Analytical approximation schemes for solving exact renormalization group equations

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Wilson-Polchinski's fixed point equation in the LPA

LPA for $d = 3$, $U(\phi)$ is the potential:

$$U'' - (U')^2 - \frac{\phi}{2}U' + 3U = 0, \quad U'(0) = 0, \quad U(0) = k,$$

$$k = k^* \Longrightarrow U^*(\phi) \underset{\phi\to\infty}{=} \frac{1}{2}\phi^2 - \frac{1}{3} + A^*\phi^{6/5} + \mathrm{O}\left[\phi^{2/5}\right]$$

Wilson-Fisher fixed point

Expansion about the origin $\phi = 0$: $U_M(\phi) = k + \sum_{n=1}^{M} a_n(k)\phi^{2n}$

$$\text{For } M = 2: \quad \left.\begin{array}{l} 2a_1 - 3k = 0 \\ 12a_2 - 4a_1^2 - 4a_1 = 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} a_1(k) = 3k/2 \\ a_2(k) = k(3k+2)/4 \end{array}\right.$$

$$a_2(k) = 0 \Longrightarrow \left\{\begin{array}{ll} k^* = 0 & \text{Gaussian} \\ k^* = -2/3 & \to \text{NOT Wilson} - \text{Fisher!}\ (k^* \simeq 0.0762) \end{array}\right.$$

MOP's method $\Longrightarrow$ $a_M(k) = 0$ $\longrightarrow$ All $k^* < 0$ for any M

Margaritis, Ódor & Patkós, 1988; Fernández & Castro, 1981; see also Biswas et al. 1973 (Hill determinant)

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The simple methods does not converge

- MOP's method works for Wegner-Houghton's, Litim's, Morris', Wetterich's equations (Average action)

  (This is independent of the radius of convergence of the series about the origin: In the LPA, the Wilson-Polchinski and Litim series have similar radius of convergence)

- It does not converge

  Fernández & Castro, 1987; Aoki, Morikawa, Souma, Sumi & Terao, 1998

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Expansion about $\phi_0 \neq 0$, $U'(\phi_0) = 0$: Tetradis & Wetterich, Alford, 1994

$U(\phi) = b_0 + \sum_{n=1}^{M} b_n \left(\phi_0 - \phi\right)^n$ is more efficient but:

- does not always work (Wilson-Polchinski)
- two conditions are needed: $b_M = 0$ and $b_{M-1} = 0$
- does not converge Aoki et al, 1998
- requires a radius of convergence of the series larger than ϕ_0 (this radius decreases when $d \to 2$).

The asymptotic behavior of the solution is not accounted for. No attempt is made to continue the solution towards large ϕ.

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The Padé-Hankel method (PAD)

- Starting with: $f_{M-1}(z) = k + \sum_{n=1}^{M-1} a_n(k) z^n$ (with $z = \phi^2$, $f(z) = U(\phi)$)
- Construct the Padé $[N_1, N_2]$: $g(z) = \frac{\sum_{j=0}^{N_1} b_j z^j}{\sum_{j=0}^{N_2} c_j z^j}$ with $N_1 + N_2 + 1 = M$ $(c_0 = 1)$
- Impose that this construction is again true at next order (for $M - 1 \to M$)
- $\Longrightarrow$ linear homogeneous system of equations for the $N_2 + 1$ coefficients c_i, $i = 0, \cdots, N2$, the determinant of which (a polynomial in k) must vanish

$$H_M(k) = \left|a_{i+j+N_1-N_2+1}(k)\right|_{i,j=0,\cdots,N_2} = 0$$

Hankel matrix: constant skew diagonals

- The zeros of this polynomial in k are candidate to give the value k^* we are looking for. The explicit Padé approximant provides an approximation of the global solution $U^*(\phi)$ $(\phi \in [0, \infty[)$

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Generalized hypergeometric functions (HFA)

