Functional renormalization for ultracold quantum gases

Stefan Floerchinger (Heidelberg)

S. Floerchinger and C. Wetterich, Phys. Rev. A **77**, 053603 (2008);
S. Floerchinger and C. Wetterich, arXiv:0805.2571.

ERG 2008
Heidelberg, 01. - 06. Jun. 2008

Microscopic action for nonrelativistic Bose gas

$$S[\phi] = \int_x \left\{ \phi^* \left(\partial_\tau - \Delta - \mu \right) \phi + \frac{1}{2} \lambda (\phi^* \phi)^2 \right\}$$

Microscopic action for nonrelativistic Bose gas

$$S[\phi] = \int_x \left\{ \phi^* \left(\partial_\tau - \Delta - \mu \right) \phi + \frac{1}{2} \lambda (\phi^* \phi)^2 \right\}$$

- natural units: $\hbar = k_B = 2M = 1$

Microscopic action for nonrelativistic Bose gas

$$S[\phi] = \int_x \left\{ \phi^* \left(\partial_\tau - \Delta - \mu \right) \phi + \frac{1}{2} \lambda (\phi^* \phi)^2 \right\}$$

- natural units: $\hbar = k_B = 2M = 1$
- μ is chemical potential

Microscopic action for nonrelativistic Bose gas

$$S[\phi] = \int_x \left\{ \phi^* \left(\partial_\tau - \Delta - \mu \right) \phi + \frac{1}{2} \lambda (\phi^* \phi)^2 \right\}$$

- natural units: $\hbar = k_B = 2M = 1$
- μ is chemical potential
- homogeneous space and Matsubara formalism

$$\int_x = \int_0^{1/T} d\tau \int d^d x$$

Microscopic action for nonrelativistic Bose gas

$$S[\phi] = \int_x \left\{ \phi^* \left(\partial_\tau - \Delta - \mu \right) \phi + \frac{1}{2} \lambda (\phi^* \phi)^2 \right\}$$

- natural units: $\hbar = k_B = 2M = 1$
- μ is chemical potential
- homogeneous space and Matsubara formalism

$$\int_x = \int_0^{1/T} d\tau \int d^d x$$

- consider here $d = 3$ and $d = 2$

Symmetries of the microscopic action

Symmetries of the microscopic action

- rotation and translation

Symmetries of the microscopic action

- rotation and translation
- global U(1) symmetry

$$\phi \to e^{i\alpha}\phi, \quad \phi^* \to e^{-i\alpha}\phi^*$$

broken spontaneously in the superfluid phase

Symmetries of the microscopic action

- rotation and translation
- global U(1) symmetry

$$\phi \to e^{i\alpha}\phi, \quad \phi^* \to e^{-i\alpha}\phi^*$$

  broken spontaneously in the superfluid phase
- Galilean invariance with analytic continuation to real time

$$\begin{aligned} \phi(t,\vec{x}) &\to e^{-i(\vec{q}^2 t - \vec{q}\vec{x})}\phi(t,\vec{x}-2\vec{q}t) \\ \phi^*(t,\vec{x}) &\to e^{i(\vec{q}^2 t - \vec{q}\vec{x})}\phi^*(t,\vec{x}-2\vec{q}t) \end{aligned}$$

  also broken spontaneously in the superfluid phase.

Symmetries of the microscopic action

- rotation and translation
- global U(1) symmetry

$$\phi \to e^{i\alpha}\phi, \quad \phi^* \to e^{-i\alpha}\phi^*$$

  broken spontaneously in the superfluid phase
- Galilean invariance with analytic continuation to real time

$$\begin{aligned} \phi(t,\vec{x}) &\to e^{-i(\vec{q}^2 t - \vec{q}\vec{x})}\phi(t,\vec{x}-2\vec{q}t) \\ \phi^*(t,\vec{x}) &\to e^{i(\vec{q}^2 t - \vec{q}\vec{x})}\phi^*(t,\vec{x}-2\vec{q}t) \end{aligned}$$

  also broken spontaneously in the superfluid phase.
- with $\mu = \mu(t)$ semilocal gauge invariance

$$\begin{aligned} \phi(t,\vec{x}) &\to e^{i\alpha(t)}\phi(t,\vec{x}), \\ \phi^*(t,\vec{x}) &\to e^{-i\alpha(t)}\phi^*(t,\vec{x}) \\ \mu &\to \mu + \partial_t\alpha. \end{aligned}$$

Grand canonical partition function

$$Z = e^{-\beta\Phi_G} = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int D\chi\, e^{-S[\chi]}$$

.

Grand canonical partition function

$$Z = e^{-\beta\Phi_G} = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int D\chi\, e^{-S[\chi]}$$

.

- from effective action $\Gamma[\phi]$

$$\beta\Phi_G = \Gamma[\phi_{\mathrm{eq}}] \qquad \text{with} \qquad \left.\frac{\delta\Gamma[\phi]}{\delta\phi}\right|_{\phi=\phi_{\mathrm{eq}}} = 0$$

Grand canonical partition function

$$Z = e^{-\beta\Phi_G} = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int D\chi\, e^{-S[\chi]}$$

.

- from effective action $\Gamma[\phi]$

$$\beta\Phi_G = \Gamma[\phi_{\mathrm{eq}}] \qquad \text{with} \qquad \left.\frac{\delta\Gamma[\phi]}{\delta\phi}\right|_{\phi=\phi_{\mathrm{eq}}} = 0$$

- use flow equation to obtain effective action (Wetterich (1993))

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}(\Gamma_k^{(2)} + R_k)^{-1}\partial_k R_k$$

Grand canonical partition function

$$Z = e^{-\beta\Phi_G} = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int D\chi\, e^{-S[\chi]}$$

.

- from effective action $\Gamma[\phi]$

$$\beta\Phi_G = \Gamma[\phi_{\mathrm{eq}}] \qquad \text{with} \qquad \left.\frac{\delta\Gamma[\phi]}{\delta\phi}\right|_{\phi=\phi_{\mathrm{eq}}} = 0$$

- use flow equation to obtain effective action (Wetterich (1993))

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}(\Gamma_k^{(2)} + R_k)^{-1}\partial_k R_k$$

- finite size system described by average action

$$\Gamma_{k_{\mathrm{ph}}} \qquad \text{instead of} \qquad \Gamma = \Gamma_{k=0}$$

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

- motivated by more systematic derivative expansion and analysis of symmetry constraints

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

- motivated by more systematic derivative expansion and analysis of symmetry constraints
- terms linear and quadratic in Matsubara frequency

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

- motivated by more systematic derivative expansion and analysis of symmetry constraints
- terms linear and quadratic in Matsubara frequency
- implicit wavefunction renormalization $\phi^* \Delta \phi = \bar{\phi}^* \bar{A} \Delta \bar{\phi}$

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

- motivated by more systematic derivative expansion and analysis of symmetry constraints
- terms linear and quadratic in Matsubara frequency
- implicit wavefunction renormalization $\phi^* \Delta \phi = \bar{\phi}^* \bar{A} \Delta \bar{\phi}$
- propagator depends on $(\mu - \mu_0)$

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

- motivated by more systematic derivative expansion and analysis of symmetry constraints
- terms linear and quadratic in Matsubara frequency
- implicit wavefunction renormalization $\phi^* \Delta \phi = \bar{\phi}^* \bar{A} \Delta \bar{\phi}$
- propagator depends on $(\mu - \mu_0)$
- similar truncations were studied at $T = 0$ by:

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

- motivated by more systematic derivative expansion and analysis of symmetry constraints
- terms linear and quadratic in Matsubara frequency
- implicit wavefunction renormalization $\phi^* \Delta \phi = \bar{\phi}^* \bar{A} \Delta \bar{\phi}$
- propagator depends on $(\mu - \mu_0)$
- similar truncations were studied at $T = 0$ by:
  - C. Castellani, C. Di Castro, F. Pistolesi, G. C. Strinati, Phys. Rev. Lett. **78**, 1612 (1997); Phys. Rev. B **69**, 024513 (2004).

