Gauge cooling in complex Langevin for QCD with heavy quarks

Dénes Sexty

Heidelberg University

Delta meeting 2013, Heidelberg

Collaborators: Gert Aarts (Swansea), Erhard Seiler (MPI München), Ion Stamatescu (Heidelberg)

Seiler, Sexty, Stamatescu arXiv:1211.3709

Non-zero chemical potential

Euclidean gauge theory with fermions: $Z = \int dU \exp(-S_E) \, det(M)$

For nonzero chemical potential, the fermion determinant is complex

Sign problem → Naïve Monte-Carlo breaks down

Methods going around the problem work for $\mu = \mu_B / 3 < T$

Multi parameter reweighting

Fodor, Katz '02

Analytic continuation of results obtained at imaginary μ

Lombardo '00; de Forcrand, Philipsen '02; D'Elia and Sanfilippo '09

Taylor expansion in $(\mu/T)^2$

de Forcrand et al. '99; Hart, Laine, Philipsen '00; Gavai and Gupta '08;
de Forcrand, Philipsen '08

Stochastic quantisation

Aarts and Stamatescu '08
Bose Gas, Spin model, etc. Aarts '08, Aarts, James '10 Aarts, James '11
QCD with heavy quarks: Seiler, Sexty, Stamatescu '12

Stochastic Quantization

Parisi, Wu (1981)

Weighted, normalized average: $$\langle O\rangle=\frac{\int e^{-S(x)}O(x)dx}{\int e^{-S(x)}dx}$$

Stochastic process for x: $$\frac{dx}{d\tau}=-\frac{\partial S}{\partial x}+\eta(\tau)$$

Gaussian noise $\langle\eta(\tau)\rangle=0 \quad \langle\eta(\tau)\eta(\tau')\rangle=2\delta(\tau-\tau')$

Averages are calculated along the trajectories:

$$\langle O\rangle=\frac{1}{T}\int_0^T O(x(\tau))d\tau$$

Fokker-Planck equation for the probability distribution of P(x):

$$\frac{\partial P}{\partial\tau}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}+P\frac{\partial S}{\partial x}\right)=-H_{FP}P$$

Real action $\rightarrow$ positive eigenvalues

for real action the Langevin method is convergent

Stochastic quantisation on the group manifold

Updating must respect the group structure:

$$\langle \eta_{ia} \rangle = 0$$

$$U'_i = \exp\left(i\lambda_a\left(\epsilon i D_{i,a} S[U] + \sqrt{\epsilon}\,\eta_{i,a}\right)\right) U_i$$

$$\langle \eta_{ia}\eta_{jb} \rangle = 2\,\delta_{ij}\,\delta_{ab}$$

Left derivative: $D_a f(U) = \left(\frac{\partial}{\partial\alpha} f\left(e^{i\lambda_a\alpha} U\right) \right)\Big|_{\alpha=0}$

λ_a Gellmann matrices

complexifed link variables

SU(N) ⟶ SL(N,C) $det(U) = 1, \quad U^{+} \neq U^{-1}$

compact ⟶ non-compact

Distance from SU(N)

$$\sum_{ij} \left| \left(U U^{+} - 1 \right)_{ij} \right|^2$$

Unitarity Norms:

$$Tr\left(U U^{+}\right) \geq N$$

$$Tr\left(U U^{+}\right) + Tr\left(U^{-1}\left(U^{-1}\right)^{+}\right) \geq 2N$$

For SU(2): $\left(I m \;\; Tr\, U\right)^2$

Gaugefixing in SU(2) one plaquette model

Berges, Sexty '08

SU(2) one plaquette model: $S = i\beta \, Tr\, U \qquad U \in SU(2)$

Langevin updating $U' = \exp\left(i \lambda_a (\epsilon \, i D_a S[U] + \sqrt{\epsilon}\, \eta_a)\right) U$

parametrized with Pauli matrices

$$U = \exp\left(i \frac{\varphi \hat{n} \hat{\sigma}}{2}\right) = \left(\cos\frac{\varphi}{2}\right) \mathbf{1} + i \left(\sin\frac{\varphi}{2}\right) \hat{n} \hat{\sigma}$$

$$U = a \mathbf{1} + i b_i \sigma_i \qquad a^2 + b_i b_i = 1$$

exact averages by numerical integration:

$$\langle f(U) \rangle = \frac{1}{Z} \int_0^{2\pi} d\phi \int d\Omega \sin^2\frac{\phi}{2} e^{i\beta \cos\frac{\phi}{2}} f(U(\phi, \hat{n}))$$

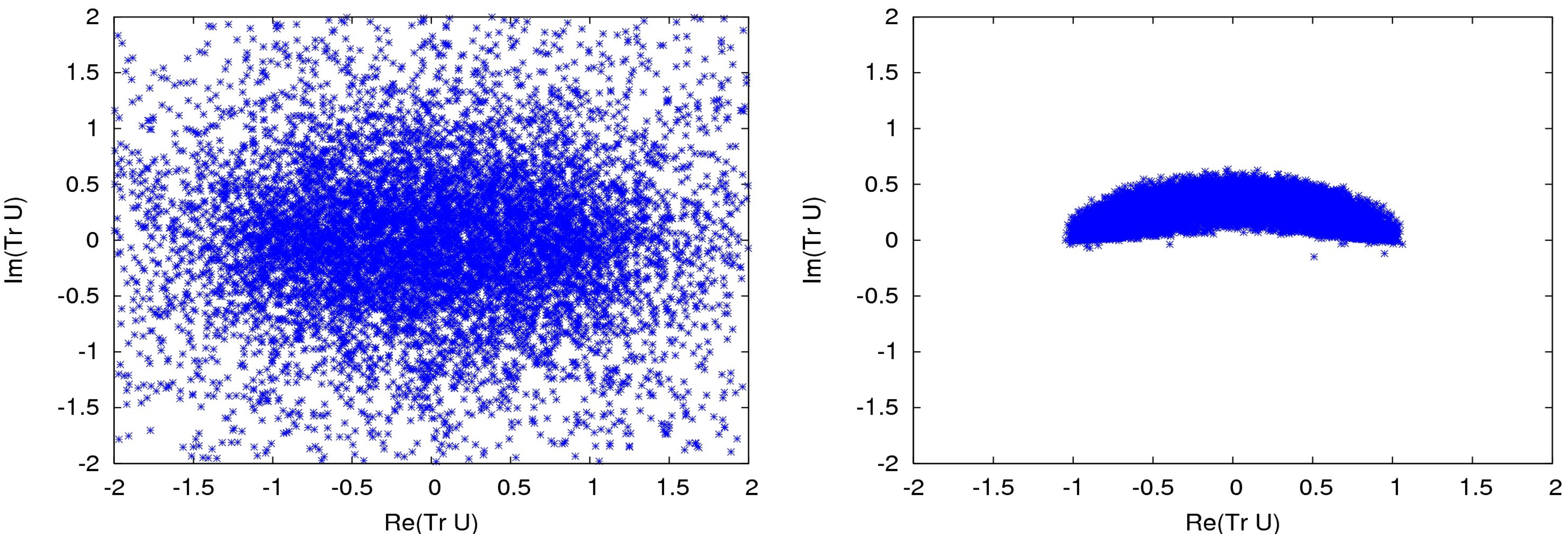
“gauge” symmetry: $U \rightarrow W U W^{-1}$ complexified theory: $U, W \in SL(2, \mathbb{C})$

