High-accuracy RG studies of scaling exponents in the LPA

ERG08
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Motivation

- full analytical solutions to QFTs are rather rare
- still:
  - make analytical predictions for a certain range of parameters
  - simplify the theory such that an analytical solution is possible
  - use numerics
- learn about tools and ways to solve the full theory

Motivation

we study $O(N)$ symmetric real scalar field theories in 3 dimensions at criticality using ERG techniques

- $N = 0$ properties of long polymer chains
- $N = 1$ liquid-vapour transition
- $N = 2$ *He* superfluid transition
- $N = 3$ ferromagnetic phase transition (Heisenberg model)
- $N = 4$ $N_f = 2$ QCD phase transition

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in particular

- $N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30$
- the fixed point potential
- leading and sub-leading corrections to scaling
- global structure of the scaling potential
- methods for high accuracy computations
- $N = 1$ has been studied and published in *[Bervillier, Litm, J., NPB 783 (2007) 213]*

ERG

- we want to trace the evolution of the effective action under an infinitesimal change of the cut-off scale - exact flow equation:

$$
\begin{aligned}
\partial_t \Gamma_k[\phi] &= \tfrac{1}{2}\int \frac{d^3q}{(2\pi)^3}\left(\frac{\delta\Gamma_k}{\delta\phi(-q)\delta\phi(q)} + R_k\right)^{-1}\partial_t R_k \\
&\equiv \tfrac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_t R_k\right\}
\end{aligned}
$$

[Wetterich, PLB301:90-94,1993]

- LPA effective action

$$
\Gamma_k = \int d^3x\left(\frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi^a + V_k(\phi^2)\right) \qquad (a = 1, ..., N)
$$

The cut-off function - optimised flow

- optimised cut-off function

$$R_k(q^2) = (k^2 - q^2)\theta(k^2 - q^2)$$

[Litim, PRD 64 (2001), PLB 486 (2000)]

  - $q^2 > k^2$: large momenta, propagation of modes fully suppressed (R=0)
  - $q^2 < k^2$: small momenta, all modes propagate with eff. mass $= k^2$ (IR-scale)
- rescaling: $u(\rho) = V_k/k^d$ and $\rho = \frac{1}{2}\phi^a\phi_a k^{2-d}$
- for general N the flow equation takes the simple form

$$\partial_t u = -3u + \rho u' + \frac{1}{6\pi^2}\frac{N-1}{1+u'} + \frac{1}{6\pi^2}\frac{1}{1+u'+2\rho u''}$$

- ODE that allows to study the potential
- the fixed point equation is $\partial_t u' = 0$
- study
  - global solution for the scaling potential
  - critical exponents

Solving the flow equation

i) Small field approximation

approximate the potential

$$u(\rho) = \sum_{n=1}^{n_{\rm trunc}} \frac{1}{n!} \lambda_n \rho^n$$

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inserting this into the fixed point equation

$$\partial_t u' = 0 = -2u' + \rho u'' - \frac{1}{6\pi^2}(N-1)\frac{u''}{(1+u')^2} - \frac{1}{6\pi^2}\frac{3u''+2\rho u'''}{(1+u'+2\rho u'')^2}$$

yields a set of $n_{\rm trunc}$ coupled ordinary differential equations

$$\partial_t \lambda_i \equiv \beta_i(\{\lambda_n\})$$

for couplings $\{\lambda_n, 1 \le n \le n_{\rm trunc}\}$

Solving the flow equation

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there is only one unknown:

$$\begin{aligned}
\lambda_0 &= \frac{1}{18}\pi^{-2}(1+\lambda_1)^{-1},\\
\lambda_2 &= 4\pi^2\lambda_1(1+\lambda_1)^{-2},\\
\lambda_3 &= \frac{72}{15}\pi^{-4}\lambda_1(1+\lambda_1)^{-3}(1+13\lambda_1),\\
&\ldots
\end{aligned}$$

