Schwinger pair creation in time-dependent fields

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Δ - Meeting, Heidelberg, 12. 1. 2012

Based on work done with
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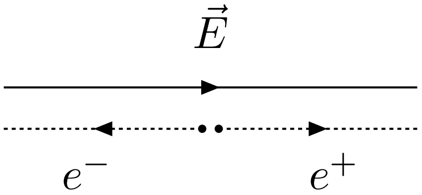
History

1931 F. Sauter: Dirac's theory of the electron predicts that an electric field of sufficient strength and extent can induce spontaneous creation of electron – positron pairs from the vacuum.
By a statistical fluctuation, a virtual pair separates out far enough to draw its rest mass energy from the field (vacuum tunneling).



Schwinger's approach

1951 F. Schwinger: For constant and not too strong fields, the total pair production probability P (probability of the decay of the vacuum) relates to the effective action $\Gamma[E]$:

$$P = 1 - \mathrm{e}^{-2\mathrm{Im}\Gamma} \approx 2\mathrm{Im}\Gamma$$

$$\begin{aligned}
\mathrm{Im}\mathcal{L}_{\mathrm{spin}}(E) &= \frac{m^4}{8\pi^3}\beta^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left[-\frac{\pi n}{\beta}\right] && \text{(Spinor QED)} \\
\mathrm{Im}\mathcal{L}_{\mathrm{scal}}(E) &= -\frac{m^4}{16\pi^3}\beta^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \exp\left[-\frac{\pi n}{\beta}\right] && \text{(Scalar QED)}
\end{aligned}$$

$(\beta = \frac{eE}{m^2})$.

$\mathrm{Im}\mathcal{L}(E)$ depends on E nonperturbatively, which is a confirmation of the tunneling picture.
The pair creation rate is exponentially small for

$$E \ll E_{\mathrm{crit}} \approx 10^{16}\mathrm{V/cm}$$

Lasers are now getting close (POLARIS, ELI, XFEL...)

Realistic laser fields are far from constant, and could have a substantially lower pair creation threshold. Many proposals have been made:

- Counterpropagating two laser beams
  (M. Ruf et al., PRL 102, 080402, 2009).
- Superimposing a plane-wave X-ray beam with a strongly focused optical laser pulse
  (G.V. Dunne et al., PRD 80:111301, 2009).
- . . . (many more).

However, the calculation of pair creation rates for generic electric fields requires approximative methods. Until recently, almost all such results were obtained using WKB (Keldysh 1965, Brezin and Itzykson 1970, Narozhnyi and Nikishov 1970, Popov 1972, Popov and Marinov 1972, . . .).

$E(t)$

Here, I will consider the simplest non-constant field: a purely time-dependent field $E(t)$, using two alternatives to WKB:

1. The worldline instanton formalism.
   I.K. Affleck, O. Alvarez, N.S. Manton, Nucl. Phys. **B 197** 509, 1982,
   G. V. Dunne and C. S., Phys. Rev. **D 72** 105004 (2005),
   G. V. Dunne, H. Gies, Q.-h. Wang and C. S., *Phys. Rev.* D **73** 065028 (2006).
2. A quantum Vlasov evolution equation.
   S. P. Kim and C. S., PRD 84, 125028 (2011),
   S. P. Kim, arXiv:1110.4684 [hep-th].

1. Worldline instantons

Feynman's worldline representation of the scalar QED effective action
R.P. Feynman, Phys. Rev. **80** (1950) 440.

$$\Gamma_{\rm scal}[A] \;=\; \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int\mathcal{D}x(\tau)\,\mathrm{e}^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4}+ieA\cdot\dot{x}\right)}$$

Here m and T are the mass and proper time of the loop scalar, and the path integral $\int\mathcal{D}x(\tau)$ is over closed trajectories in Euclidean spacetime.
Rescaling $\tau = Tu$, this becomes

$$\Gamma_{\rm scal}[A] \;=\; \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int\mathcal{D}x\,\mathrm{e}^{-\left(\frac{1}{T}\int_0^1 du\,\dot{x}^2+ie\int_0^1 du A\cdot\dot{x}\right)}$$

The T integral has a stationary point at $T_{\rm stat}=\sqrt{\int_0^1 du\,\dot{x}^2}/m$.