After each Langevin timestep: fix gauge condition

$$U = a \mathbf{1} + i \sqrt{1 - a^2} \sigma_3 \qquad b_i = (0, 0, \sqrt{1 - a^2})$$

SU(2) one-plaquette model

Distributions of Tr(U) on the complex plane



Without gaugefixing

With gaugefixing

Exact result from integration: $\langle Tr\,U \rangle = i\,0.2611$

From simulation:

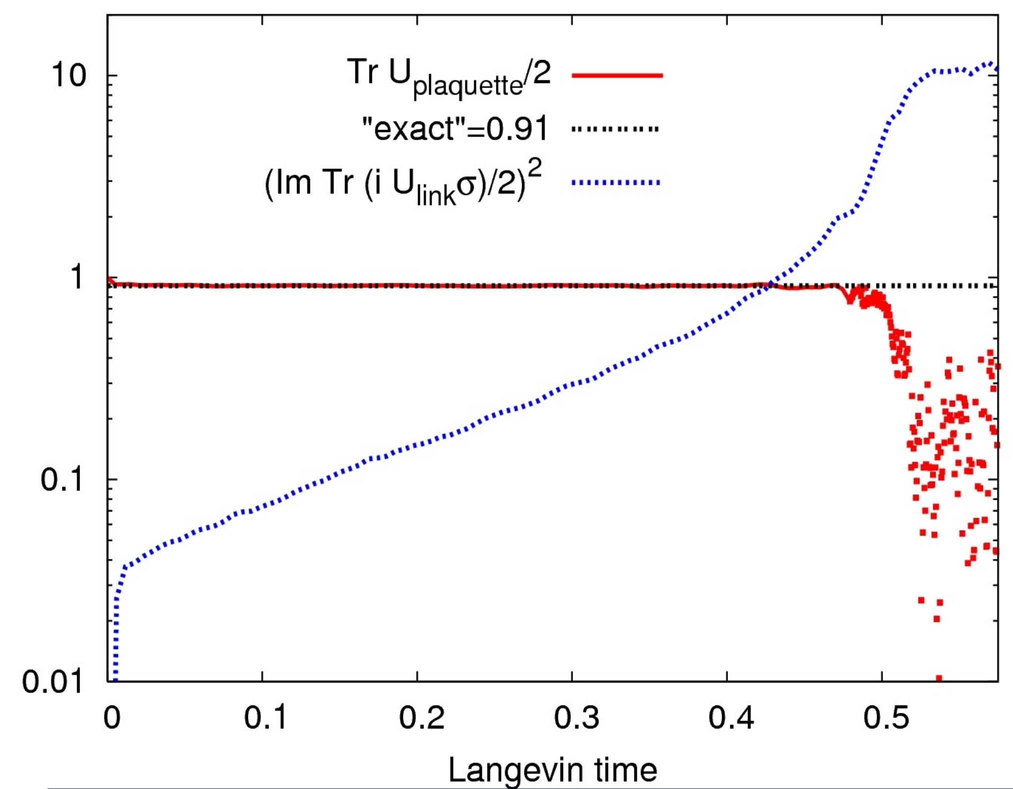
$(-0.02 \pm 0.02) + i\,(-0.01 \pm 0.02)$

$(-0.004 \pm 0.006) + i\,(0.260 \pm 0.001)$

With gauge fixing, all averages are correctly reproduced

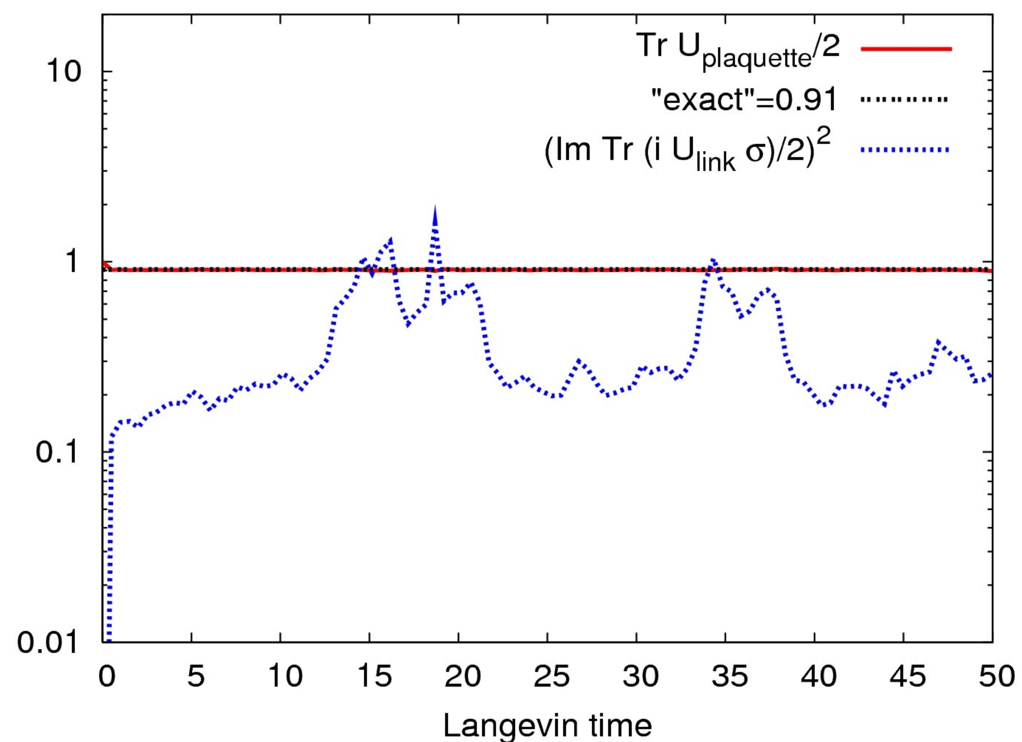
SU(2) field theory on real time contour

$(Im\,Tr\,U)^2$ measures size of distribution



Without gauge fixing

non-physical averages



Gauge fixing on maximal axial tree

Correct result stabilizes

However:

Lattice coupling $g = 0.5$

(Scaling region $g \geq 1$)