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

- motivated by more systematic derivative expansion and analysis of symmetry constraints
- terms linear and quadratic in Matsubara frequency
- implicit wavefunction renormalization $\phi^* \Delta \phi = \bar{\phi}^* \bar{A} \Delta \bar{\phi}$
- propagator depends on $(\mu - \mu_0)$
- similar truncations were studied at $T = 0$ by:
  - C. Castellani, C. Di Castro, F. Pistolesi, G. C. Strinati, Phys. Rev. Lett. **78**, 1612 (1997); Phys. Rev. B **69**, 024513 (2004).
  - C. Wetterich, Phys. Rev. B **77**, 064504 (2008).

Truncation of average action

$$\Gamma_k = \int_x \Big\{ \phi^* \left(S\partial_\tau - V\partial_\tau^2 - \Delta \right) \phi + 2V(\mu - \mu_0)\, \phi^* \left(\partial_\tau - \Delta \right) \phi + U(\rho, \mu) \Big\}$$

- motivated by more systematic derivative expansion and analysis of symmetry constraints
- terms linear and quadratic in Matsubara frequency
- implicit wavefunction renormalization $\phi^* \Delta \phi = \bar{\phi}^* \bar{A} \Delta \bar{\phi}$
- propagator depends on $(\mu - \mu_0)$
- similar truncations were studied at $T = 0$ by:
  - C. Castellani, C. Di Castro, F. Pistolesi, G. C. Strinati, Phys. Rev. Lett. **78**, 1612 (1997); Phys. Rev. B **69**, 024513 (2004).
  - C. Wetterich, Phys. Rev. B **77**, 064504 (2008).
  - N. Dupuis, K. Sengupta, Europhys. Lett. **80**, 50007 (2007).

Effective Potential, Cutoff

$$
\begin{aligned}
U(\rho,\mu) \quad = \quad & U(\rho_0,\mu_0) - n_k(\mu-\mu_0) \\
& + \left(m^2 + \alpha(\mu-\mu_0)\right)(\rho-\rho_0) \\
& + \frac{1}{2}\left(\lambda + \beta(\mu-\mu_0)\right)(\rho-\rho_0)^2 .
\end{aligned}
$$

Effective Potential, Cutoff

$$
\begin{aligned}
U(\rho,\mu) \quad = \quad & U(\rho_0,\mu_0) - n_k(\mu-\mu_0) \\
& + \left(m^2 + \alpha(\mu-\mu_0)\right)(\rho-\rho_0) \\
& + \frac{1}{2}\left(\lambda + \beta(\mu-\mu_0)\right)(\rho-\rho_0)^2 .
\end{aligned}
$$

- minimum ρ_0 is k-dependent

$$(\partial_\rho U)(\rho_0,\mu_0) = 0 \quad \text{for all}\, k$$

Effective Potential, Cutoff

$$
\begin{aligned}
U(\rho,\mu) \quad = \quad & U(\rho_0,\mu_0) - n_k(\mu-\mu_0) \\
& + \left(m^2 + \alpha(\mu-\mu_0)\right)(\rho-\rho_0) \\
& + \frac{1}{2}\left(\lambda + \beta(\mu-\mu_0)\right)(\rho-\rho_0)^2 .
\end{aligned}
$$

- minimum ρ_0 is k-dependent

$$(\partial_\rho U)(\rho_0,\mu_0) = 0 \quad \text{for all}\, k$$

- k-independent μ_0 determines particle number

$$-(\partial_\mu U)(\rho_0,\mu_0) = n \quad \text{at } k = k_{\text{ph}}$$

Effective Potential, Cutoff

$$
\begin{aligned}
U(\rho,\mu) \quad = \quad & U(\rho_0,\mu_0) - n_k(\mu-\mu_0) \\
& + \left(m^2 + \alpha(\mu-\mu_0)\right)(\rho-\rho_0) \\
& + \frac{1}{2}\left(\lambda + \beta(\mu-\mu_0)\right)(\rho-\rho_0)^2 .
\end{aligned}
$$

- minimum ρ_0 is k-dependent

$$(\partial_\rho U)(\rho_0,\mu_0) = 0 \quad \text{for all}\, k$$

- k-independent μ_0 determines particle number

$$-(\partial_\mu U)(\rho_0,\mu_0) = n \quad \text{at } k = k_{\text{ph}}$$

- use cutoff (Litim (2001))

$$R_k = \bar{A}(k^2 - \vec{p}^2 - m^2)\theta(k^2 - \vec{p}^2 - m^2)$$

Effective Potential, Cutoff

$$
\begin{aligned}
U(\rho,\mu) \quad = \quad & U(\rho_0,\mu_0) - n_k(\mu-\mu_0) \\
& + \left(m^2 + \alpha(\mu-\mu_0)\right)(\rho-\rho_0) \\
& + \frac{1}{2}\left(\lambda + \beta(\mu-\mu_0)\right)(\rho-\rho_0)^2 .
\end{aligned}
$$

- minimum ρ_0 is k-dependent

$$(\partial_\rho U)(\rho_0,\mu_0) = 0 \quad \text{for all}\, k$$

- k-independent μ_0 determines particle number

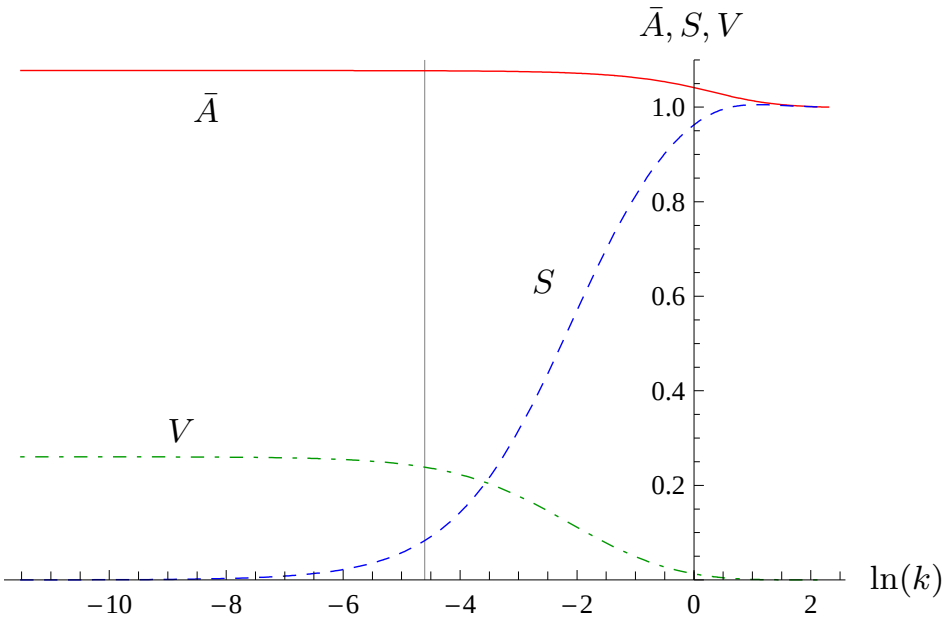
$$-(\partial_\mu U)(\rho_0,\mu_0) = n \quad \text{at } k = k_{\text{ph}}$$

- use cutoff (Litim (2001))