Solving the flow equation

ii) Large field behaviour

- $\rho \to \infty$ asymptotic behaviour known analytically: expansion in Laurent series

$$u' = \gamma\rho^2 + \gamma_3\rho^{-3} + \gamma_5\rho^{-5} + \sum_{n=7}\gamma_n\rho^{-n}$$

again the coefficients are related algebraicly, e.g. $N = 1$:

$$\begin{aligned} \gamma_3 &= -\frac{2}{25\gamma} \\ \gamma_5 &= \frac{4}{175\gamma^2} \\ \gamma_7 &= -\frac{2}{375\gamma^3} \\ \ldots \end{aligned}$$

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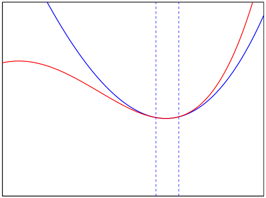
ideally: convergence radii of small and large field approximations overlap

Solving the flow equation

iii) intermediate to large field behaviour

- start integration from matching point with series approximation



Solving the flow equation

here: approximate the potential about the minimum

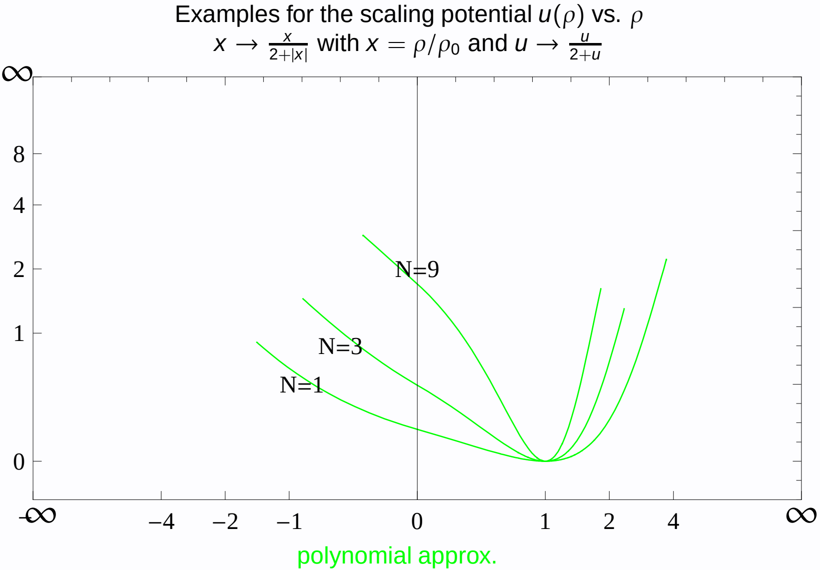
$$u(\rho) = \sum_{n=2}^{n_{\text{trunc}}} \frac{1}{n!} \lambda_n (\rho - \lambda_1)^n$$

- λ_1 is potential minimum - $u'(\lambda_1) = 0$
- λ_n is nth order coupling at the potential minimum
- for $u(\rho) = \sum_{n=2}^{n_{\text{trunc}}} \frac{1}{n!} \lambda_n (\rho - \lambda_1)^n$ we employ "boundary conditions": $\lambda_{n_{\text{trunc}}+1} = 0 = \lambda_{n_{\text{trunc}}+2}$

Our study

- Litim optimized flow equation
- $n_{\text{trunc}} \leq 32$ in our computations
- $N = 1, 2, 3, \ldots, 10, 20, 30$

