Precise NPRG calculation of critical exponents of the O(N) model.

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Introduction I

Question : Can Non-Perturbative Renormalization Group (NPRG) methods compete in precision with standard perturbative RG methods in simple quantities as critical exponents of the $O(N)$ model?

- Of course, NPRG is a good starting point:
  - Exact equations, UV and IR finites;
  - Flow preserve analyticity in intermediate steps and decouples high momentum modes;
  - Allows for more flexible formulation of approximations schemes;

Introduction II

- At low momenta, Derivative Expansion (DE) seems to have many good properties:
  - It gives qualitatively good results for all tested situations;
  - Reasonable quantitative results are obtained at order ∂^2;
  - Only known result at order ∂^4: on the Ising universality class. It seems to compete with best field theoretical estimates;
- But Derivative Expansion :
  - Does not reproduce exactly perturbative results at two loop orders;
  - Applies only for momenta smaller that smallest mass;

Introduction III

- We have found an approximation scheme that allows to calculate correlations functions at arbitrary momenta:
  - It includes simultaneously an infinite number of vertices;
  - Reproduce DE, perturbative expansion and large N for $O(N)$ models in corresponding limits.
  - It can be improved systematically.
  - Testing ground: critical regimes of $O(N)$ models.
  - In this presentation the focus will be on the low momenta regime.
  - It will be shown that including the leading order for the 2-point function already gives critical exponents of the same precision that the best field theoretical estimates.

Refs: J. P. Blaizot, R. Mendez-Galain and NW '06
J. P. Blaizot, R. Mendez-Galain and NW '07
F.Benitez, R. Mendez-Galain and NW '08
F.Benitez, J. P. Blaizot, B. Delamotte,H. Chaté, R. Mendez-Galain and NW (in preparation)

Outline

- A point of view on the derivative expansion;
- Another approximation scheme;
- Application for the $O(N)$ model;
- Results.

The NPRG equation

- Consider the NPRG equation:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_k R_k(q^2) \left[\Gamma_k^{(2)} + R_k \right]^{-1}_{q,-q} .$$

Wetterich '93

- Equations for vertices are obtained by taking functional derivatives.
- The difficutly is that the equation for a vertex of order n requires vertices up to order $n+2$.
- Very important properties:
  - Internal momentum q is bounded: $q \lesssim k$;
  - In general, the regulator ensures analyticity of vertices for $k > 0$.

Derivative Expansion I

- Let's consider a problem which requires the knowledge of vertices (or derivatives of them) at zero momenta.
  Examples: Phase diagrams in homogeneous problems, critical exponents, gap existence, etc.
- Then, external momenta verifies :

$$p_i^2 \ll k^2$$

- But, because of the structure of equations, internal momenta are limited by :

$$q^2 \lesssim k^2$$

Derivative Expansion II

- Then, the sector

$$p_i^2, q^2 \lesssim k^2$$

is insensitive to other momenta: decoupling property).

- Then it makes sense to formulate an approximation for this sector alone.
- One approximation very used : Expand vertices in this sector as a polynomial in momenta (Derivative Expansion).
- This is equivalent to say that the set:

$$\{\Gamma_k^{(2)}(0,0), \Gamma_k^{(4)}(0,0,0,0), \Gamma_k^{(6)}(0,0,0,0,0,0), \dots\}$$

is approximatelly 'closed' on the NPRG flow.