Gauge cooling

complexified distribution with slow decay → convergence wrong results

Minimize unitarity norm: $\sum_i Tr\left(U_i U_i^+\right)$

Using gauge transformations in SL(N,C)

$$U_\mu(x) \rightarrow V(x) U_\mu(x) V^{-1}(x+a_\mu) \qquad V(x) = \exp\left(i \lambda_a v_a(x)\right)$$

$v_a(x)$ is imaginary (for real $v_a(x)$, unitarity norm is not changed)

Gradient of the unitarity norm gives steepest descent

$$G_a(x) = 2\,Tr\left[\lambda_a\left(U_\mu(x) U_\mu^+(x) - U_\mu^+(x-a_\mu) U_\mu(x-a_\mu)\right)\right]$$

Gauge transformation at x changes 2d link variables

$$U_\mu(x) \rightarrow \exp(-\alpha \epsilon \lambda_a G_a(x)) U_\mu(x)$$

$$U_\mu(x-a_\mu) \rightarrow U_\mu(x-a_\mu) \exp(\alpha \epsilon \lambda_a G_a(x))$$

Dynamical steps are interspersed with several gauge cooling steps

The strength of the cooling is determined by
- cooling steps
- gauge cooling parameter α

During cooling, unitarity norm decays to a minimum with a power law behaviour

Possible extension: adaptive cooling with $\alpha = f(|G|)$

Polyakov chain model

exactly solvable toy model with gauge symmetry

$$S = -\beta_1 \, Tr \, U_1 \ldots U_N - \beta_2 \, Tr \, U_N^{-1} \ldots U_1^{-1} \qquad U_i \in SU(3)$$