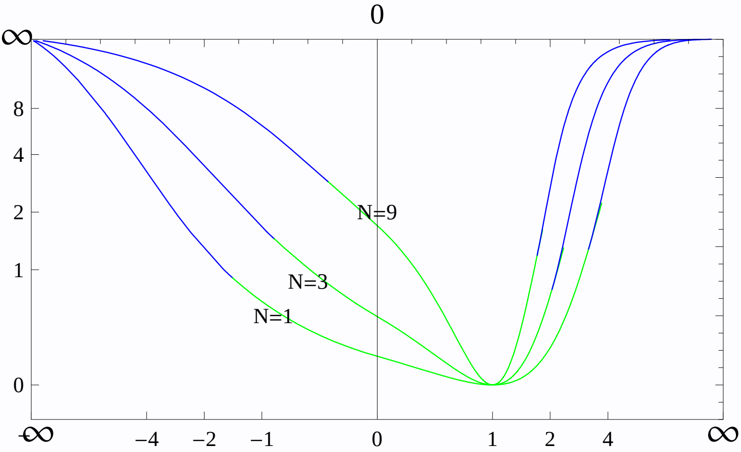
Results - scaling potential



Results - scaling potential

Examples for the scaling potential $u(\rho)$ vs. ρ

$x \rightarrow \frac{x}{2+|x|}$ with $x = \rho/\rho_0$ and $u \rightarrow \frac{u}{2+u}$



polynomial approx., boundary value problem, integrated

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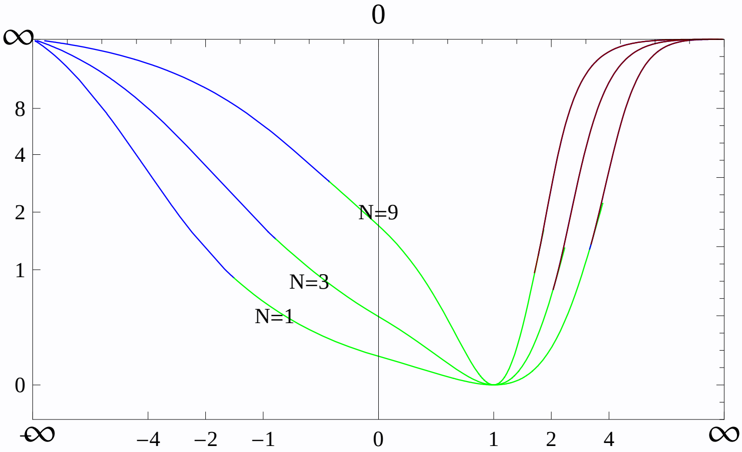

polynomial approx., boundary value problem, integrated, large field asymptotic

for large enough n_{trunc} the small and large field asymptotics overlap

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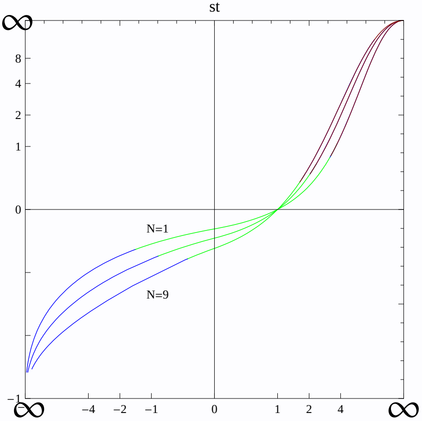
The first derivative of the scaling potential: $u'(\rho)$ vs. ρ



Results - scaling potential

Examples for the scaling potential $u(\rho)$ vs. ρ

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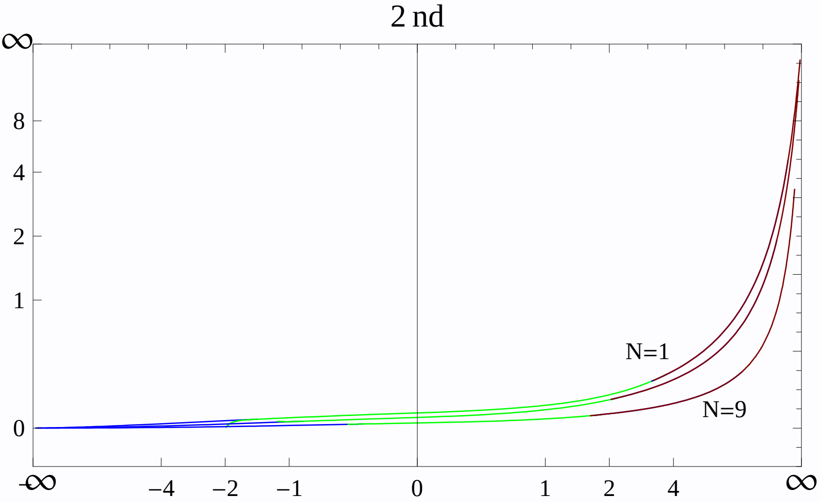
The second derivative of the scaling potential: $u''(\rho)$ vs. ρ