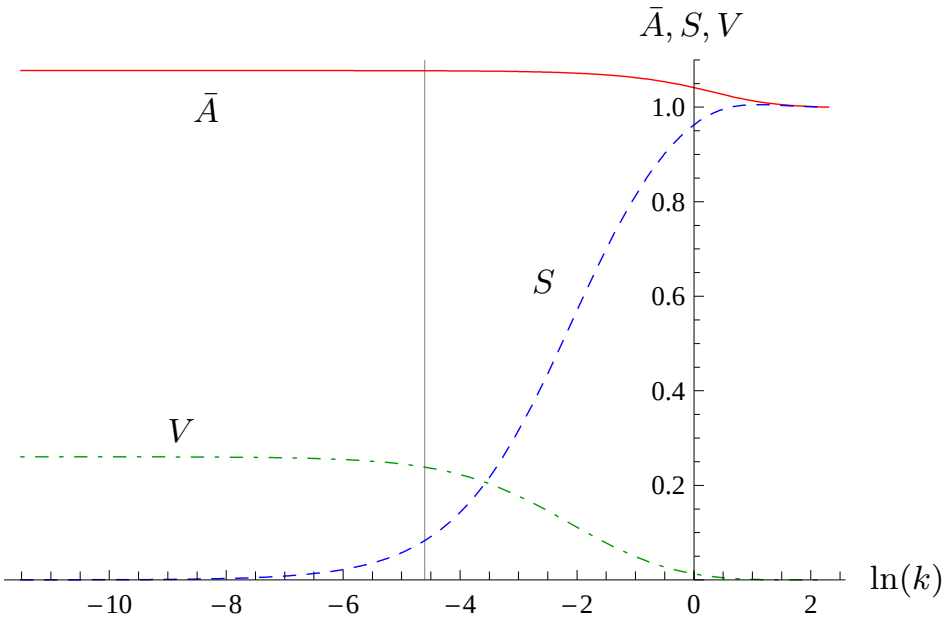
$$R_k = \bar{A}(k^2 - \vec{p}^2 - m^2)\theta(k^2 - \vec{p}^2 - m^2)$$

- all momentum integrations and Matsubara sums are performed analytically

Flow of kinetic coefficients



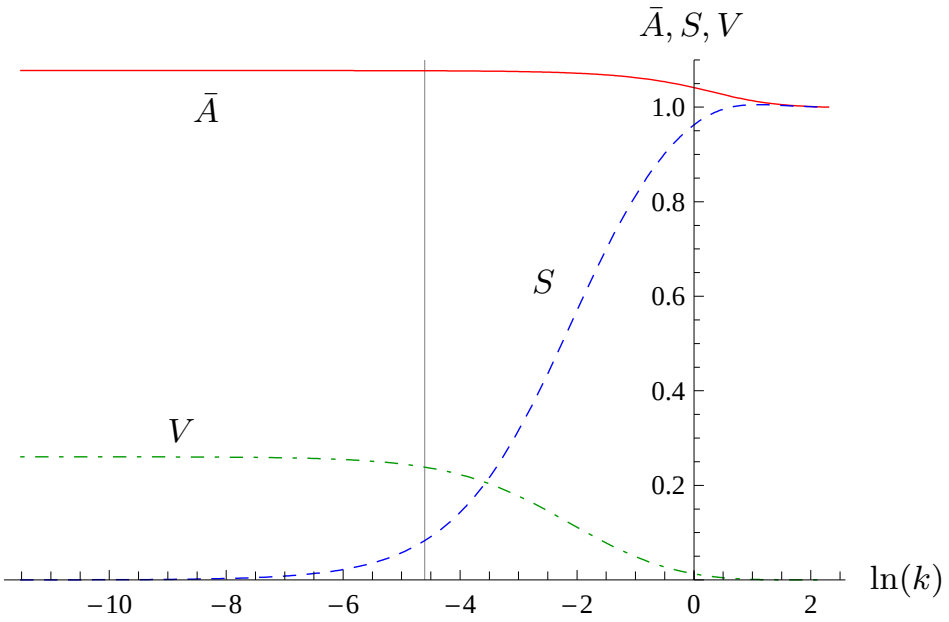
Flow of kinetic coefficients



- wavefunction renormalization $\bar{A}$ conected to condensate depletion

$$\bar{A} = \frac{\rho_0}{\bar{\rho}_0} = \frac{n_S}{n_C} \quad \left(= \frac{n}{n_C} \quad \text{at} \quad T = 0\right)$$

Flow of kinetic coefficients

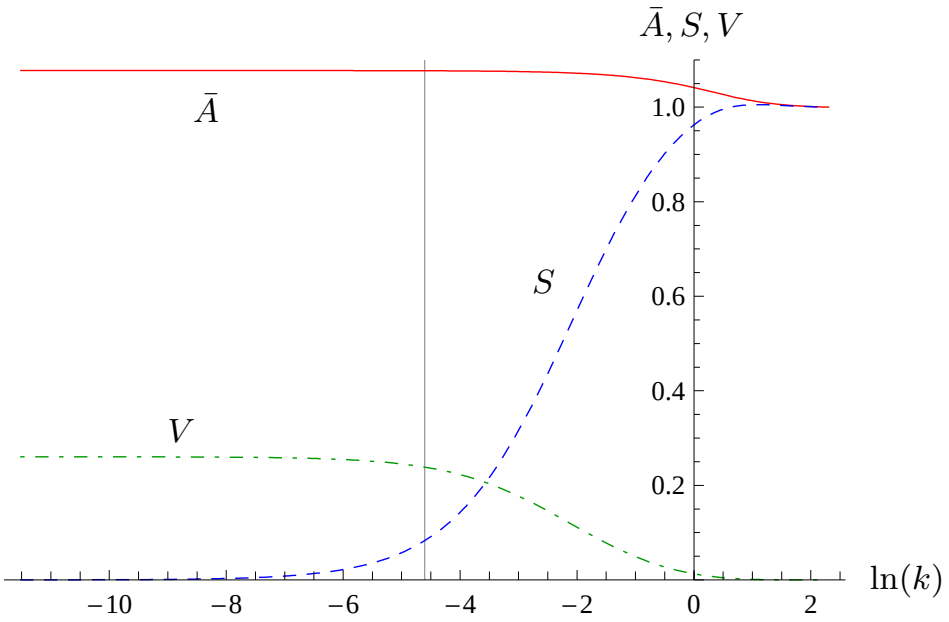


- wavefunction renormalization $\bar{A}$ conected to condensate depletion

$$\bar{A} = \frac{\rho_0}{\bar{\rho}_0} = \frac{n_S}{n_C} \quad \left(= \frac{n}{n_C} \quad \text{at} \quad T = 0\right)$$

- linear frequency term S goes to zero
  ($d = 3$ logarithmically, $d = 2$ linearly)

Flow of kinetic coefficients

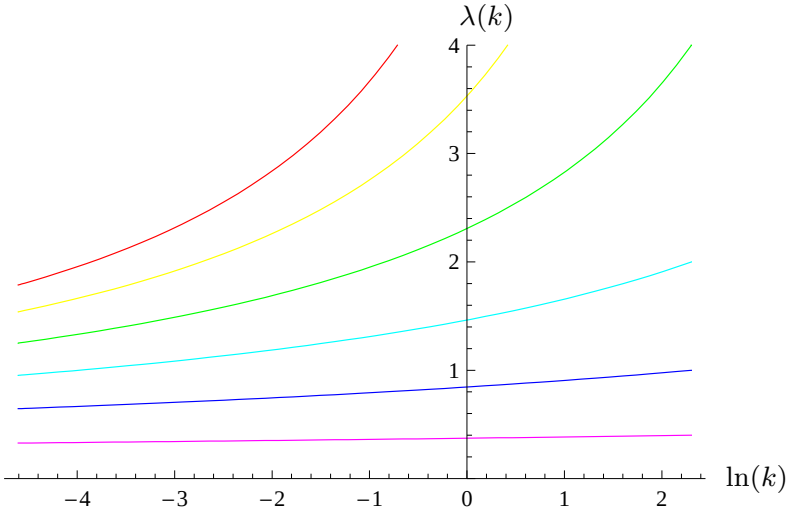


- wavefunction renormalization $\bar{A}$ conected to condensate depletion

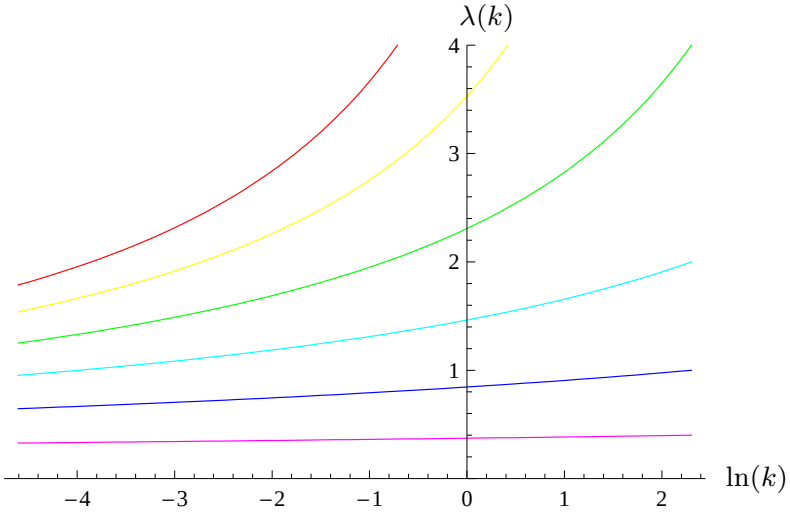
$$\bar{A} = \frac{\rho_0}{\bar{\rho}_0} = \frac{n_S}{n_C} \quad \left(= \frac{n}{n_C} \quad \text{at} \quad T = 0\right)$$