Order ∂^0: Local Potential Approximation

In practice: one encodes all vertices at zero momenta by a single function: the effective potential using:

$$V_k(\phi) = \sum_n \frac{1}{n!}\Gamma_k^{(n)}(0, 0, \ldots, 0)\phi^n$$

The exact equation for the potential is then :

$$\partial_k V_k(\phi) = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\partial_k R_k(q^2) G_k(q^2),$$

where

$$\Gamma_k^{(2)}(q^2) + R_k(q^2) = G_k^{-1}(q^2).$$

Given the fact that $q \lesssim k$ one can, in a first approximation neglect the non-bare dependence on q on the 2-point function:

$$\Gamma_k^{(2)}(q^2) \sim q^2 + \Gamma_k^{(2)}(0) = q^2 + V''(\phi).$$

Order ∂^0: Local Potential Approximation II

- This gives the zero order of the derivative expansion (LPA):

$$\partial_k V_k(\phi) = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\frac{\partial_k R_k(q^2)}{q^2 + V''(\phi) + R_k(q^2)}.$$

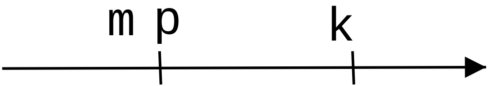
- It is equivalent to the ansatz :

$$\Gamma_k = \int d^d x \left\{\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V_k(\phi)\right\} + \mathcal{O}(\partial^2)$$

- In the $O(N)$ case, the LPA potential is exact at large N.
- One can consider polynomials of higher orders in momenta (Derivative Expansion at highers orders).

Parameter of Derivative Expansion I

- It is not an expansion in a small parameter:
  $p \leftrightsquigarrow$ order of magnitude of external momenta.
  $q \leftrightsquigarrow$ momenta circulating in the loop.
  $m_k \leftrightsquigarrow$ smallest mass.

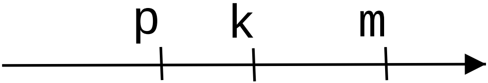


- If $p, m_k \ll k$, the expansion is on:

$$\sim \frac{q^2}{k^2} = \mathcal{O}(1)$$

$$\sim \frac{q^2}{k^2} = \mathcal{O}(\eta_k)$$

(still to be proved)



- When $k \ll m_k$ the regulator do not play any role $\Rightarrow$ the flow essentially stops. $\Rightarrow$ for arbitrary masses, the expansion parameter is

$$\sim \frac{q^2}{k^2 + m_k^2} = \begin{cases} \mathcal{O}(1) \text{ or } \mathcal{O}(\eta_k)? \text{ when } k \gg m_k \\ \mathcal{O}\left(\frac{q^2}{m_k^2}\right) \text{ when } k \ll m_k \end{cases}$$

- In a rough approximation, when $k \ll m_0$,

$$\Gamma_k^{(n)}(p, m_k) \sim \Gamma_{m_0}^{(n)}(p, m_0).$$

Parameter of Derivative Expansion II

- For arbitrary momenta p, if $m_k \ll k$, in the Derivative Expansion one expands also in

$$\sim \frac{p^2}{k^2}$$

- In the physical limit $k \to 0$, the result may only be valid if $p \sim 0$.
- Exception: when $p \lesssim m_0$. Then the flow stops before $k \sim p$ $\Rightarrow$ the expansion parameters are $\sim \frac{p^2}{k^2+m_k^2}$ and $\sim \frac{q^2}{k^2+m_k^2}$
- In the perturbative regime: Convergence to exact two loop expressions is fast. However, at any order, derivative expansion misses exact perturbative results beyond one loop.

Another approximation scheme

- The original interest in the new approximation scheme to be presented was to calculate correlation functions at arbitrary momenta.
- For finite momenta the derivative expansion can only work for external momenta $p^2 \lesssim m^2$, where m^2 is the smallest physical mass.
- To look at arbitrary momenta another strategy must be followed.
- We choose one where, as in the derivative expansion, an infinite number of vertices are included at once.
- To do so, we observed, that the set:

$$\{\Gamma_k^{(2)}(p,-p), \Gamma_k^{(4)}(p,-p,0,0), \Gamma_k^{(6)}(p,-p,0,0,0,0), \dots\}$$

is approximatelly closed on the NPRG flow.