If we are only interested in the imaginary part of the effective action at large mass, we can use this stationary point to eliminate the T integral, giving

$$\mathrm{Im}\Gamma_{\mathrm{scal}}[A] \approx \frac{1}{m}\sqrt{\frac{2\pi}{T_{\mathrm{stat}}}}\,\mathrm{Im}\int \mathcal{D}x\ \mathrm{e}^{-\left(m\sqrt{\int du\,\dot{x}^2}+ie\int_0^1 du A\cdot\dot{x}\right)}$$

The new worldline action,

$$S \quad = \quad m\sqrt{\int_0^1 du\,\dot{x}^2} + ie\int_0^1 duA\cdot\dot{x}$$

is stationary if

$$m\frac{\ddot{x}_\mu}{\sqrt{\int_0^1 du\,\dot{x}^2}} \quad = \quad ieF_{\mu\nu}\dot{x}_\nu$$

Contracting this equation with $\dot{x}^\mu$ shows that $\dot{x}^2 = \mathrm{const.} \equiv a^2$, so that $m\ddot{x}_\mu = ieaF_{\mu\nu}\dot{x}_\nu$. Thus the extremal action trajectory $x^{\mathrm{cl}}(u)$, to be called *worldline instanton*, is simply a periodic solution of the Lorentz force equation.

The constant field case

For a constant field

$$\mathbf{E} = (0, 0, E)$$

the worldline instanton turns out to be a circle in the $x_3 - x_4$ plane, of radius m/eE and winding number n:

$$\begin{aligned} x^{\mathrm{cl}}(u) &= \frac{m}{eE}\Big(x_1, x_2, \cos(2n\pi u), \sin(2n\pi u)\Big) \\ S[x^{\mathrm{cl}}] &= n\pi\,\frac{m^2}{eE} \end{aligned}$$

Thus the instanton action for winding number n reproduces the nth exponent in Schwinger's representation of $\mathrm{Im}\mathcal{L}_{\mathrm{scal}}(E)$.

$E(t)$

For a time-dependent field $\mathbf{E}(t) = (0, 0, E(t))$ we choose the gauge

$$A_3 = A_3(x_4) \quad ; \quad A_\mu = 0 \ \text{ for } \ \mu \neq 3 \tag{1}$$

Since $F_{\mu 1} = F_{\mu 2} = 0$, the stationarity conditions imply that

$$\ddot{x}_1 = \ddot{x}_2 = 0 \quad \Rightarrow \quad \dot{x}_1 = \text{constant} \quad , \quad \dot{x}_2 = \text{constant}$$

For $x_1(u)$ and $x_2(u)$ to be periodic, we require $\dot{x}_1 = \dot{x}_2 = 0$.

Using (1) the Lorentz force equation

$$\begin{aligned} \ddot{x}_3 &= \frac{iea}{m} F_{34}\, \dot{x}_4 \\ \ddot{x}_4 &= -\frac{iea}{m} F_{34}\, \dot{x}_3 \end{aligned}$$

can be simplified to a first-order ordinary DGL,

$$\begin{aligned} \dot{x}_3 &= -\frac{iea}{m} A_3(x_4) \\ \dot{x}_4 &= a\sqrt{1 + \left(\frac{e\, A_3(x_4)}{m}\right)^2} \end{aligned}$$

At least numerically, this DGL can always be solved easily. Then

$$\mathrm{Im}\mathcal{L}(E(t)) \overset{m\,\mathrm{large}}{\approx} N\, \mathrm{e}^{-S[x^{\mathrm{cl}}]}$$

The prefactor N involves a fluctuation determinant that is much harder to calculate, but there is a systematic procedure (based on the Gelfand-Yaglom theorem).
All this applies to Spinor QED unchanged (but for a factor of 2).

Sauter-like field

Example: $E(t) = E\,\mathrm{sech}^2(\omega\, t)$.

$$
\begin{aligned}
x_3(u) &= -\frac{1}{\omega}\,\frac{1}{\sqrt{1+\gamma^2}}\,\mathrm{arcsinh}\left[\gamma\ \cos\left(2n\pi u\right)\right] \\
x_4(u) &= \frac{1}{\omega}\ \arcsin\left[\frac{\gamma}{\sqrt{1+\gamma^2}}\ \sin\left(2n\pi\ u\right)\right]
\end{aligned}
$$

$$
\gamma \equiv \frac{m\omega}{eE}
$$

$$
a = \frac{\gamma}{\omega\,\sqrt{1+\gamma^2}}\,2\pi n \quad , \quad n \in \mathbf{Z}^+
$$

Stationary action:

$$
S_0 = n\,\frac{m^2\pi}{eE}\left(\frac{2}{1+\sqrt{1+\gamma^2}}\right)
$$

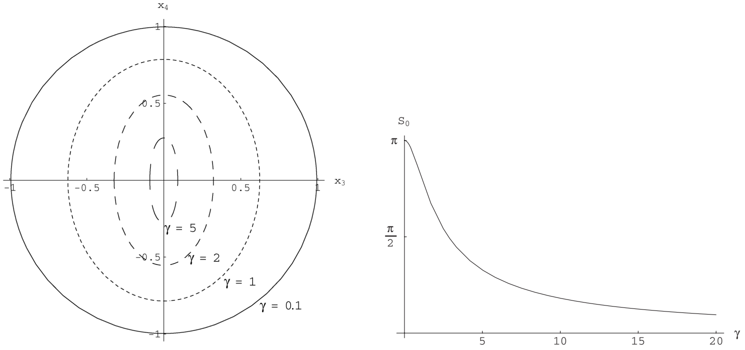


Figure: Worldline instanton and its action (in units of nm^2/eE)

General rule: time dependence lowers the worldline action and thus increases the pair production rate.