- linear frequency term S goes to zero ($d = 3$ logarithmically, $d = 2$ linearly)
- quadratic frequency term V takes over

Vacuum flow of interaction strength λ



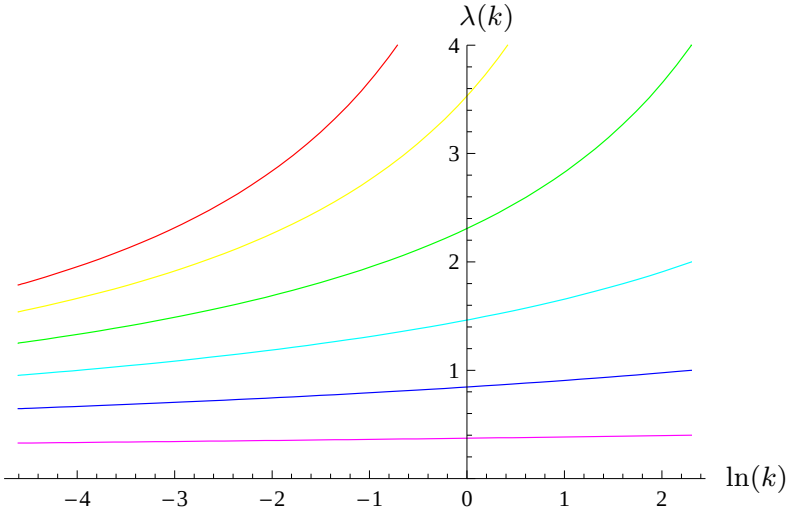
Vacuum flow of interaction strength λ



- in $d = 2$ logarithmic running

$$\partial_t \lambda = \frac{1}{4\pi}\lambda^2, \qquad \lambda(k) = \frac{1}{\frac{1}{\lambda_\Lambda} + \frac{1}{4\pi}\ln(\Lambda/k)}, \qquad \lambda = \lambda(k_{\rm ph}) \leq \frac{4\pi}{\ln(\Lambda/k_{\rm ph})}$$

Vacuum flow of interaction strength λ



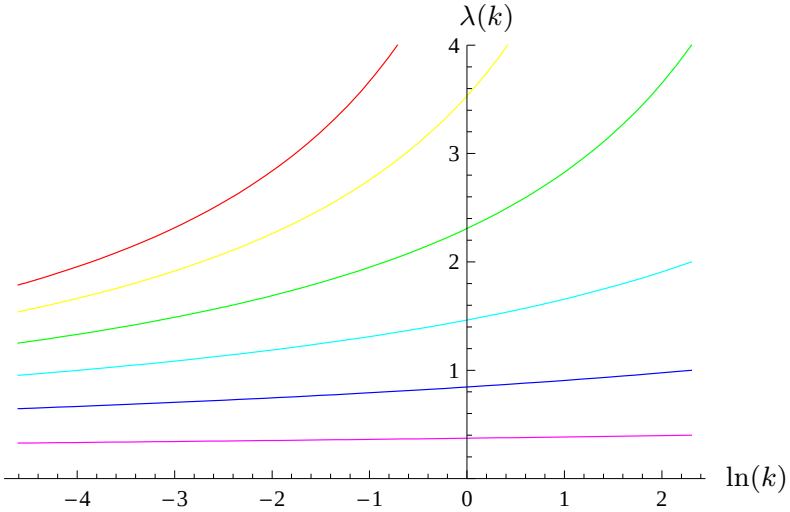
- in $d = 2$ logarithmic running

$$\partial_t \lambda = \frac{1}{4\pi}\lambda^2, \qquad \lambda(k) = \frac{1}{\frac{1}{\lambda_\Lambda} + \frac{1}{4\pi}\ln(\Lambda/k)}, \qquad \lambda = \lambda(k_{\rm ph}) \leq \frac{4\pi}{\ln(\Lambda/k_{\rm ph})}$$

- in $d = 3$ linear running

$$\partial_t \lambda = \frac{k}{6\pi^2}\lambda^2, \qquad \lambda(k) = \frac{1}{\frac{1}{\lambda_\Lambda} + \frac{1}{6\pi^2}(\Lambda - k)}, \qquad \lambda = \lambda(0) \leq \frac{6\pi^2}{\Lambda}$$

Vacuum flow of interaction strength λ



- in $d = 2$ logarithmic running

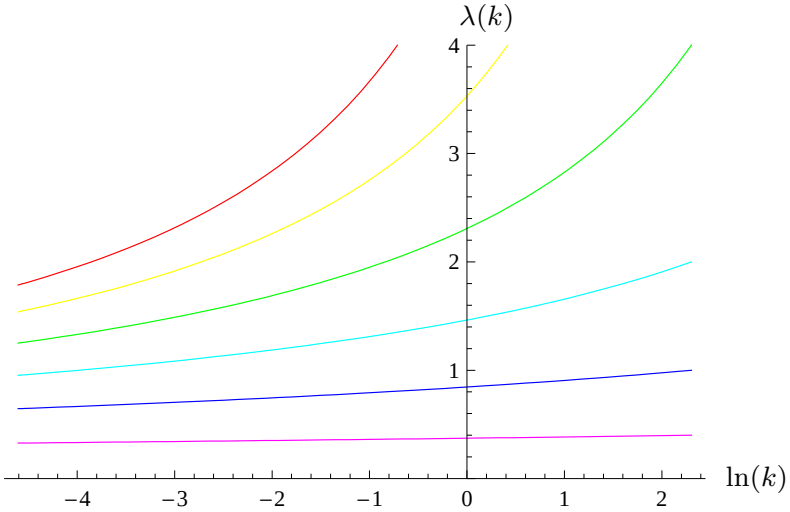
$$\partial_t \lambda = \frac{1}{4\pi}\lambda^2, \qquad \lambda(k) = \frac{1}{\frac{1}{\lambda_\Lambda} + \frac{1}{4\pi}\ln(\Lambda/k)}, \qquad \lambda = \lambda(k_{\rm ph}) \leq \frac{4\pi}{\ln(\Lambda/k_{\rm ph})}$$

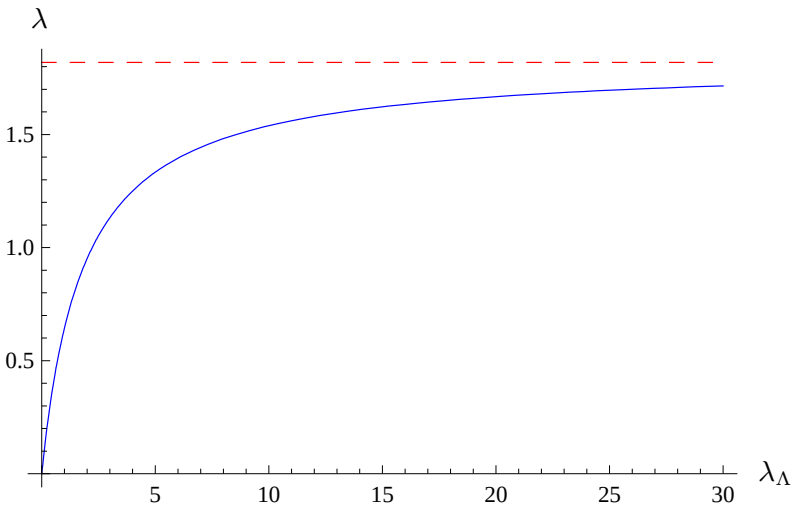
- in $d = 3$ linear running

$$\partial_t \lambda = \frac{k}{6\pi^2}\lambda^2, \qquad \lambda(k) = \frac{1}{\frac{1}{\lambda_\Lambda} + \frac{1}{6\pi^2}(\Lambda - k)}, \qquad \lambda = \lambda(0) \leq \frac{6\pi^2}{\Lambda}$$

- triviality bound on interaction strength!