$$\beta_1 = \beta + \kappa e^{\mu} \qquad \beta_2 = \beta^* + \kappa e^{-\mu}$$

Complex action for $\kappa, \mu > 0$

Observables: $Tr \, P^k$ *with* $P = U_1 \ldots U_N$

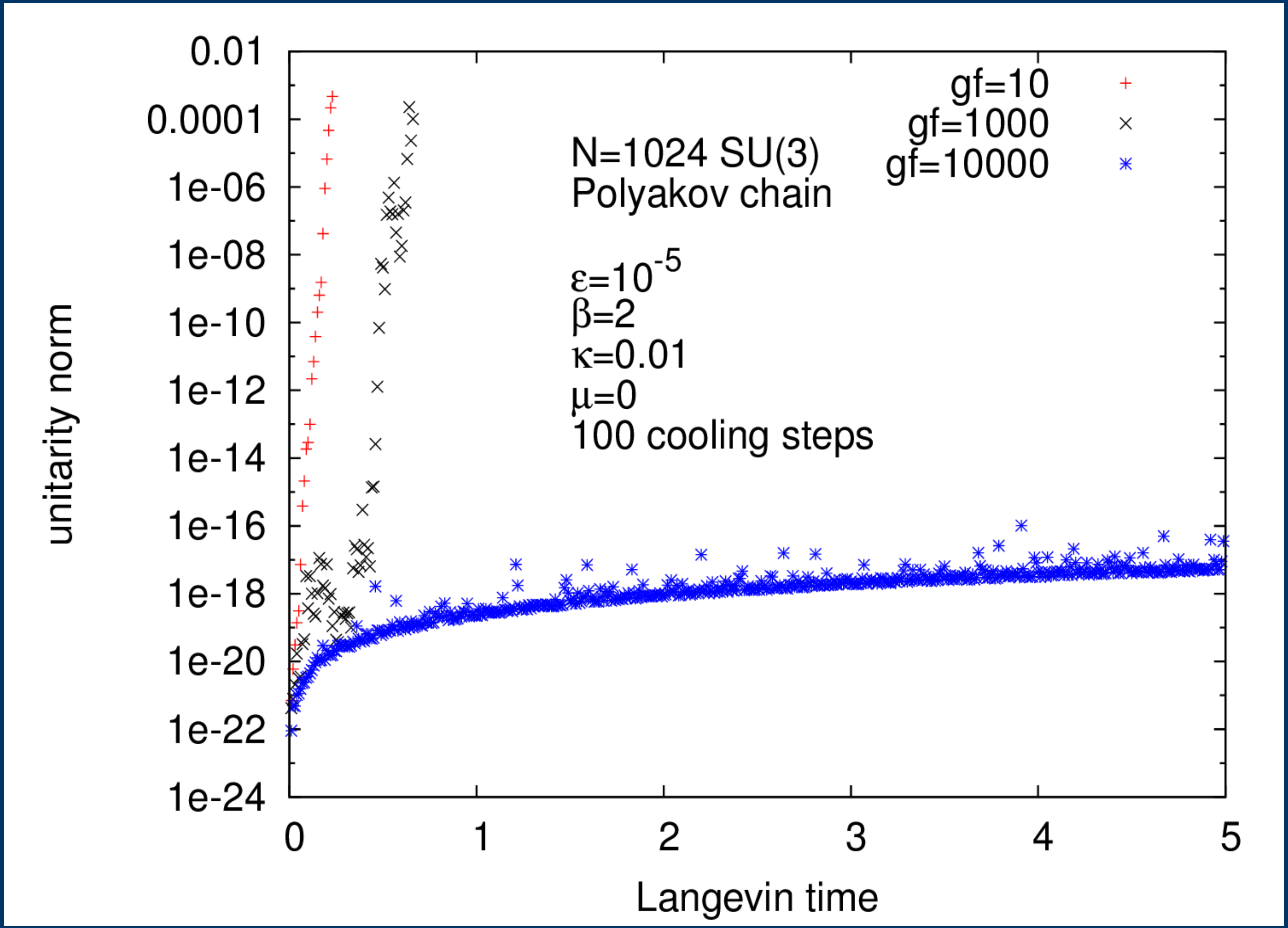
Averages independent of N

Calculated with numerical integration at $N = 1$

Gauge symmetry

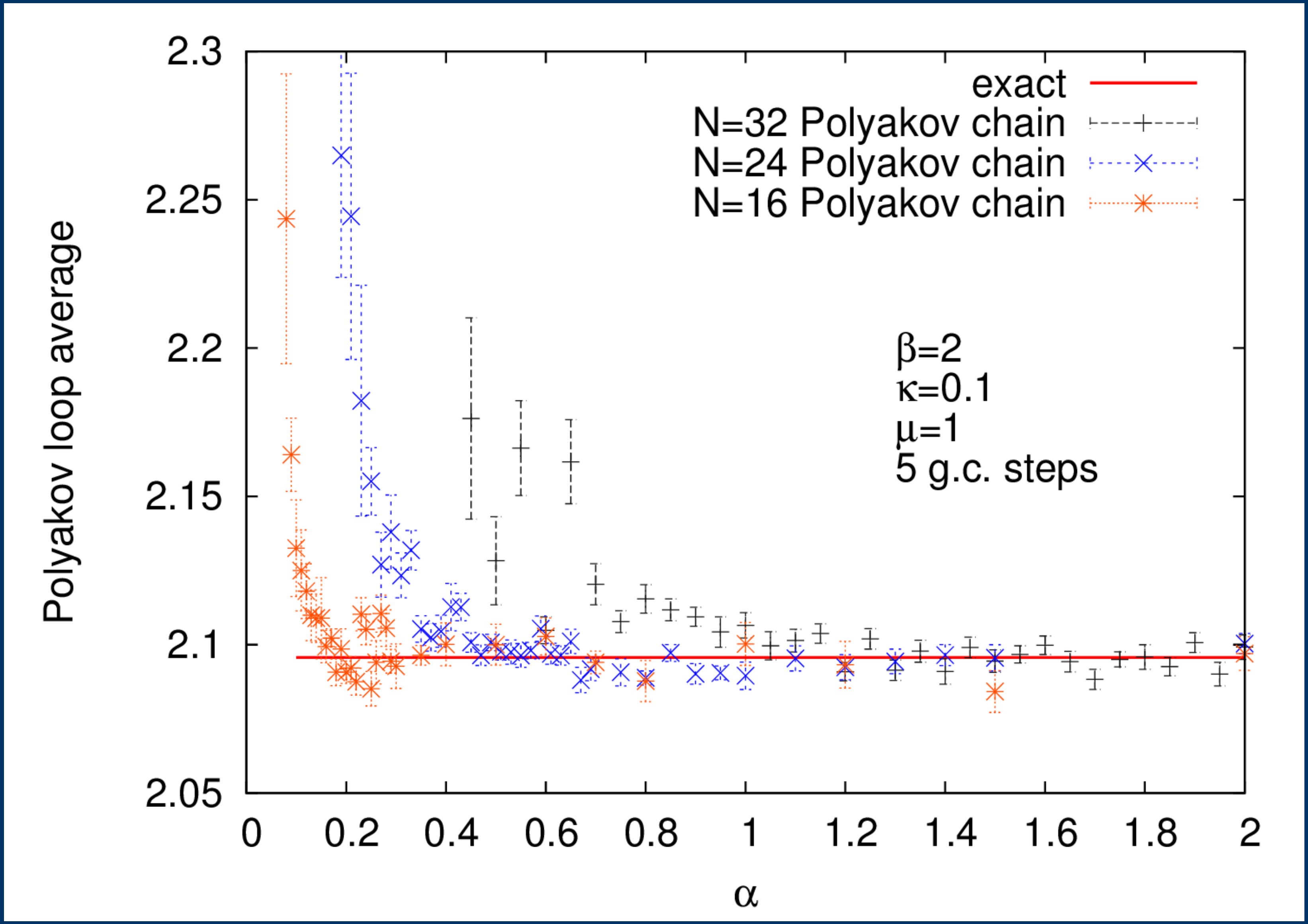
$$U_i \rightarrow V_i U_i V_{i+1}^{-1}$$

Check for "real" simulation $(\text{e.g. } \mu=0)$



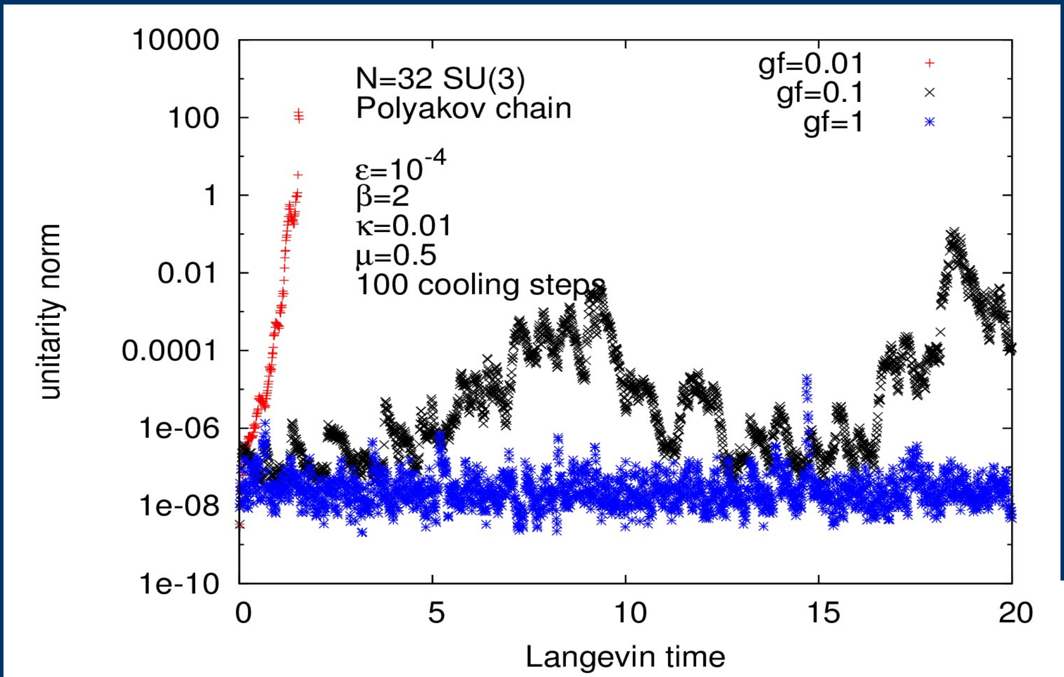
Without cooling, the SU(3) manifold is unstable

Turning on the chemical potential...



