The BV Master Equation for the Gauge Wilson Action

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arXiv:0709.1522 [hep-th]

Prog. Theor. Phys. 118:1115-1125,2007.

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2008/7/4, ERG2008 @ Heidelberg

1.Non-perturbative (Wilsonian) renormalization group equation

We study the gauge (BRS) invariant renormalization group flows.

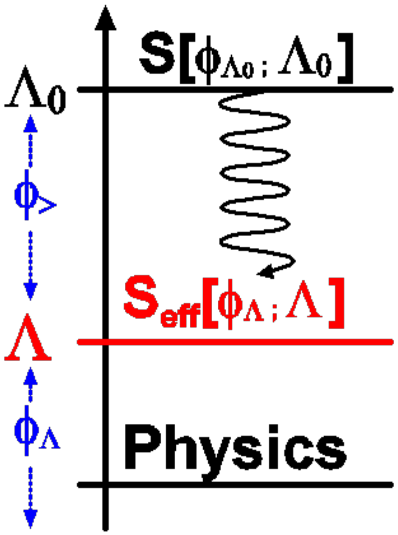
(The existence of gauge invariant flow.)

- Large coupling region
- the perturbatively nonrenormalizable interaction terms

  (higher dim. operators, beyond 4-dim.)
- Non-perturbative phenomena for SYM (nonrenormalization theorem)

Non-perturbative renormalization group

$$Z_\phi[J] = \int D\phi \exp(-\mathcal{S}[\phi] + J \cdot \phi)$$



1.We introduce the cutoff scale in momentum space.

2.We divide all fields Φ into two groups,

(high frequency modes and low frequency modes).

3.We integrate out all high frequency modes.

$$\boxed{e^{-S_{\mathrm{eff}}[\phi_\Lambda,\Lambda]} = \int^{\Lambda_0} [d\phi_>] e^{-S[\phi_\Lambda+\phi_>,\Lambda_0]}}$$

➢Infinitesimal change of cutoff $\Lambda \to e^{-\delta t}\Lambda = \Lambda - \delta\Lambda$

The partition function does not depend on Λ .

WRG equation for the Wison effection action

There are some Wilsonian renormalization group equations.

- Wegner-Houghton equation (sharp cutoff)

  K-I. Aoki, H. Terao, K.Higashijima…

  local potential, Nambu-Jona-Lasinio, NLσM

- Polchinski equation (smooth cutoff)

  T.Morris, K. Itoh, Y. Igarashi, H. Sonoda, M. Bonini,…

  YM theory, QED, SUSY…

- Exact evolution equation (for 1PI effective action)

  C. Wetterich, M. Reuter, N. Tetradis, J. Pawlowski,…

  quantum gravity, Yang-Mills theory,

  higher-dimensional gauge theory…

Gauge invariance and renormalization group

Cutoff vs Gauge invariance

Gauge transformation: $\delta A_\mu(p) = -ig \int_k A_\mu(p-k)c(k)$

Mix UV fields and IR fields

- Wegner-Houghton equation (sharp cutoff)
- Polchinski equation (smooth cutoff)
- Exact evolution equation (for 1PI effective action)

Identity for the BRS invariance

- Master equation for BRS symmetry
- modified Ward-Takahashi identity (Ellwanger `94

  Sonoda `07)

2. BV formalism and Master equation

Batalin-Vilkoviski（BV） formalism：

Local (global) symmetry

⇩

Action（S）satisfies the quantum Master equation

Introduce the anti-field $\hat{S} \equiv S[\phi] + \boxed{\phi^* \delta_Q \phi}$

classical

Master equation: $(\hat{S}, \hat{S}) = 0 \Rightarrow \dfrac{\partial^r S}{\partial \Phi^A} \delta \Phi^A = 0$ Ward-Takahashi id.

quantum

Master equation：

$$\boxed{\Sigma[\Phi, \Phi^*] \equiv \frac{1}{2}(S, S) - \Delta S = 0}$$

This action is the Master action.

$$(X, Y) \equiv \frac{\partial^r X}{\partial \Phi^A} \frac{\partial^l Y}{\partial \Phi^*_A} - \frac{\partial^r X}{\partial \Phi^*_A} \frac{\partial^l Y}{\partial \Phi^A}, \ \Delta \equiv (-)^{\epsilon_A + 1} \frac{\partial^r}{\partial \Phi^A} \frac{\partial^r}{\partial \Phi^*_A}$$

r (l) denote right (left) derivative for fermionic fields.