- $S(z) = \sum_{n=0}^{\infty} h_n z^n$ such that $\frac{h_n}{h_{n+1}} = \frac{P(n)}{Q(n)}$, where P and Q are polynomials. $S(z)$: generalized hypergeometric function
- construct the ratio of two polynomials in n: $(\sum_{i=1}^{m1} b_i n^{i-1})/(\sum_{i=1}^{m2} c_i n^{i-1}))$, so that they match the $M-2$ ratios $a_{n+1}(k)/a_n(k)$ for $n = 1, \cdots, M-2$.
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Auxiliary differential equation (ADE)

- Determine the coefficients $G_i(k)$ so that, at order M, $f_M(z) = k + \sum_{n=1}^{M} a_n(k) z^n$ be also solution of the differential equation:

$$G_1 + G_2\, f + G_3\, f' + G_4\, f^2 + G_5\, f\, f' + G_6 f'^2 + \cdots + G_m\, f^{s-q}\, f'^q = 0$$

$m = s(s+1)/2 + q + 1.$

- Impose the conditions at infinity; e.g.: $f(z) \underset{z\to\infty}{=} 1$, $f'(z) \underset{z\to\infty}{=} 0$

$$G_1(k) + G_2(k) + G_4(k) + \cdots + G_s(k) = 0$$

- Again a polynomial in k, the zeros of which are candidate to give the value k^* we are looking for. More general, but no direct explicit approximation of the global solution $U^*(\phi)$ ($\phi \in [0, \infty[$)

Boisseau, Forgacs & Giacomini, 2007

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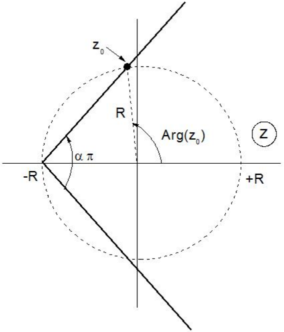
Conformal mapping (MAP)



- Perform the conformal mapping of the angular sector onto the unit circle centered at the origin: $w = \frac{(1+z/R)^{1/\alpha}-1}{(1+z/R)^{1/\alpha}+1}$
- $f_M(z) \to g_M(w) = \sum_{n=0}^{M} b_n(k) w^k$, this series converges onto the whole disc $|w| < 1$.
  The condition at infinity may be imposed: $g_M(1) = \sum_{n=0}^{M} b_n(k) = 1$
  or simply $b_M(k) = 0$
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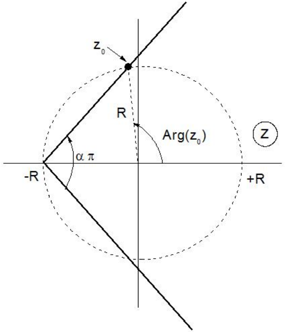
CB, Boisseau, & Giacomini, 2008

Conformal mapping (MAP)



- Perform the conformal mapping of the angular sector onto the unit circle centered at the origin: $w = \frac{(1+z/R)^{1/\alpha}-1}{(1+z/R)^{1/\alpha}+1}$
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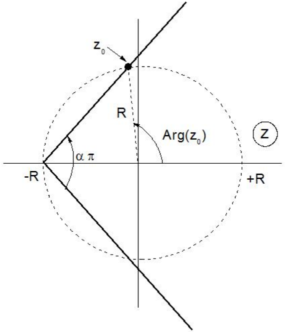

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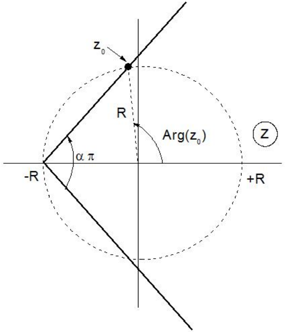

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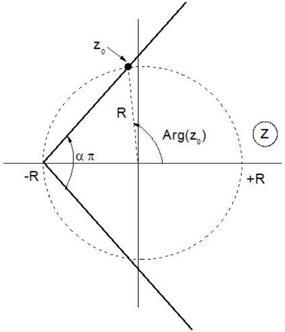

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CB, Boisseau, & Giacomini, 2008

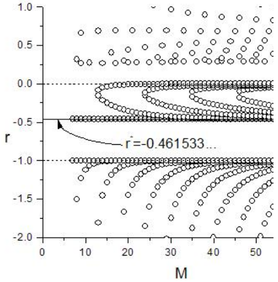
Comparison of the efficiency of the four methods