With enough cooling, exact results are recovered

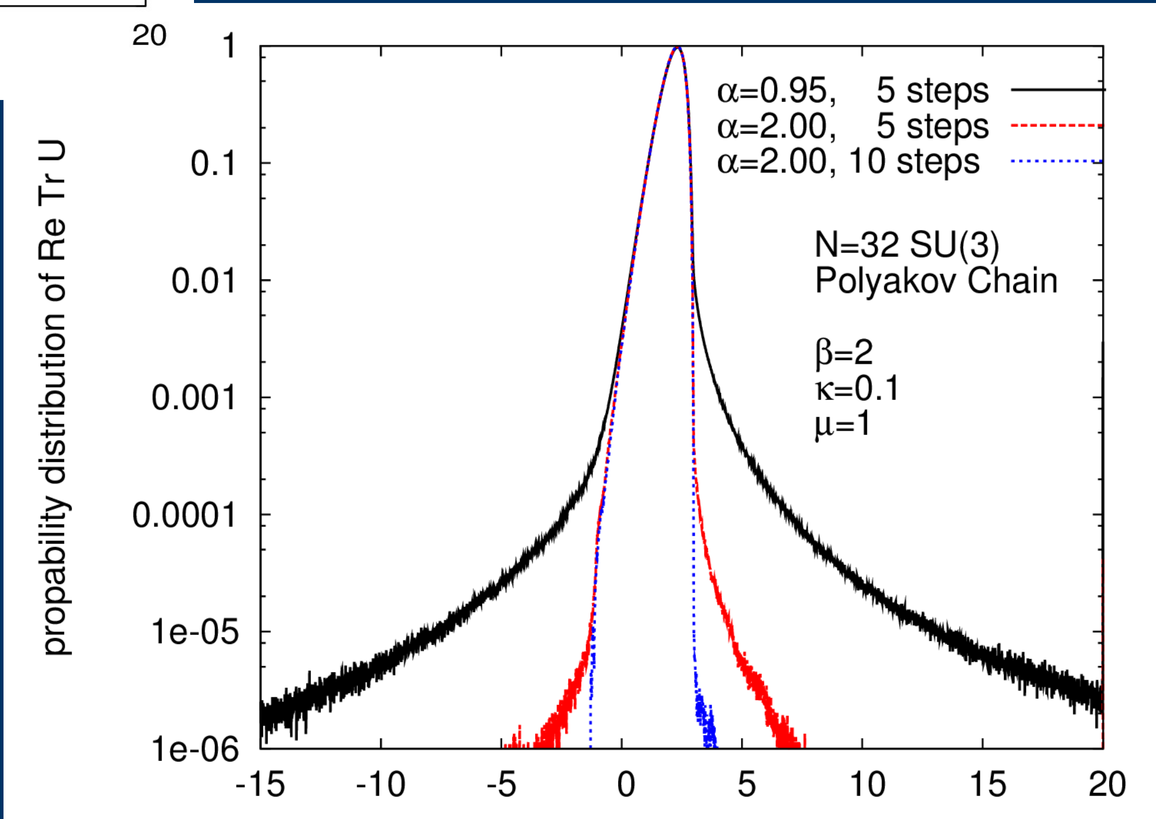
Longer chaing requires more cooling

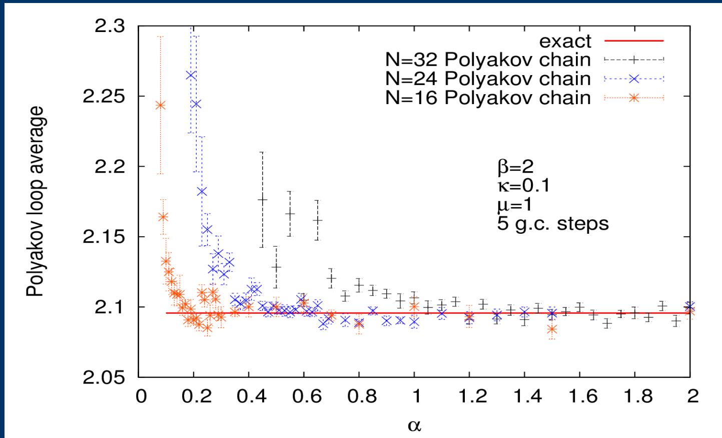


Smaller cooling → excursions

"Skirt" develops

small skirt gives correct result





Heavy Quark QCD

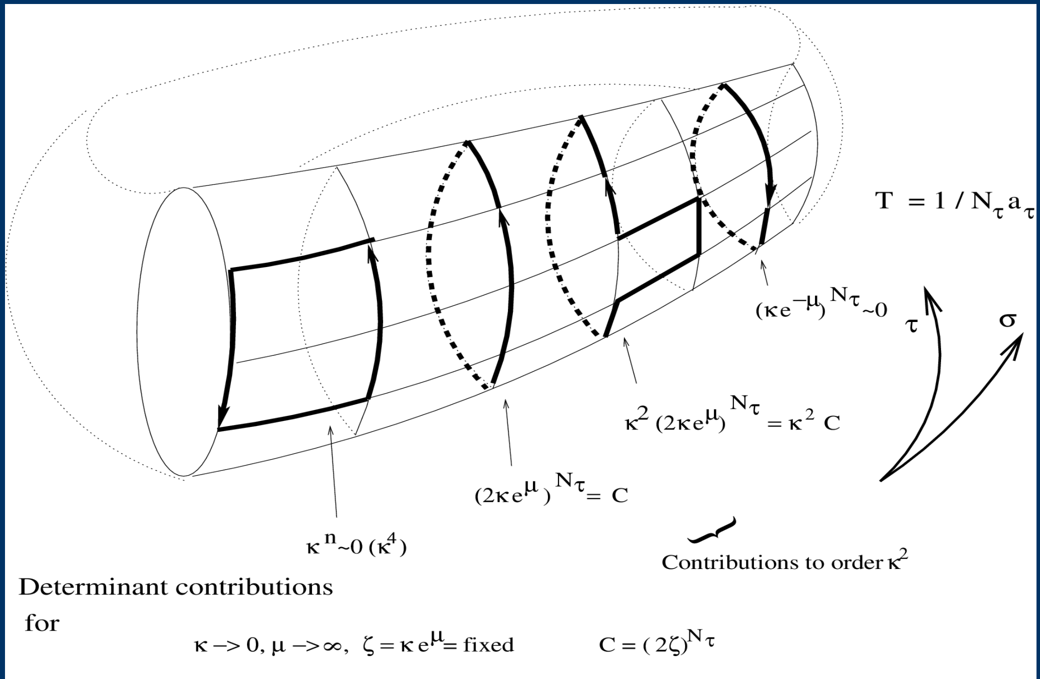
Hopping parameter expansion of the fermion determinant
Spatial hoppings are dropped