quantum BRS transformation in anti-field formalism

- Classical (usual) BRS transformation

$$\delta_Q X = (X, \widehat{S})$$

- quantum BRS transformation

$$\delta_Q X \equiv (X, S_M) - \Delta X$$

- Master action is invariant under the quantum BRS transf.
- Nilpotency $\delta_Q^2 X = (X, \Sigma[\Phi, \Phi^*]) = 0$
- The BRS tr. depends on the Master action

Polchinski eq for the Master action:

- Master eq.
- Polchinski eq.

quantum Master equation:

$$\Sigma[\Phi, \Phi^*] \equiv \frac{1}{2}(S, S) - \Delta S = 0$$

The scale dependence of the Master action is

$$\begin{aligned} \partial_t \Sigma &= (\partial_t S_M, S_M) - \Delta \partial_t S_M \\ &= \delta_Q \partial_t S_M = 0 \end{aligned}$$

quantum BRS invariant

Problem : "Can we solve the Master equation?"

3.Review of IIS (QED)

Igarashi, Itoh and Sonoda （IIS）

Prog.Theor.Phys.118:121-134,2007.

Outline of the IIS’s paper

Modified Ward-Takahashi identity （W-T id. for the Wilson action）

⇓ J. Phys. A40 (2007) 9675, Sonoda

Read off the modified BRS transformation from MWT identity

⇓

The modified BRS transformation does not have a nilpotency.

⇓ Extend to the Master action

They introduce the anti-field as a source of the modified BRS transformation.

⇓

They construct the Master action order by order of the anti-fields.

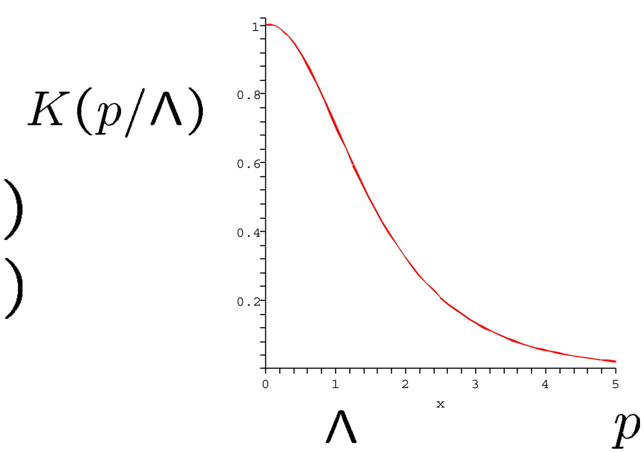
The modified BRS tr. for the IR fields as follow:

$$\begin{aligned}
\delta A_\mu(k) &= -ik_\mu c(k),\ \delta\bar{c} = iB(k),\ \delta c(k) = \delta B(k) = 0, \\
\delta\psi(p) &= ie\int_k c(k)\left[\frac{K(p)}{K(p-k)}\psi(p-k) - U(-p,p-k)\frac{\partial^l S}{\partial\bar{\psi}(-p+k)}\right], \\
\delta\bar{\psi}(-p) &= ie\int_k \frac{K(p)}{K(p+k)}\bar{\psi}(-p-k)c(k)
\end{aligned}$$

- the BRS tr. depends on the action itself.
- It is not nilpotent. $\delta\delta\psi \neq 0$

Smooth Cutoff fn.

$$K(p/\Lambda) \quad \rightarrow \quad \begin{cases} 1 & (p^2 < \Lambda^2) \\ 0 & (p^2 \rightarrow \infty) \end{cases}$$



Extend the Wilson action to the Master action order by order of the anti-fields.

$$S_M[\Phi.\Phi^*] \;=\; S[\Phi] + \Phi^*_A \delta\Phi^A + \Phi^*_A \Phi^*_B C^{AB}[\Phi] + \cdots$$

the anti-field is the source for modefied BRS tr.