Solving the flow equation - critical exponents

- at the fixed point $\partial_t u' = 0$ the flow of the couplings vanishes

$$\partial_t \lambda_i = \beta_i(\{\lambda_i\}) = 0$$

- critical exponents are determined as eigenvalues of the stability matrix

$$M_{ij} = \frac{\partial \beta_i}{\partial \lambda_j}|_{\partial_t u' = 0}$$

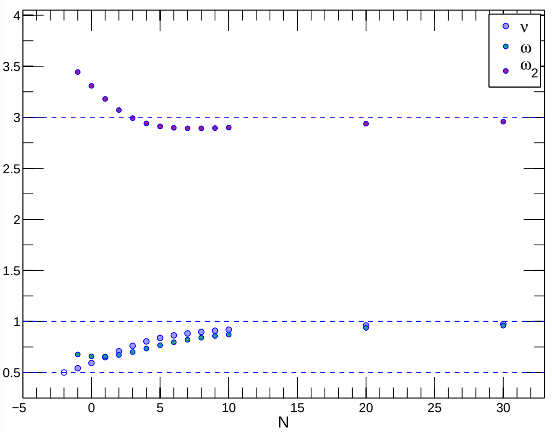
Results - scaling exponents

Results for $n_{\text{trunc}} = 32$

N	λ_1	ν	ω
1	1.814898403687	0.6495617739	0.65574594
2	2.5839400502815	0.70821090749	0.6712211945
3	3.425833929683	0.761123137153	0.699837318
4	4.320912329980	0.8043476966	0.7337529264
5	5.25099710744	0.8377407110	0.766735296
6	6.20317562546	0.8630761595	0.795814494
7	7.169312432296	0.88238895680	0.82031640463
8	8.14447579888273	0.897337664626	0.840612280535
9	9.12566549143533	0.9091281394508	0.857383970451
10	10.111014379319933	0.9186051231544	0.871310976591
20	20.050543706705838076	0.960678346035376052	0.93674237197849997
30	30.032591962018148734614	0.974173017876101998702	0.958441374389100042 56

Results

ν, ω and ω_2 for $N \to \infty$ ($\omega_n = 2N - 1 + O(1/N)$)

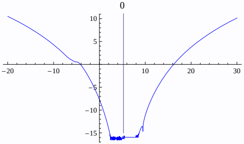


Error control

polynomial expansion:

- for a given solution compute $|\partial_t u'(\rho)|$ and then define $N_{\rm acc}$

$$10^{-N_{\rm acc}} = \max_{\rho\in[0,\rho_{\rm max}]} |\partial_t u'(\rho)|$$



- number of significant digits N_X in a coupling or critical index X at order n in the expansion:

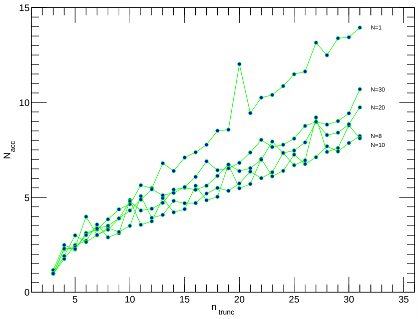
$$10^{-N_X} = \left|1 - \frac{X_n}{X}\right|$$

where $X \equiv X_{n_{\rm trunc}}$

Error control

Numerical precision

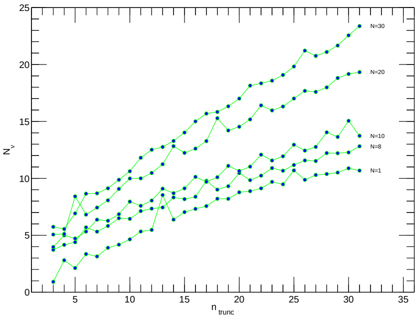
$$10^{-N_{\rm acc}} = \max_{\rho\in[0,\rho_{\rm max}]} |\partial_t u'(\rho)|$$