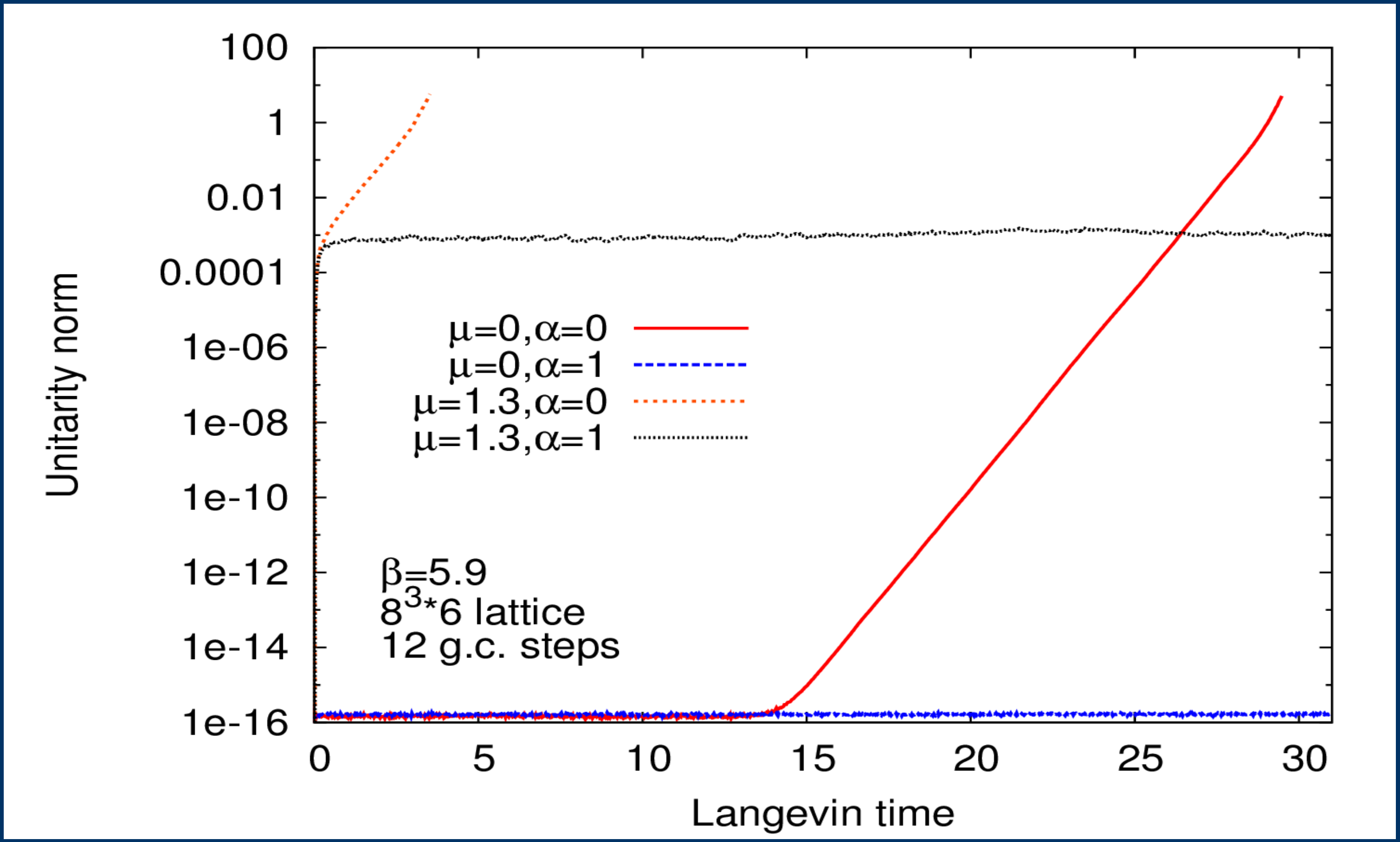
$$\mathrm{Det}\, M(\mu) = \prod_x \mathrm{Det}\left(1 + C P_x\right)^2 \mathrm{Det}\left(1 + C' P_x^{-1}\right)^2$$

$$P_x = \prod_\tau U_0(x + \tau a_0) \qquad C = [2\kappa \exp(\mu)]^{N_\tau} \qquad C' = [2\kappa \exp(-\mu)]^{N_\tau}$$

$$S = S_W[U_\mu] + \ln \mathrm{Det}\, M(\mu)$$

Studied with reweighting De Pietri, Feo, Seiler, Stamatescu '07





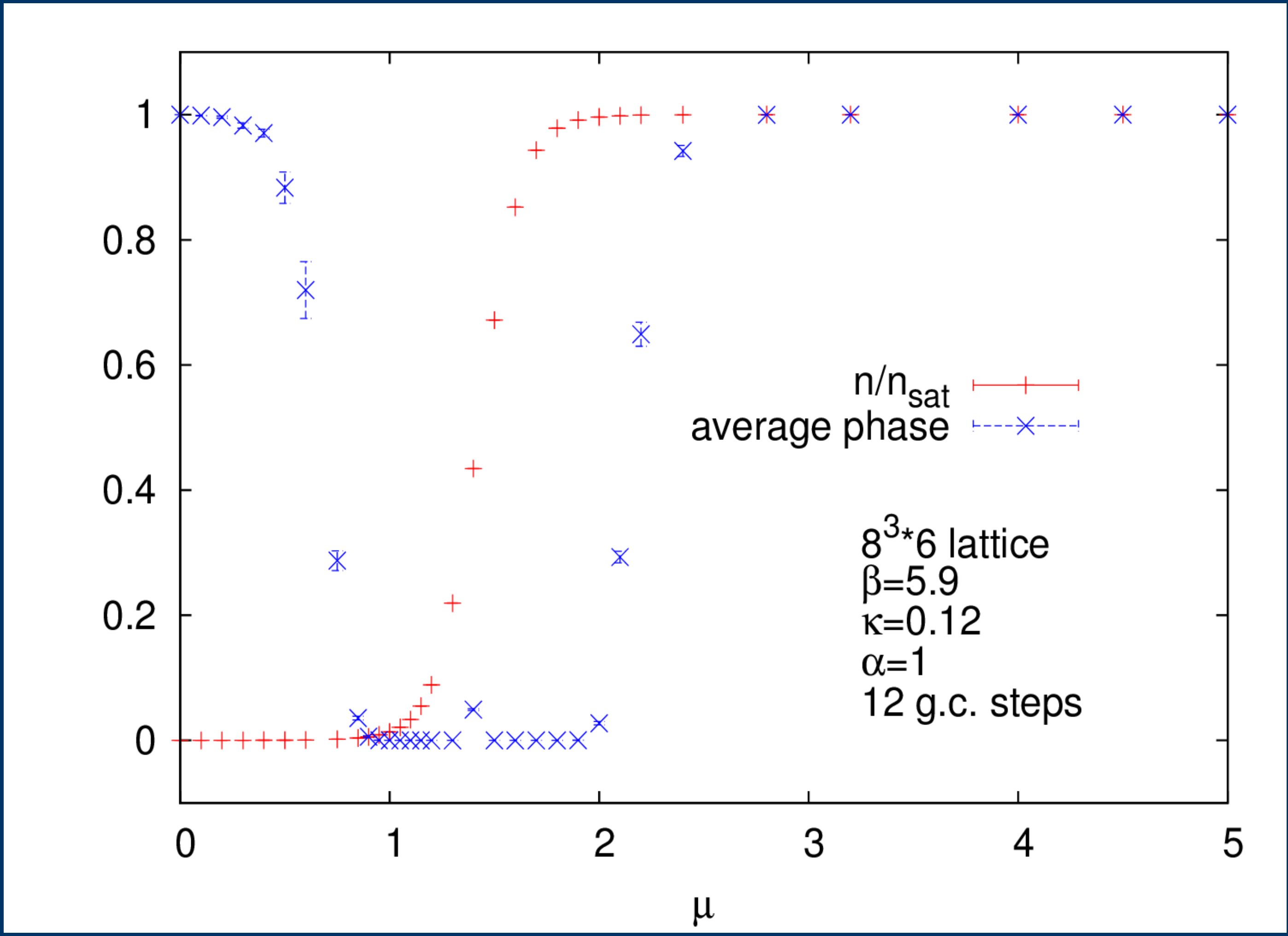
Gauge cooling stabilizes the distribution
SU(3) manifold instable even at $\mu = 0$