Vacuum flow of interaction strength λ





- in $d = 2$ logarithmic running

$$\partial_t \lambda = \frac{1}{4\pi}\lambda^2, \qquad \lambda(k) = \frac{1}{\frac{1}{\lambda_\Lambda} + \frac{1}{4\pi}\ln(\Lambda/k)}, \qquad \lambda = \lambda(k_{\rm ph}) \leq \frac{4\pi}{\ln(\Lambda/k_{\rm ph})}$$

- in $d = 3$ linear running

$$\partial_t \lambda = \frac{k}{6\pi^2}\lambda^2, \qquad \lambda(k) = \frac{1}{\frac{1}{\lambda_\Lambda} + \frac{1}{6\pi^2}(\Lambda - k)}, \qquad \lambda = \lambda(0) \leq \frac{6\pi^2}{\Lambda}$$

- triviality bound on interaction strength!

Dispersion relation

- follows from propagator with analytic continuation ($T = 0$)

$$\det \begin{pmatrix} \vec{p}^2 - V\omega^2 + U' + 2\rho U'', & -iS\omega \\ iS\omega, & \vec{p}^2 - V\omega^2 + U' \end{pmatrix} = 0$$

$$S^2\omega^2 + (\vec{p}^2 - V\omega^2 + U' + 2\rho U'')(\vec{p}^2 - V\omega^2 + U') = 0$$

Dispersion relation

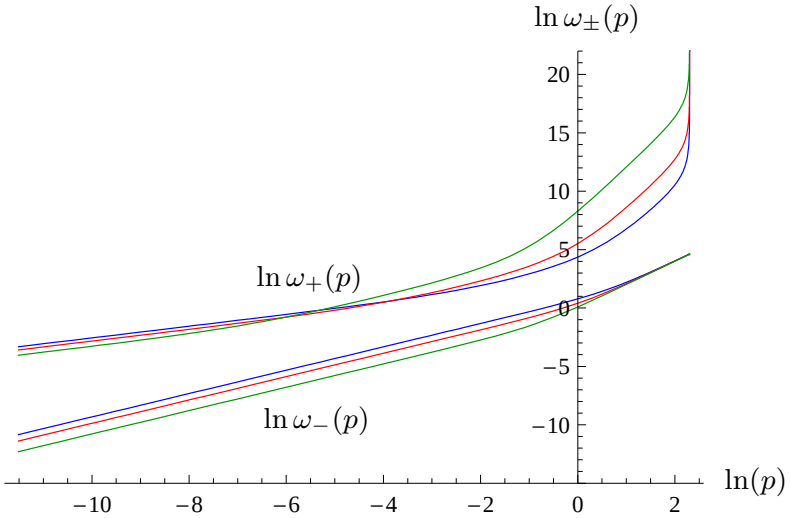
- follows from propagator with analytic continuation ($T = 0$)

$$\det \begin{pmatrix} \vec{p}^2 - V\omega^2 + U' + 2\rho U'', & -iS\omega \\ iS\omega, & \vec{p}^2 - V\omega^2 + U' \end{pmatrix} = 0$$

$$S^2\omega^2 + (\vec{p}^2 - V\omega^2 + U' + 2\rho U'')(\vec{p}^2 - V\omega^2 + U') = 0$$

- solution for ω has two branches

Dispersion relation



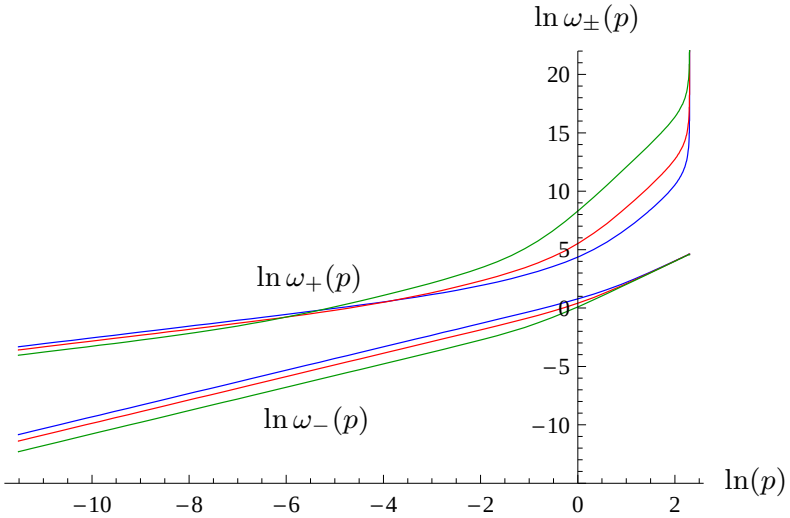
- follows from propagator with analytic continuation ($T = 0$)

$$\det \begin{pmatrix} \vec{p}^2 - V\omega^2 + U' + 2\rho U'', & -iS\omega \\ iS\omega, & \vec{p}^2 - V\omega^2 + U' \end{pmatrix} = 0$$

$$S^2\omega^2 + (\vec{p}^2 - V\omega^2 + U' + 2\rho U'')(\vec{p}^2 - V\omega^2 + U') = 0$$

- solution for ω has two branches

Dispersion relation



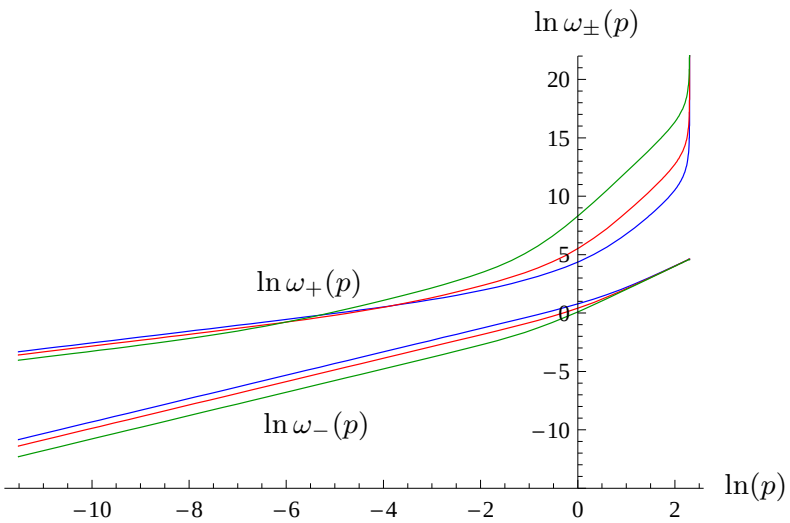
- follows from propagator with analytic continuation ($T = 0$)

