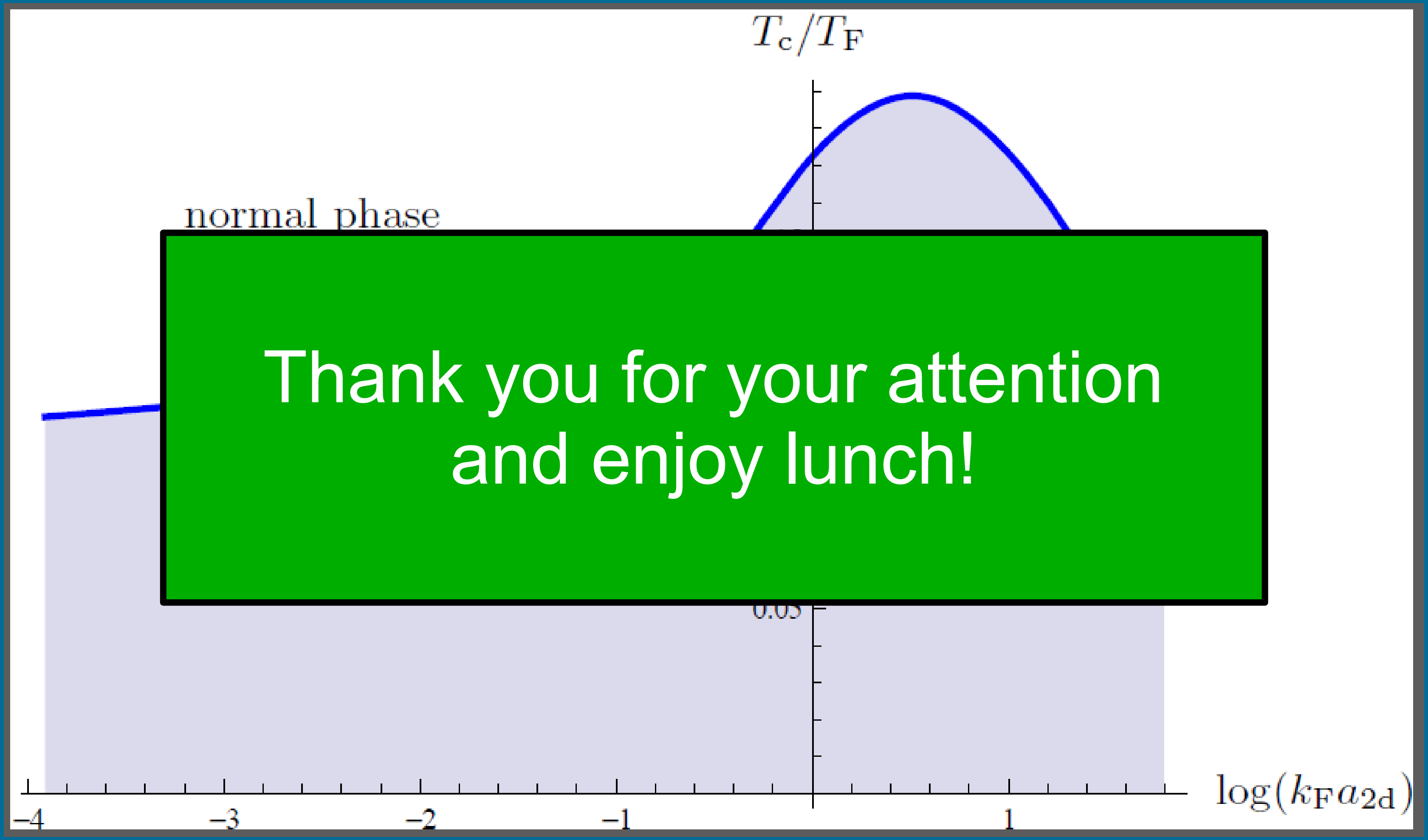
$$(\mu - \varepsilon_B/2)/\varepsilon_F = 0.874 \quad \text{for} \quad \log(k_F a_{2d}) = 0$$

Superfluid phase transition



$T_c/T_F = 0.172$ for $\log(k_F a_{2d}) = 0$

Superfluid phase transition



$T_c/T_F = 0.172$ for $\log(k_F a_{2d}) = 0$