To show this: observe first that one can encode all these vertices by a single function, the two point function in a constant external field, using:

$$\Gamma_k^{(2)}(p,-p;\phi) = \sum_n \frac{1}{n!} \Gamma_k^{(2+n)}(p,-p,0,0,\ldots,0)\phi^n$$

where we used

$$\Gamma_k^{(n+2)}(p,-p,0,\ldots,0;\phi) = \frac{\partial^n \Gamma_k^{(2)}(p,-p;\phi)}{\partial \phi^n}.$$

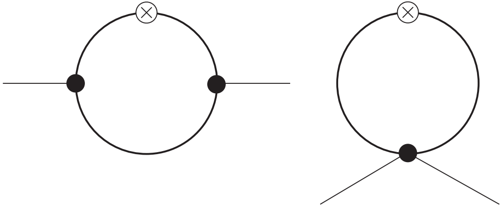
- Its exact flow equation is:

$$\partial_k \Gamma_k^{(2)}(p,-p;\phi) = \int \frac{d^d q}{(2\pi)^d} \partial_k R_k(q) \Big\{ G_k(q^2;\phi)\Gamma_k^{(3)}(p,q,-p-q;\phi) \times G_k((q+p)^2;\phi)\Gamma_k^{(3)}(-p,p+q,-q;\phi) G_k(q^2;\phi) - \frac{1}{2} G_k(q^2;\phi)\Gamma_k^{(4)}(p,-p,q,-q;\phi) G_k(q^2;\phi) \Big\},$$

where

$$G_k^{-1}(q^2;\phi) = \Gamma_k^{(2)}(q,-q;\phi) + R_k(q^2).$$

- The corresponding diagrams are



- Consider an approximation of the same nature that derivative expansion:

$$\Gamma_k^{(n)}(p_1, p_2, \ldots, p_{n-1}+q, p_n-q; \phi) \sim \Gamma_k^{(n)}(p_1, p_2, \ldots, p_{n-1}, p_n; \phi).$$

where $q^2 < k^2$.

- The equation for the two point functions becomes at zero order in q $(q \to 0)$:

$$\begin{aligned}\partial_k \Gamma_k^{(2)}(p, -p; \phi) \; = \int \frac{d^d q}{(2\pi)^d} \partial_k R_k(q^2) \Big\{ & G_k(q^2; \phi) \Gamma_k^{(3)}(p, 0, -p; \phi) \\ & \times G_k((q+p)^2; \phi) \Gamma_k^{(3)}(p, -p, 0; \phi) G_k(q^2; \phi) \\ & -\frac{1}{2} G_k(q^2; \phi) \Gamma_k^{(4)}(p, -p, 0, 0; \phi) G_k(q^2; \phi) \Big\}\end{aligned}$$

- But:

$$\Gamma_k^{(n+1)}(p_1, p_2, \ldots, p_n, 0; \phi) = \frac{\partial \Gamma_k^{(n)}(p_1, p_2, \ldots, p_n; \phi)}{\partial \phi}.$$

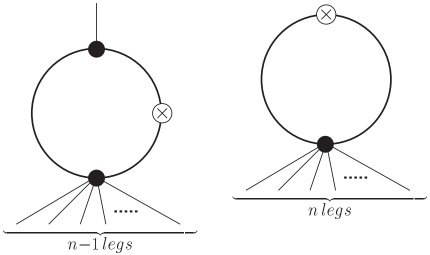
- Replacing in $\Gamma_k^{(2)}(p, -p; \phi)$ equation:

$$\partial_k \Gamma_k^{(2)}(p, -p; \phi) = \int \frac{d^d q}{(2\pi)^d} \, \partial_k R_k(q^2) \; G_k^2(q^2; \phi)$$
$$\times \left\{ \left(\frac{\partial \Gamma_k^{(2)}(p, -p; \phi)}{\partial \phi} \right)^2 G_k((p+q)^2; \phi) \; - \; \frac{1}{2} \frac{\partial^2 \Gamma_k^{(2)}(p, -p; \phi)}{\partial \phi^2} \right\}.$$

- A closed equation is obtained.