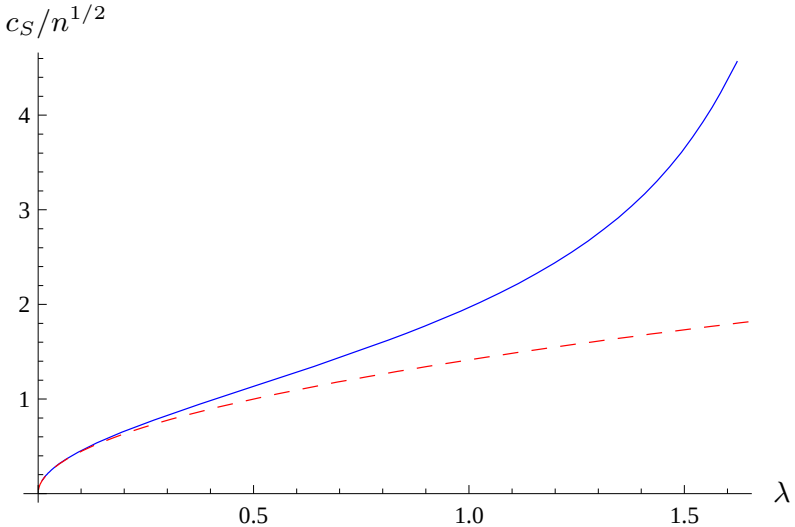
$$\det\begin{pmatrix} \vec{p}^2 - V\omega^2 + U' + 2\rho U'', & -iS\omega \\ iS\omega, & \vec{p}^2 - V\omega^2 + U' \end{pmatrix} = 0$$

$$S^2\omega^2 + (\vec{p}^2 - V\omega^2 + U' + 2\rho U'')(\vec{p}^2 - V\omega^2 + U') = 0$$

- solution for ω has two branches
- lower branch describes phase fluctuations (sound mode)

Dispersion relation





- follows from propagator with analytic continuation ($T = 0$)

$$\det\begin{pmatrix} \vec{p}^2 - V\omega^2 + U' + 2\rho U'', & -iS\omega \\ iS\omega, & \vec{p}^2 - V\omega^2 + U' \end{pmatrix} = 0$$

$$S^2\omega^2 + (\vec{p}^2 - V\omega^2 + U' + 2\rho U'')(\vec{p}^2 - V\omega^2 + U') = 0$$

- solution for ω has two branches
- lower branch describes phase fluctuations (sound mode)
- sound velocity deviates from mean field for large λ

Sound at $T > 0$

Landau (1941): For temperatures $0 < T < T_c$ two fluid hydrodynamics: *First and Second sound*

Sound at $T > 0$

Landau (1941): For temperatures $0 < T < T_c$ two fluid hydrodynamics: *First and Second sound*

- can be calculated in grand canonical ensemble

Sound at $T > 0$

Landau (1941): For temperatures $0 < T < T_c$ two fluid hydrodynamics: *First and Second sound*

- can be calculated in grand canonical ensemble
- use flow equation for $p = -U(\rho_0, \mu_0)$

Sound at $T > 0$

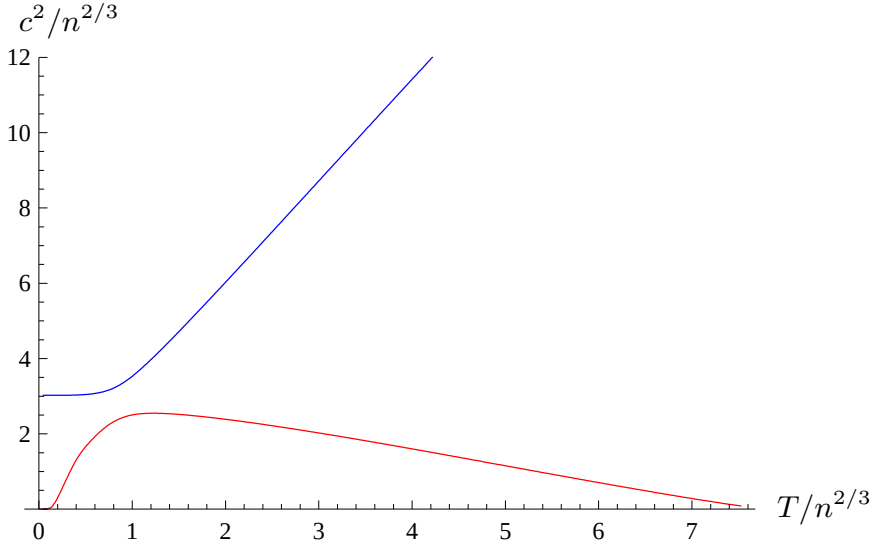
Landau (1941): For temperatures $0 < T < T_c$ two fluid hydrodynamics: *First and Second sound*

- can be calculated in grand canonical ensemble
- use flow equation for $p = -U(\rho_0, \mu_0)$
- perform various derivatives with respect to T and μ nummerically

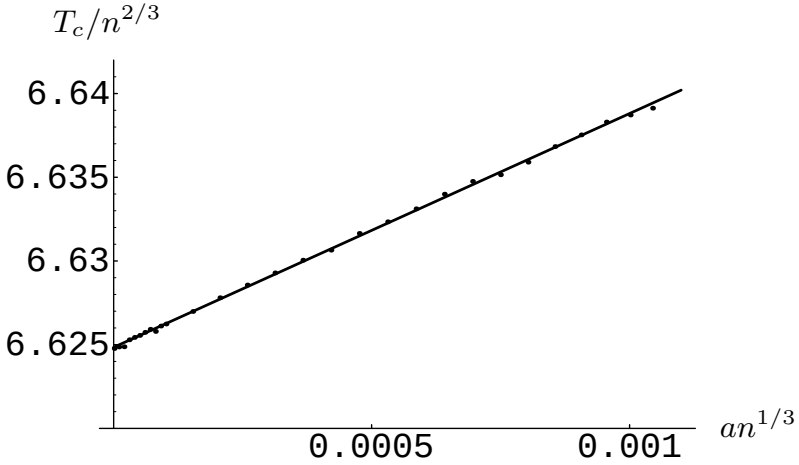
Sound at $T > 0$

Landau (1941): For temperatures $0 < T < T_c$ two fluid hydrodynamics: *First and Second sound*

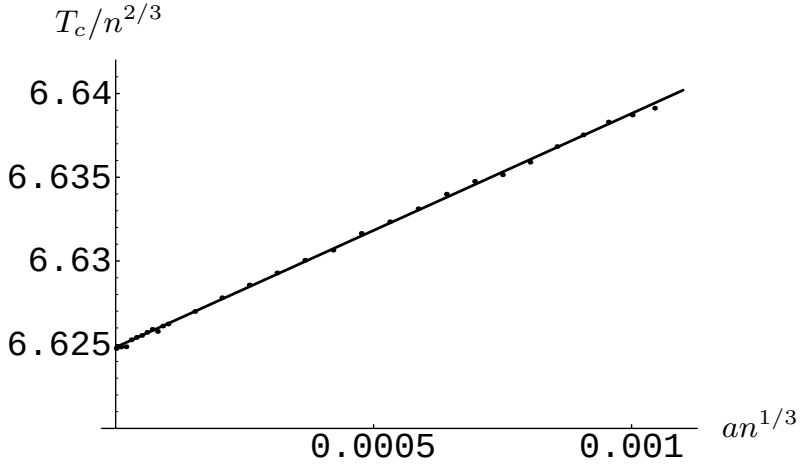
- can be calculated in grand canonical ensemble
- use flow equation for $p = -U(\rho_0, \mu_0)$
- perform various derivatives with respect to T and μ nummerically



Critical temperature in $d = 3$

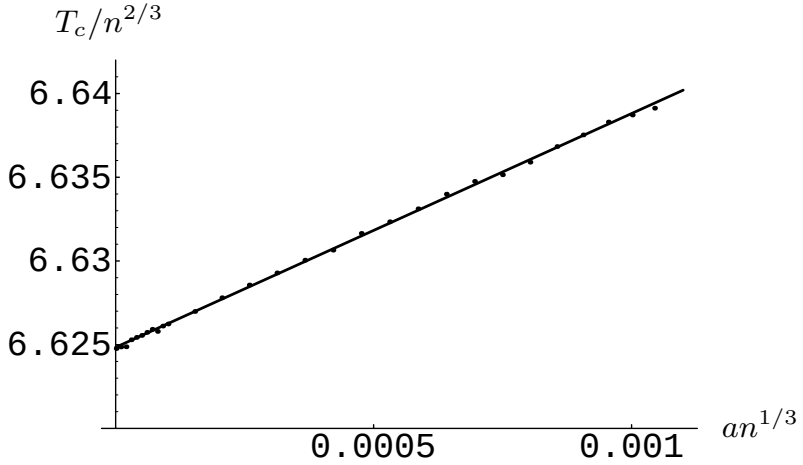


Critical temperature in $d = 3$



- for $a = \frac{\lambda}{8\pi} = 0$ free Bose gas: $T_c/(n^{2/3}) = 4\pi/\zeta(3/2)^{2/3} = 6.6250$