The solution of the quantum Master equation in Abelian gauge theory

$$\begin{aligned} S_M[\Phi,\,\Phi^*] \;&=\; \frac{1}{2}\Phi'\cdot K^{-1}D\cdot\Phi' + S'_I[\Phi'] \\ &+ \int_k \left(A^*_\mu(-k)(-ik^\mu C(k)) + \bar{C}^*(-k)iB(k)\right) \\ &+ ie\int_{p,k}\left(K(p)\Psi^*(-p)C(k)\frac{\Psi(p-k)}{K(p-k)} + \frac{\bar{\Psi}(p-k)}{K(p-k)}K(p)\bar{\Psi}^*(-p)C(k)\right) \end{aligned}$$

$$\begin{aligned} \Phi'^A \;&=\; \{A_\mu,\; B,\; C,\; \bar{C},\; \Psi,\; \bar{\Psi}'\}\,, \\ \bar{\Psi}'(-p) \;&=\; \bar{\Psi}(-p) - ie\int_k \Psi^*(-p-k)C(k)U(-p-k,\,p) \end{aligned}$$

Remarks of IIS Master action

- only the anti-fermion field is shifted
- only linear dependence of the anti-field

4.Our method

T.Higashi,E.I and T.Kugo :

Prog. Theor. Phys. 118:1115-1125,2007

$$\mathcal{Z}_\phi[J,\ \phi^*] = \int \mathcal{D}\phi\ \mathrm{exp}\left(-\mathcal{S}[\phi] + J\cdot\phi - \phi^*\cdot F(\phi)\right)$$

$$\delta_Q\phi^A = F^A(\phi)$$

The anti-field is the source of usual BRS tr.

The action is the Yang-Mills action.

To decompose the IR and UV fields, we insert the gaussian integral.

$$\boxed{\int D\theta \exp - \left\{ \frac{1}{2}(\theta - J(1-K)D^{-1}) \cdot \frac{D}{K(1-K)} \cdot (\theta - (-)^J D^{-1}(1-K)J) \right\} = const.}$$

$$\phi = \Phi + \tilde{\phi}$$

$$\theta = (1-K)\Phi - K\tilde{\phi}$$

- **IR field** $K(p)D^{-1}(p)$
- **UV field** $(1-K(p))D^{-1}(p)$

$$\mathcal{Z}_\phi[J,\, \phi^*] \;=\; N_J \int \mathcal{D}\Phi \mathcal{D}\tilde{\phi} \exp -\Big(\frac{1}{2}\Phi \cdot K^{-1} D \cdot \Phi + \frac{1}{2}\tilde{\phi} \cdot (1-K)^{-1} D \cdot \tilde{\phi} + \mathcal{S}_I[\Phi + \tilde{\phi}] + \phi^* \cdot F(\Phi + \tilde{\phi}) - J \cdot K^{-1}\Phi\Big)$$

The partition fn. for IR field

$$\boxed{Z_\Phi[K^{-1}J,\, \Phi^*] = \int \mathcal{D}\Phi \exp\left(-S[\Phi,\, \Phi^*] + K^{-1}J \cdot \Phi\right)}$$

$$S[\Phi,\, \Phi^*] \equiv \frac{1}{2}\Phi \cdot K^{-1} D \cdot \Phi + \underline{\mathcal{S}_I[\Phi,\, \Phi^*]}$$

Now $S[\Phi,\, \Phi^*]$ is the Wilsonian action which includes the anti-fields.

Ward-Takahashi identity

The action and the anti-field term are BRS invariant, then the external source term is remained.

$$\langle 0|[Q_B, \exp(-S + J\cdot\phi - \phi^*\delta_B\phi)]|0\rangle = 0$$

$$J\cdot K^{-1}\frac{\delta^l}{\delta\Phi^*}Z_\Phi[K^{-1}J,\ \Phi^*] = \langle J\cdot K^{-1}\frac{\delta^l S}{\delta\Phi^*}\rangle_{K^{-1}J,\Phi^*} = 0$$

Act the total derivative on the identity.

$$0 \;=\; \int \mathcal{D}\Phi\,\frac{\delta^r}{\delta\Phi^A}\left(\frac{\delta^l S}{\delta\Phi^*_A}e^{\left(-S[\Phi,\Phi^*]+K^{-1}J\cdot\Phi\right)}\right)$$

$$\boxed{\langle\frac{\delta^r\delta^l S}{\delta\Phi^A\delta\Phi^*_A} - \frac{\delta^r S}{\delta\Phi^A}\frac{\delta^l S}{\delta\Phi^*_A}\rangle_{K^{-1}J,\ \Phi^*} = 0}$$

The Wilsonian action satisfies the Master equation.