Higher vertices

- Also applies to any n-point vertex in an external field.
- Diagrams for a n-point vertex with higher order vertices:



Higher orders of the approximation

- Improving the approximation:
  - Consider exact equations for vertices (in a constant external field) up to $\Gamma_k^{(n-2)}$,
  - Perform the leading order approximation *only* in equations for $\Gamma_k^{(n-1)}$ and $\Gamma_k^{(n)}$ vertices.
- In the perturbative regime, this scheme is exact at m loops for a n-point function if all the equations up to that for the $n+2m-2$-point function are included.
- It also reproduces derivative expansion at any desired order.

A technical point

- The equation for $\Gamma_k^{(2)}(p^2;\phi)$ contains also information about the potential:

$$\Gamma_k^{(2)}(p^2=0;\phi) = \frac{\partial^2 V_k(\phi)}{\partial\phi^2}.$$

- In order to respect this relation and do not calculate twice redundant information, it is better instead of using $\Gamma_k^{(2)}(p^2;\phi)$, to use the set

$$\begin{cases} V_k(\phi), \\ \Delta_k^{(2)}(p^2;\phi) = \Gamma_k^{(2)}(p^2;\phi) - \Gamma_k^{(2)}(p^2=0;\phi) - p^2. \end{cases}$$

- $\Gamma_k^{(2)}(p^2;\phi)$ can then be recovered by:

$$\Gamma_k^{(2)}(p^2;\phi) = p^2 + V_k''(\phi) + \Delta_k^{(2)}(p^2;\phi)$$

- The same idea can be applied for higher vertices.

Example: Instead of $\Gamma_k^{(3)}(p_1,p_2,p_3;\phi)$ it is convenient to use:

$$\phi\Delta_k^{(3)}(p_1,p_2,p_3;\phi) = \Gamma_k^{(3)}(p_1,p_2,p_3;\phi) - \frac{1}{2}\frac{\partial\Gamma_k^{(2)}(p_1^2;\phi)}{\partial\phi} - \frac{1}{2}\frac{\partial\Gamma_k^{(2)}(p_2^2;\phi)}{\partial\phi} - \frac{1}{2}\frac{\partial\Gamma_k^{(2)}(p_3^2;\phi)}{\partial\phi} + \frac{1}{2}V'''(\phi)$$

Consequence: $\Delta_k^{(3)}(p_1,p_2,p_3;\phi) \to 0$ if, $p_1, p_2, or\ p_3 \to 0$.

Application to the $O(N)$ field theory

- Here the complete numerical solution for these equations for $O(N)$ models at criticality is presented.
- In this case, $\Gamma_k^{(2)}(p,-p;\phi)$ is a NxN matrix with the structure:

$$\Gamma_{ij}^{(2)}(p,-p;\phi;k) = \delta_{i,j}\Gamma_A(p,-p;\phi;k) + \phi_i\phi_j\Gamma_B(p,-p;\phi;k)$$

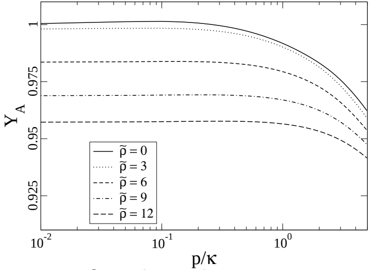
- Numerical difficulty: all different p are coupled!
- We used mainly $R_k(q^2) = \alpha \frac{Z_k q^2}{\exp(q^2/k^2)-1)}$
- We used the dependence on the shape of the regulator, by varying α in order to: Canet et al. '02
  - optimize results by choosing the minimal sensitivity of physical quantities as critical exponents.
  - estimate the error of a given result by studying its dependence on α.