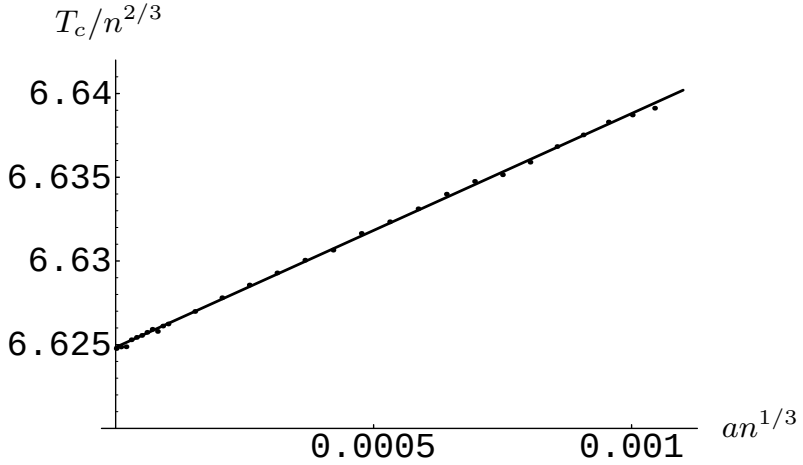
Critical temperature in $d = 3$



- for $a = \frac{\lambda}{8\pi} = 0$ free Bose gas: $T_c/(n^{2/3}) = 4\pi/\zeta(3/2)^{2/3} = 6.6250$
- for $a > 0$ shift in T_c:

$$\Delta T_c/T_c = \kappa\, a n^{1/3}$$

Critical temperature in $d = 3$

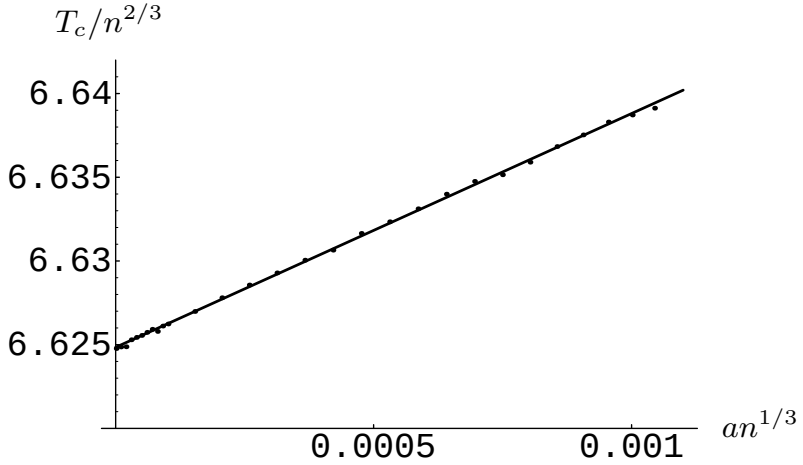


- for $a = \frac{\lambda}{8\pi} = 0$ free Bose gas: $T_c/(n^{2/3}) = 4\pi/\zeta(3/2)^{2/3} = 6.6250$
- for $a > 0$ shift in T_c:

$$\Delta T_c/T_c = \kappa\, a n^{1/3}$$

- we find $\kappa = 2.1$

Critical temperature in $d = 3$



- for $a = \frac{\lambda}{8\pi} = 0$ free Bose gas: $T_c/(n^{2/3}) = 4\pi/\zeta(3/2)^{2/3} = 6.6250$
- for $a > 0$ shift in T_c:

$$\Delta T_c/T_c = \kappa\, a n^{1/3}$$

- we find $\kappa = 2.1$
- Monte-Carlo and RG studies in the high-T limit find $\kappa = 1.3$ (including only the $n = 0$ Matsubara frequency)
  (Arnold, Moore (2001); Kashurnikov et al. (2001); Baym et al. (1999); Ledowski et al. (2004); Blaizot et al. (2006).)

Kosterlitz-Thouless physics in $d = 2$

Kosterlitz-Thouless physics in $d = 2$

- in the infinite size limit ($k \to 0$) bare order parameter $\bar{\rho}_0$ vanishes (Mermin-Wagner theorem)

Kosterlitz-Thouless physics in $d = 2$

- in the infinite size limit ($k \to 0$) bare order parameter $\bar{\rho}_0$ vanishes (Mermin-Wagner theorem)
- for small temperatures $T < T_c$ the renormalized order parameter $\rho = \bar{A}\bar{\rho}$ (superfluid density) remains finite

Kosterlitz-Thouless physics in $d = 2$

- in the infinite size limit ($k \to 0$) bare order parameter $\bar{\rho}_0$ vanishes (Mermin-Wagner theorem)
- for small temperatures $T < T_c$ the renormalized order parameter $\rho = \bar{A}\bar{\rho}$ (superfluid density) remains finite
- crititcal temperature T_c depends on λ, not computable from Kosterlitz-Thouless theory

Kosterlitz-Thouless physics in $d = 2$

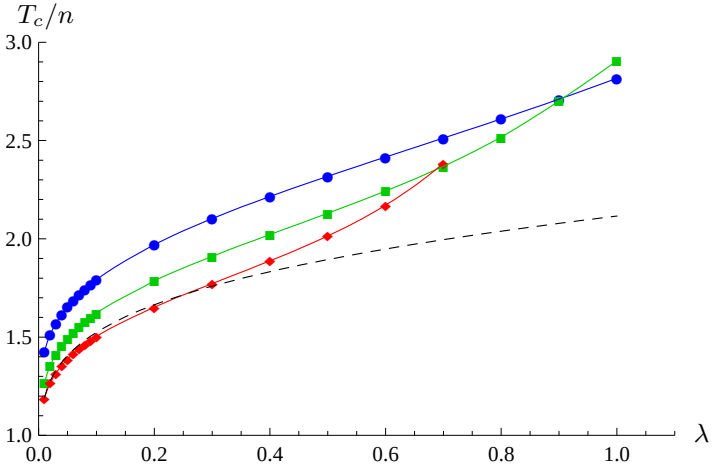
- in the infinite size limit ($k \to 0$) bare order parameter $\bar{\rho}_0$ vanishes (Mermin-Wagner theorem)
- for small temperatures $T < T_c$ the renormalized order parameter $\rho = \bar{A}\bar{\rho}$ (superfluid density) remains finite
- crititcal temperature T_c depends on λ, not computable from Kosterlitz-Thouless theory
- perturbative treatment (Popov (1983)) gives $T_c/n = 4\pi/\ln(\zeta/\lambda)$ and Monte-Carlo (Prokof'ev et al. (2001)) determines $\zeta = 380$

Kosterlitz-Thouless physics in $d = 2$

- in the infinite size limit ($k \to 0$) bare order parameter $\bar{\rho}_0$ vanishes (Mermin-Wagner theorem)
- for small temperatures $T < T_c$ the renormalized order parameter $\rho = \bar{A}\bar{\rho}$ (superfluid density) remains finite
- crititcal temperature T_c depends on λ, not computable from Kosterlitz-Thouless theory
- perturbative treatment (Popov (1983)) gives $T_c/n = 4\pi/\ln(\zeta/\lambda)$ and Monte-Carlo (Prokof'ev et al. (2001)) determines $\zeta = 380$
- we find that ζ depends on the size of system k_{ph}^{-1}

Kosterlitz-Thouless physics in $d = 2$

- in the infinite size limit ($k \to 0$) bare order parameter $\bar{\rho}_0$ vanishes (Mermin-Wagner theorem)
- for small temperatures $T < T_c$ the renormalized order parameter $\rho = \bar{A}\bar{\rho}$ (superfluid density) remains finite
- crititcal temperature T_c depends on λ, not computable from Kosterlitz-Thouless theory
- perturbative treatment (Popov (1983)) gives $T_c/n = 4\pi/\ln(\zeta/\lambda)$ and Monte-Carlo (Prokof'ev et al. (2001)) determines $\zeta = 380$
- we find that ζ depends on the size of system $k_{\rm ph}^{-1}$



circles: $k_{\rm ph} = 10^{-4}$, $\zeta = 100$
boxes: $k_{\rm ph} = 10^{-6}$, $\zeta = 225$
diamonds: $k_{\rm ph} = 10^{-8}$, $\zeta = 424$

Conclusions

Conclusions

- Functional renormalization for the average action yields a unified description of ultracold bosons.