2. The Quantum Vlasov Equation

For a purely time-dependent field the spatial momentum $\mathbf{k}$ is a good quantum number, so that one has a mode decomposition (for a scalar particle at one loop)

$$2\mathrm{Im}\mathcal{L}(t) = \sum_{\mathbf{k}} \ln\big(1 + \mathcal{N}_{\mathbf{k}}(t)\big)$$

The $\mathcal{N}_{\mathbf{k}}(t)$ are densities of created pairs of momentum $\mathbf{k}$. Using the in-out formalism and a standard Bogoliubov transformation, one can show that they obey a Quantum Vlasov Equation

Quantum Vlasov Equation

$$\frac{d}{dt}(1+2\mathcal{N}(t)) = \Omega^{(-)}(t)\int_{t_0}^{t} dt'\Big[\Omega^{(-)}(t')(1+2\mathcal{N}(t')) \times \cos\Big(\int_{t'}^{t} dt''\Omega^{(+)}(t'')\Big)\Big]$$

(t_0 is the initital time, usually $-\infty$).

$$\Omega_k^{(\pm)}(t) = \frac{\omega_k^2(t) \pm \omega_k^2(t_0)}{\omega_k(t_0)}$$

$$\omega_k^2(t) = (k_{\parallel} - qA_{\parallel}(t))^2 + \mathbf{k}_{\perp}^2 + m^2$$

Open question

This is a nasty integro-differential equation, but quite suitable to numerical evaluation.

Open question: Strictly speaking the interpretation of $\mathcal{N}_{\mathbf{k}}(t)$ as the actual density of created pairs is valid only asymptotically for $t \to \infty$. Numerical evaluation in some cases shows a $\mathcal{N}_{\mathbf{k}}(t)$ at intermediate times that is much larger than the asymptotic value. What would happen if we could make a measurement at intermediate times? Is this an artefact?

Solitonic gauge fields

Try to find closed-form solutions to the Vlasov equation: Inspection shows, that its general solution can be parameterized by a function $f(t)$ fulfilling the integral equation (we now omit the index $\mathbf{k}$).

$$\dot{f}(t) = \frac{\Omega^{(-)}(t)}{\omega_0} - 2\int_{t_0}^{t} dt' f(t')(\omega^2(t) + \omega^2(t')) \tag{2}$$

with the initial condition $f(t_0) = \dot{f}(t_0) = 0$.
Knowing $f(t)$, $\mathcal{N}(t)$ can be recovered as

$$1 + 2\mathcal{N} = 1 + \omega_0 \int_{t_0}^{t} dt' f(t')\Omega^{(-)}(t')$$

Ansatz:

$$f(t) = \frac{(\omega^2)^{\cdot}(t)}{8\omega_0^4}, \quad F(t) = \frac{\omega^2(t) - \omega_0^2}{8\omega_0^4}$$

Defining $r(t) := \omega^2(t)/\omega_0^2$ and then $u(x,t) := -r(x - 10t)$, one can show that for (2) to be fulfilled u must solve the Korteweg-de Vries equation,

$$u_{xxx} - 6uu_x + u_t = 0$$

Thus we can use certain solutions of the KdV equation to calculate pair creation rates for the corresponding electric fields.

Solitonic example

Example: choose the following soliton-type solution of the KdV equation

$$u(x,t) = -1 - \frac{2}{\cosh^2(x - 10t)}$$

which corresponds to

$$r(t) = \frac{\omega^2(t)}{\omega_0^2} = 1 + \frac{2}{\cosh^2(\omega_0 t)}, \quad F(t) = \frac{1}{4\omega_0^2 \cosh^2(\omega_0 t)}$$

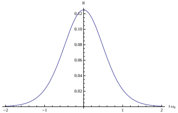
The gauge potential is

$$qA(t) = k_{\parallel} - \sqrt{k_{\parallel}^2 + \frac{2\omega_0^2}{\cosh^2(\omega_0 t)}}.$$

Pair non-creation

The exact pair creation rate:

$$\mathcal{N}(t) = \frac{1}{8\cosh^4(\omega_0 t)}$$



No pair creation for $t \to \infty$. Would we find something at finite time?

Thank You!