Construction of the Master action (QED)

$$\mathcal{Z}_\phi[J,\,\phi^*] \;=\; N_J \int \mathcal{D}\Phi\mathcal{D}\tilde{\phi}\, \exp -(\frac{1}{2}\Phi\cdot K^{-1}D\cdot\Phi + \frac{1}{2}\tilde{\phi}\cdot(1-K)^{-1}D\cdot\tilde{\phi} + \mathcal{S}_I[\Phi+\tilde{\phi}] + \phi^*\cdot F(\Phi+\tilde{\phi}) - J\cdot K^{-1}\Phi)$$

The linear term of UV fields can be absorbed into the kinetic terms by shifting the integration variables:

$$\boxed{\Phi \to \Phi' = \Phi - (f(\Phi)\cdot\Phi^*)}$$

$$S[\Phi,\,\Phi^*] \;=\; \frac{1}{2}\Phi'\cdot K^{-1}D\cdot\Phi' + S_I'[\Phi'] + (\text{linear terms in } \Phi^*) + (\text{quadratic terms in } \Phi^*)$$

$$\begin{cases} A'_\mu(k) &= A_\mu(k) + \dfrac{k_\mu}{k^2}(1-K(k))K(k)\bar{C}^*(k), \\ \Psi'(p) &= \Psi(p) - ie\dfrac{1-K(p)}{p^\mu\gamma_\mu+m}\displaystyle\int_k K(p-k)\bar{\Psi}^*(p-k)C(k), \\ \bar{\Psi}'(-p) &= \bar{\Psi}(-p) - ie\displaystyle\int_k K(p+k)\Psi^*(-p-k)C(k)\dfrac{1-K(p)}{p^\mu\gamma_\mu+m} \end{cases}$$

(linear terms in Φ^*)

$$
\begin{aligned}
&= K\Phi^* \cdot F(\Phi') + \Phi' \cdot K^{-1} D \cdot (f(\Phi) \cdot \Phi^*) \\
&= \int_k (K(k) A^*_\mu(-k)(-ik^\mu C(k)) + \bar{C}^*(-k) iB(k)) \\
&\quad + ie \int_{p,k} \left(K(p)\Psi^*(-p)C(k) \frac{\Psi'(p-k)}{K(p-k)} + \frac{\bar{\Psi}'(p-k)}{K(p-k)} K(p)\bar{\Psi}^*(-p)C(k) \right)
\end{aligned}
$$

(quadratic terms in Φ^*)

$$
= K\Phi^* \cdot F'(\Phi')(f(\Phi) \cdot \Phi^*) + \frac{1}{2}(f(\Phi) \cdot \Phi^*) \cdot \left[-\frac{1}{1-K} + \frac{1}{K} \right] D \cdot (f(\Phi) \cdot \Phi^*)
$$

- the gauge field and fermion field are also shifted.
- there are quadratic term of the anti-field.

Relation between IIS's and our Master action

Master action is defined by the solution of the Master equation.

$\Downarrow$

there is a freedom of doing the canonical transformation in the field and anti-field space.

We found the following functional gives the canonical transformation.

$$
\begin{aligned}
W[\Phi,\,\Phi^*_{IIS}]
&= \int_k \Big[A^{*\mu}_{IIS}(-k)(A_\mu(k) + \frac{k_\mu}{k^2}K(k)(1-K(k))\bar{C}^*_{IIS}(k)) + \bar{C}^*_{IIS}(-k)\bar{C}(k)\Big] \\
&\quad + \int_p \Big[\Psi^*_{IIS}(-p)(\Psi(p) - ie(1-K(p))\int_k (p^\mu\gamma_\mu + m)^{-1}K(p-k)\bar{\Psi}^*_{IIS}(p-k)C(k)) \\
&\qquad\qquad + \bar{\Psi}(p)\bar{\Psi}^*_{IIS}(-p)]
\end{aligned}
$$

5. Summary

- Using BV formalism, if there is a Master action, the flow eq. of the Master action is quantum BRS invariant.
- We introduce the anti-field as the source term for the usual BRS transformation.
- We show the Wilsonian effective action satisfies the Master eq.
- In the case of abelian gauge theory, we can solve the Master equation.
- We show that our Master action equals to IIS action via the canonical transformation.
- The BRS invariant RG flows exist.

Discussion

- To solve the Master eq. for the non-abelian gauge theory

  Because of the non-trivial ghost interaction terms, the quadratic terms of UV field cannot be eliminated.

  The Master action cannot be represented by the shift of the fields.
- Approximation method (truncate the interaction terms)
- To find the explicit form of the quantum BRS invariant operators.
- 1PI evolution equation version.(Legendre transf.)