Conclusions

- Functional renormalization for the average action yields a unified description of ultracold bosons.
- Varios thermodynamic observables in equilibrium have been calculated.

Conclusions

- Functional renormalization for the average action yields a unified description of ultracold bosons.
- Varios thermodynamic observables in equilibrium have been calculated.
- A simple truncation describes the essential features of superfluidity in $d = 3$ and $d = 2$.

Conclusions

- Functional renormalization for the average action yields a unified description of ultracold bosons.
- Varios thermodynamic observables in equilibrium have been calculated.
- A simple truncation describes the essential features of superfluidity in $d = 3$ and $d = 2$.
- Deviations from Mean-Field results are found for large interaction strength λ and close to T_c.

Conclusions

- Functional renormalization for the average action yields a unified description of ultracold bosons.
- Varios thermodynamic observables in equilibrium have been calculated.
- A simple truncation describes the essential features of superfluidity in $d = 3$ and $d = 2$.
- Deviations from Mean-Field results are found for large interaction strength λ and close to T_c.
- Extentions of the truncation are straight forward and should lead to quantitative improvements for large λ.

Conclusions

- Functional renormalization for the average action yields a unified description of ultracold bosons.
- Varios thermodynamic observables in equilibrium have been calculated.
- A simple truncation describes the essential features of superfluidity in $d = 3$ and $d = 2$.
- Deviations from Mean-Field results are found for large interaction strength λ and close to T_c.
- Extentions of the truncation are straight forward and should lead to quantitative improvements for large λ.
- Thank you for your attention!

First and second sound

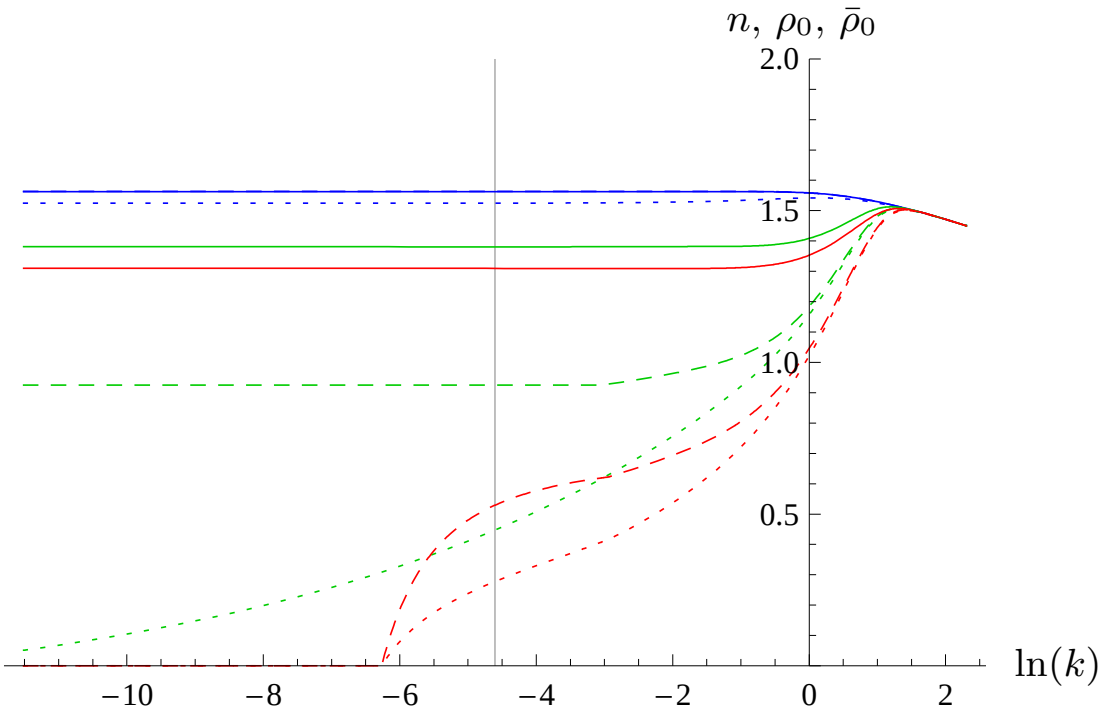
- sound velocities c follow are solutions of the equation

$$(Mc^2)^2 - \left(\left.\frac{\partial p}{\partial n}\right|_{\frac{s}{n}} + \frac{n_s T s^2}{(n - n_S) c_v n^2} \right) (Mc^2) + \frac{n_s T s^2}{(n - n_s) c_v n^2} \left.\frac{\partial p}{\partial n}\right|_T = 0$$

- use here specific heat per particle

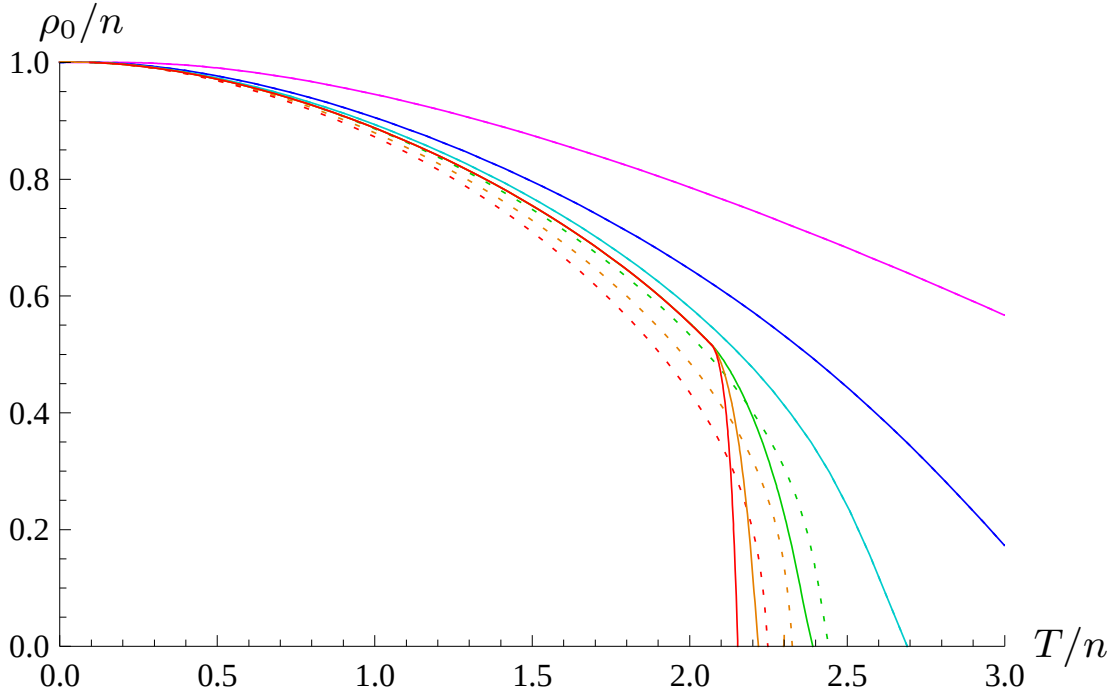
$$c_v = \frac{T}{n} \left.\frac{\partial s}{\partial T}\right|_n$$

Order in $d = 2$



Flow of the density n (solid), the superfluid density ρ_0 (dashed) and the condensate density $\bar{\rho}_0$ (dotted) for temperatures $T = 0$ (top), $T = 2.4$ (middle) and $T = 2.8$ (bottom).

Jump in superfluid density



Superfluid fraction of the density ρ_0/n at different scales $k_{\rm ph}$.
dotted: simple truncation
solid: improved truncation