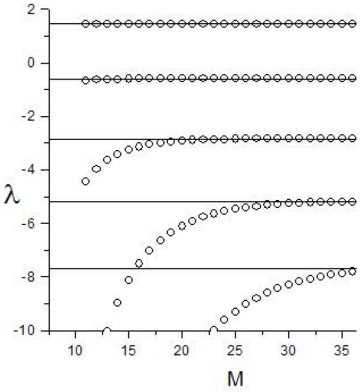
method	k^*	time
ADE	0.07619940081236**5**	1523.84
PAD	0.07619940081205	1364.73
HFA	0.07619940081234**0**	138.58
MAP	0.0761994008160	2.00

Table: Comparison between estimates of the connection parameter k^*, of the Wilson-Polchinski RG equation in the LPA ($d = 3$), obtained using different efficient analytical methods at order $M = 25$ of the Taylor polynomial. The "time" given in the third column is a CPU time (in seconds) corresponding to the calculation, on the same computer, using each method.

Wegner-Houghton $d = 3$, LPA (MAP)

$R = 9.7344$, $\alpha \simeq 1/2$ Morris, 1994





Fixed point: $r^* = -0.461533720116207119965757 6484$
with $M = 145$ ($r = U''(0)$)

Eigenvalues: $\nu = 0.689459056162134 84062727$
with $M = 104$

Aoki, Morikawa, Souma, Sumi & Terao, 1998

Wilson-Polchinski $d = 3$, LPA (MAP)

- Fixed point ($d = 3$): $R = 5.72167$, $\alpha = 5/2$

  $k^* = 0.0761994008123406414578853691323490628080181433621 4 \pm 6 \times 10^{-50}$

  for $M = 120$

- Eigenvalues:
  - for $M = 75$ in the even case ($d = 3$):

$$
\begin{aligned}
\nu &= 0.649561773880648017614299724015827 \pm 2 \times 10^{-33}\,, \\
\omega_1 &= 0.6557459391933387407836879749684 \pm 2 \times 10^{-31}\,, \\
\omega_2 &= 3.1800065120591675323 14140242\,, \\
\omega_3 &= 5.9122306127477010263 51105\,, \\
\omega_4 &= 8.7960928254139036 43907\,, \\
\omega_5 &= 11.798087658336857239\,.
\end{aligned}
$$

  - for $M = 69$ in the odd case ($d = 3$):

$$
\begin{aligned}
\breve{\omega}_1 &= 1.8867038380914203710417873172 \pm 5.3 \times 10^{-28}\,, \\
\breve{\omega}_2 &= 4.524390733670772780436353\,, \\
\breve{\omega}_3 &= 7.3376506433543135387526\,, \\
\breve{\omega}_4 &= 10.283900724025958 1722\,, \\
\breve{\omega}_5 &= 13.3361699643459431\,.
\end{aligned}
$$

CB, Jüttner & Litim, 2007

Morris' equations, LPA and $O\left(\partial^2\right)$ (MAP)

The reparametrization invariance has been imposed within the derivative expansion by choosing a particular cutoff function, (lack of optimisation, see Litim's talk).

	LPA ($M = 82,\ 60$) $R = 2$, $\alpha = 1/2$	$O\left(\partial^2\right)$ $M = 17$ $R = 2.5$, $\alpha = 2$
k^*	0.2753644064810282 0.275364406	0.258216 0.2582144
η	0	0.053941 0.05393208
ν	0.660389431 0.660389	0.618063 0.6181
ω_1	0.6285575 0.6285	0.8964 0.8975
$\breve{\omega}_1$	1.8124863608 —	0.86562 —

Morris, 1994 & 1997

Best values: $\nu \simeq 0.63$ (0.639) $\omega \simeq 0.78$ (0.763)

Problem with $\breve{\omega}_1$

$\breve{\omega}_1 = 2.34 \pm 0.49$ Zhang, Zia, 1982 (ϵ-exp.)
$\breve{\omega}_1 = 2.4 \pm 0.4$ Newman, Riedel, 1984 (scal. field)
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MERCI !