Fermion density:

$$n = \frac{1}{N_\tau} \frac{\partial \ln Z}{\partial \mu}$$

average phase:

$$\langle \exp(2 i \varphi) \rangle = \left\langle \frac{\det M(\mu)}{\det M(-\mu)} \right\rangle$$



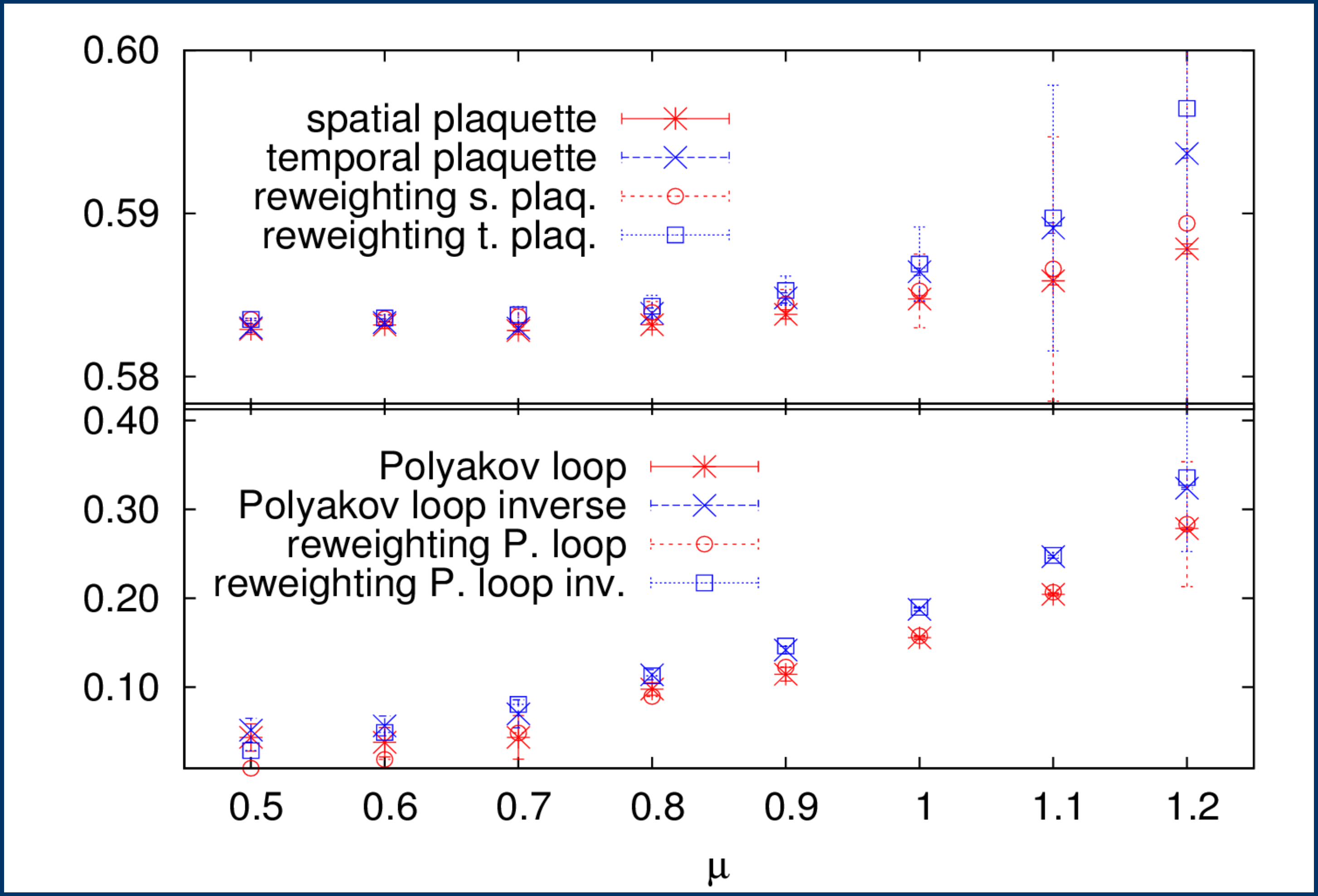
$$\det(1 + C P)^2 = 1 + C^3 + C \,\mathrm{Tr}\, P + C^2 \,\mathrm{Tr}\, P^{-1}$$