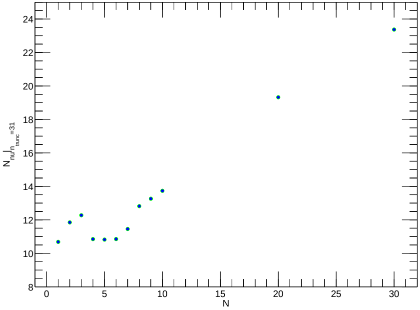


Error control

Numerical precision

$$10^{-N_\nu} = \left|1 - \frac{X_\nu}{X}\right|$$





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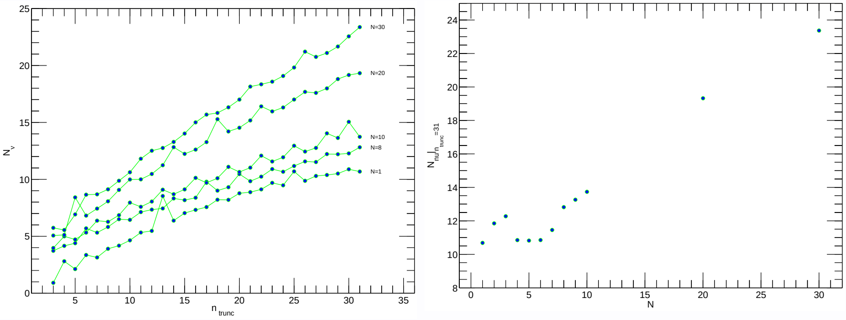
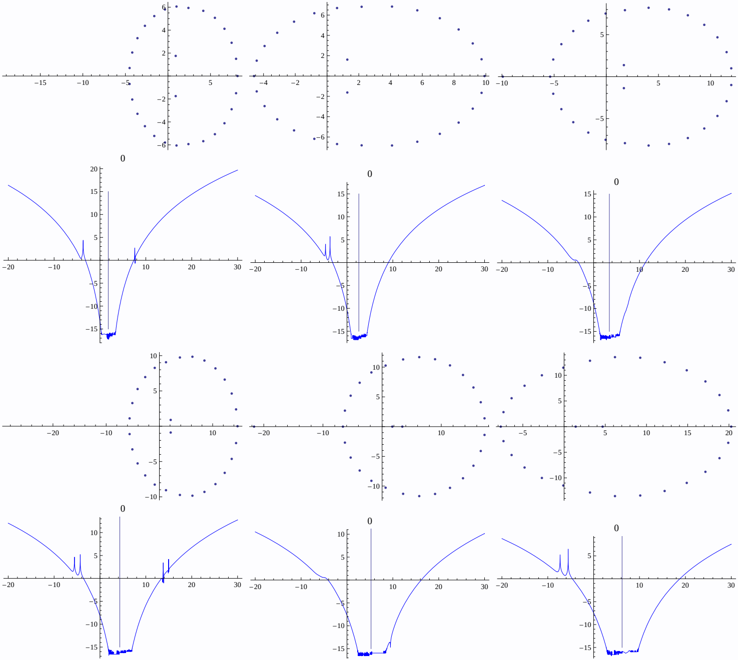


- with $n_{\mathrm{trunc}} = 32$ we confirm $\approx$11 significant digits for ν at $N = 1$
- less significant with each increasing order in the sub-leading corrections to scaling

Summary

- Analytical and numerical methods for studies of potential at criticality well under control
  - all couplings $\lambda_n(\lambda_1)$ known explicitly
  - λ_1: here numerical determination (RG-flow)
  - λ_1: algebraical determination $\rightarrow$ talk by Claude Bervillier
- high accuracy possible in the LPA
- polynomial approximation systematically improveable
- indications that small and large field expansions overlap
- mapped out universality classes $N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30$
- we have a very good picture of the scaling potential in the LPA

Summary



Summary