Results for $O(N)$ critical exponents

N	η	η (other)	ν	ν (other)	ω (prelim.)	ω (other)
0	0.033(3)	0.028(3) [1]	0.588	0.588(1) [1]	0.80	
1	0.039(3)	0.0364(2) [2]	0.6298(4)	0.6301(2) [2]	0.78	0.79(1) [1]
		0.0368(2) [3]		0.6302(1) [3]		
		0.033(3) [1]		0.630(1) [1]		
2	0.041(3)	0.0381(2) [4]	0.6719(4)	0.6717(1) [4]	0.78	0.79(1) [1]
		0.035(3) [1]		0.670(2) [1]		
3	0.040(3)	0.0375(5) [5]	0.709	0.7112(5) [5]	0.73	
		0.036(3) [1]		0.707(4) [1]		
4	0.038(3)	0.035(5)[1]	0.738	0.741(6) [1]	0.74	0.77(2) [1]
		0.037(1) [6]		0.749(2) [6]		
5	0.035(3)	0.031(3) [8]	0.768	0.764(4) [8]	0.73	0.77(2) [1]
		0.034(1) [7]		0.779(3) [7]		
10	0.022(2)	0.024 [9]	0.860	0.859 [9]	0.81	
20	0.012(1)	0.014 [9]	0.929	0.930 [9]	0.94	
100	0.0023(2)	0.0027 [10]		0.989 [10]	0.99	

[1] R. Guida and J. Zinn-Justin '98. [2] M. Campostrini, A. Pelissetto, P. Rossi, E. Vicari '02.
[3] Y. Deng and H. W. J. Blote '03. [4] M. Campostrini, M. Hasenbusch, A. Pelissetto, E. Vicari '06.
[5] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, E. Vicari '02. [6] M. Hasenbusch '01.
[7] M. Hasenbusch, A. Pelissetto, E. Vicari '05. [8] A. Butti and F. Parisen Toldin '05.
[9] S. A. Antonenko and A. I. Sokolov '95. [10] M. Moshe and J. Zinn-Justin '03.

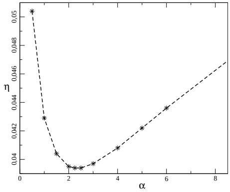
- Scailing property: We obtain *the same* exponent η from the momentum dependence of $\Gamma_k^{(2)}(p,-p;\phi)$
- Even more: the full dimensionless function $Y_A(p/k) = \Gamma_k^{(2)}(p,-p;\phi)/(Z_k\, p^2)$ approach a fixed form in the IR:



- Precision results also for $d = 2$! $eta_{d=2} = 0.26$ (preliminary).
- Last but not least: the method also allows for the calculation of correlation function at arbitrary momenta. (see Blaizot's talk).

Error estimate

- To estimate errors, we studied the dependence on the regulator by varying α.
- If no approximation $\Rightarrow$ physical quantities independent on α.
- A dependence on α is a measure of the error introduced by approximations. Canet et al. '02
- In given an approximation, it is reasonable to choose the value of α_{PMS} that gives a minimum sensitivity to this parameter.
- Rough estimate of the error: consider the variation from $\alpha_{PMS}/2$ to $2\alpha_{PMS}$.



Conclusion

- NPRG equations are an ideal context to perform non perturbative approximations.
- A very successful approximation scheme in this context is Derivative Expansion.
- It exploits specific properties of the NPRG.
- However:
    - It does not reproduce exact at two loops expressions,
    - It only works for very small momenta.
- We exploited the same properties of NPRG and obtained an approximation scheme that:
    - Allows to calculate correlations functions at arbitrary momenta.
    - Reproduces Derivative Expansion for small momenta.
    - Reproduces Perturbation Theory at any order.
    - For $O(N)$ models it is exact at LO for all vertices if $N \to \infty$.
    - Reproduces correctly the cross-over region (see Blaizot's talk).

- In progress or near future:
  - Numerical analysis of improved version of LO;
  - Improving the error estimate;
  - Finite temperature (see Ipp's talk);
  - Correlation functions in an external magnetic field;
  - Spectrum in different phases;
  - Numerical analysis of NLO;
  - Out of equilibrium phenomena (see Canet's talk);
  - Yang-Mills theory.