Sign problem is absent at small or large μ

Reweigthing is impossible at $1 \leq \mu \leq 2$

CLE works all the way to saturation

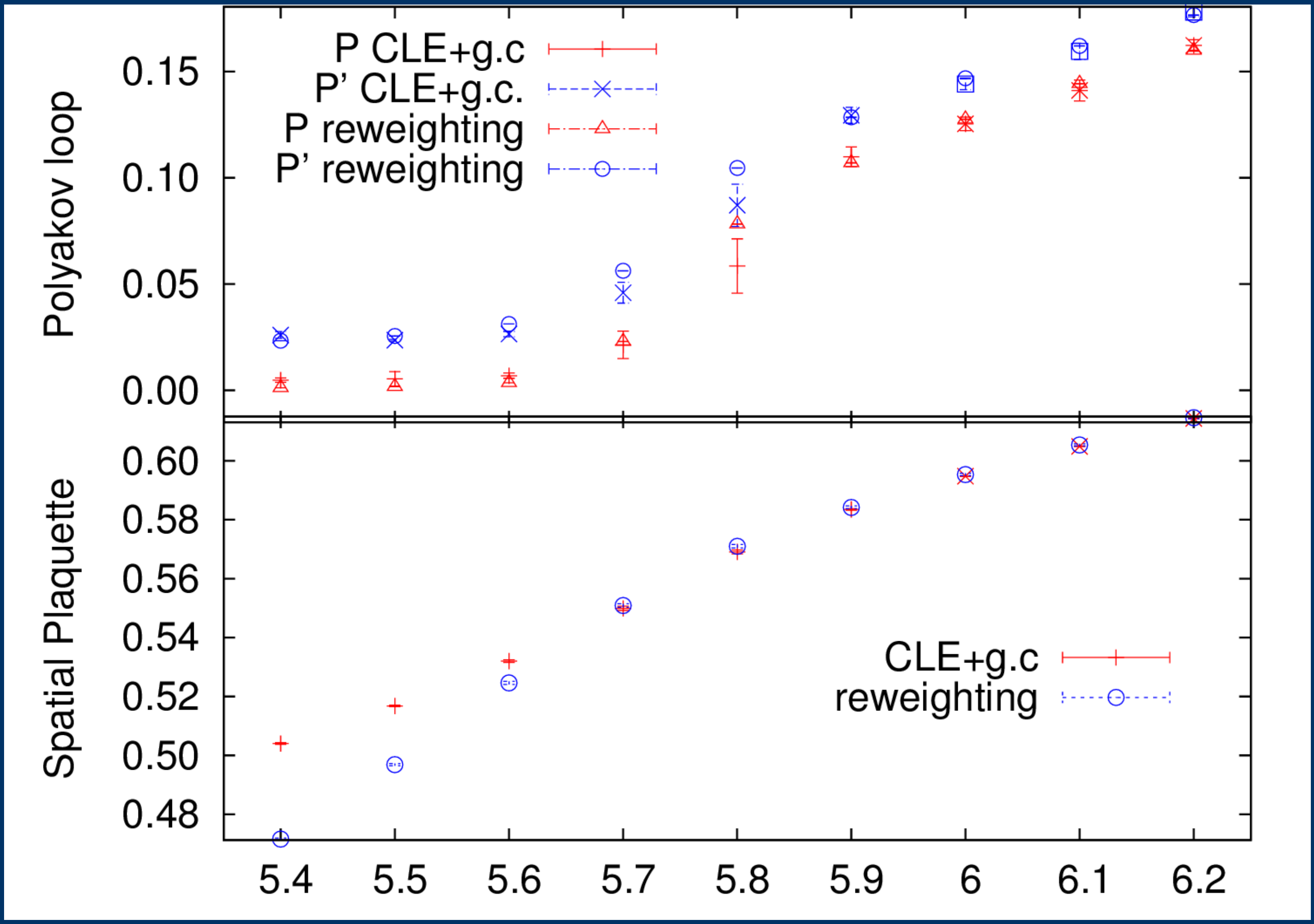
Comparison to reweighting



6^4 lattice, $\beta = 5.9$, $\alpha = 1$, 12 gaugecooling steps

Reweighting errors start to blow up at $\mu \approx 1.1$

Comparison to reweighting



6^4 lattice, $\mu = 0.85$, $\alpha = 1$, adaptive step size

Discrepancy of plaquettes at $\beta \leq 5.6$
a skirted distribution develops

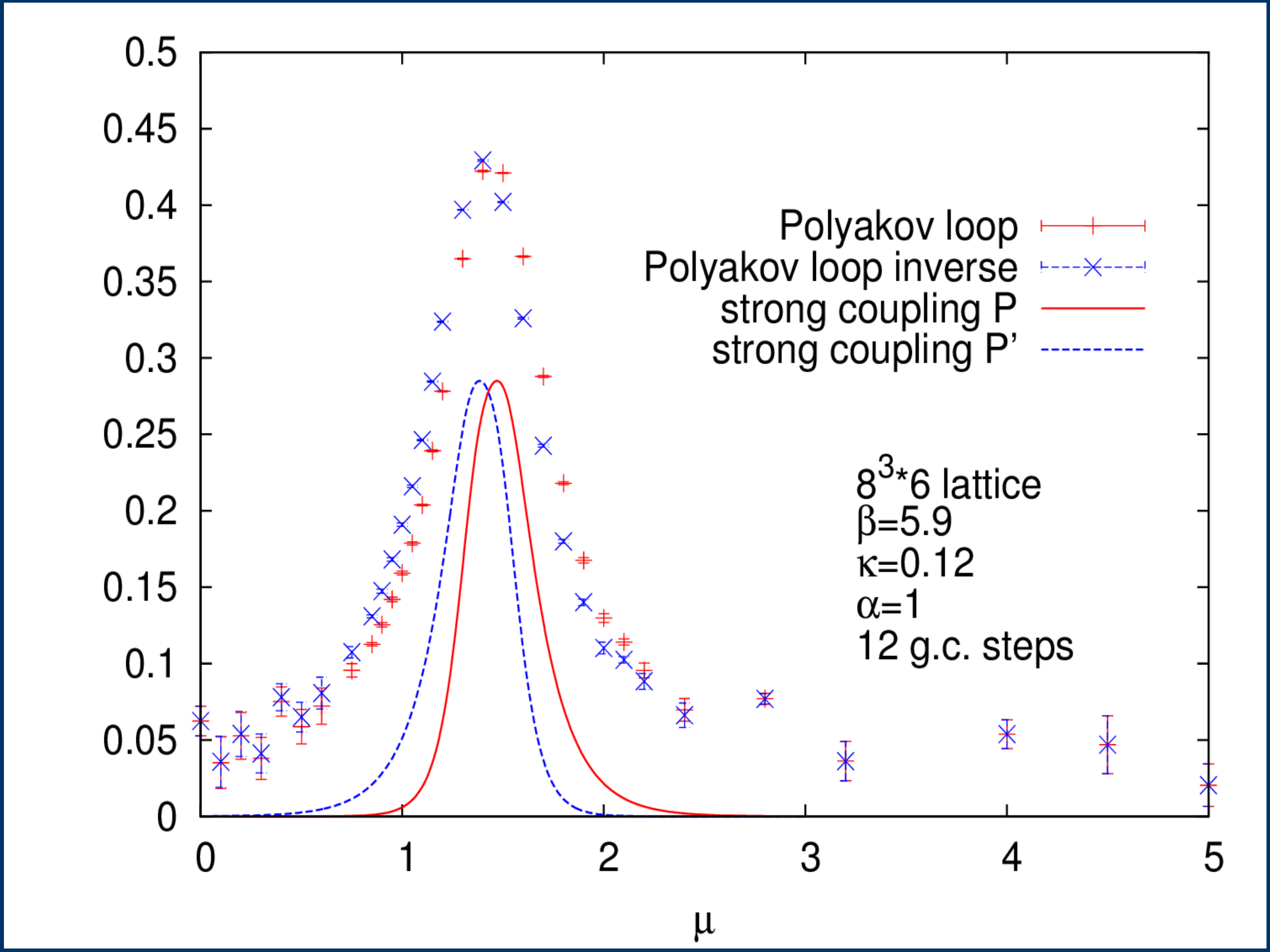
Nonzero value when: colorless bound states formed with P or P'

1 quark:
 meson with P'

2 quark:
 Barion with P

↓

P' has a peak before P



Large chemical potential: all quark states are filled
 No colorless state can be formed

↓

P and P' decays again

Conclusions

New algorithm for Complex Langevin of gauge theories:
- Gauge cooling

Tested on exactly solvable toy model
- Polyakov chain

Results for QCD with heavy quarks with chemical potential
- No sign or overlap problem
- CLE works all the way into the saturation region
- Validated with reweighting
- Phase transition line at nonzero chemical potential is visible