Linear and Nonlinear aspects of Finite Size Scaling

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Outline

1. Models with discrete finite size scaling (Hierarchical Model, MK)
2. How to control nonlinear effects in Binder cumulants (0712.1190)
3. The zero volume limit (as a way to learn about the infinite volume limit)
4. Attempts to reach continuous scaling (connection with Polchinski's equation and Litim's optimization)
5. Finite size effects for the density of states in lattice gauge theory (0807.0185)
6. Conclusions

Models with discrete Finite Size Scaling

Consider a lattice model in D dimensions, with lattice spacing a, linear size N, volume $V = N^D$ and nonlinear scaling variables u_i.

Under a RG transformation

$$a \to \ell a;\; N \to N/\ell \;; u_i \to \ell^{y_i} u_i$$

with ℓ a fixed value (e.g. 2) that cannot be shrunk to 1

For scalar models with average magnetization m

$$V_{eff}(\ell^{y_m} m, \ell^{y_i} u_i, N/\ell) = \ell^D V_{eff}(m, u_i, N)$$

For gauge models ($SU(2)$ hereafter) with $\mathcal{N}_p = \frac{D(D-1)}{2} V$ plaquettes

$$Z(\beta, \{\beta_i\}) = \int_0^{2\mathcal{N}_p} dS\ n(S, \{\beta_i\}) \mathrm{e}^{-\beta S} \ ,$$

$$n(S, \{\beta_i\}) = \prod_l \int dU_l \delta(S - \sum_p (1 - (1/N) ReTr(U_p))) \mathrm{e}^{-\sum_i \beta_i (1 - \chi_i(U_p)/d_i)}$$

$$f(s, \{\beta_i\}, \mathcal{N}_p) \equiv ln(n(s\mathcal{N}_p, \{\beta_i\}, \mathcal{N}_p))/\mathcal{N}_p \ .$$

can be used as the effective potential if we can find a RG transformation for the $\{\beta_i\}$ associated with the characters χ_i (e.g. Migdal-Kadanoff)

$$lim_{\mathcal{N}_p \to \infty} f(s, \{\beta_i\}, \mathcal{N}_p) = f(s, \{\beta_i\})$$

Dyson Hierarchical Model

2^n sites Labeled with n indices $x_n, ...x_1$, each index being 0 or 1 (think about a tree with n branching levels).

Kinetic term (sum over blocks of all 2^l sizes; not renormalized):

$$S = -\frac{1}{2}\sum_{l=1}^{n}(\frac{c}{4})^l \sum_{x_n,\dots,x_{l+1}} (\sum_{x_l,\dots,x_1} \phi_{(x_n,\dots,x_1)})^2$$

If $c = 2^{(D-2)/D}$, Gaussian fields scale like in D-dimensions

$\ell = 2^{\frac{1}{D}}$: "linear" scale factor (block spin: 2 sites$\rightarrow$1 site). $D = 3$ hereafter

Exact RG transformation affects only the local potential

Recursion Formula

Initial local measure: $W_0(\phi) = \delta(\phi^2 - 1)$ (Ising) or $W_0(\phi) = e^{-A\phi^2 - B\phi^4}$

Block spin transformation:

$$W_{n+1}(\phi) = C_{n+1} e^{\frac{\beta}{2}(\frac{c}{4})^{n+1}\phi^2} \int d\phi' W_n(\frac{(\phi-\phi')}{2}) W_n(\frac{(\phi+\phi')}{2}) \ ,$$

Fourier Representation of the RG transformation ($c = 2^{1-2//D}$)

$$R_{n+1}(k) = C_{n+1} exp(-\frac{1}{2}\beta\frac{\partial^2}{\partial k^2})(R_n(\frac{\sqrt{c}k}{2}))^2$$

M_n: the total field $\sum \phi_x$ inside blocks of side 2^n ;

$$R_n(k) = \sum_{q=0}^{\infty} \frac{(-ik)^{2q}}{(2q)!} \frac{<(M_n)^{2q}>_n}{(4/c)^{qn}}$$

Remarks

Polynomial truncations of $R_n(k)$: very accurate in the symmetric phase

You can calculate the RG flows, exponents, nonlinear scaling variables etc.. in the symmetric phase with any desired accuracy.

The model has an ERG but the model is not exactly what you want, so the hierarchical approximation needs to be improved.

For details see the "post-Lefkada" review article:

YM, *Nonlinear Aspects of the Renormalization Group Flows of Dyson's Hierarchical Model*, J. Phys. A 40 R39-102.

2.Nonlinear effects in Binder cumulants (0712.1190)

We consider the fourth order Binder cumulant

$$B_4 \equiv \frac{< m^4 >}{< m^2 >^2} = f(u_\kappa N^{1/\nu}, u_1 N^{-\omega_1}, u_2 N^{-\omega_2}, \dots)$$

u_k is the relevant scaling variable $u_k \simeq \kappa \equiv (\beta - \beta_c)/\beta_c$ in the linear approximation.

$< m^{2l} >$ are unsubtracted moments of the average spin.

At β_c and infinite volume, B_4 is a universal quantity.

Remarks

- For finite T $SU(N_{col.})$ lattice Gauge theory with $\beta = 2N_{col.}/g^2$:

$$(T - T_c)/T_c \simeq (\beta - \beta_c)12\pi^2/11N_{col.}^2 \ .$$

  as for 3 D spin models the ordered phase corresponds to $\beta > \beta_c$.

- For gauge theories, the calculations at large volume are expensive and require a good strategy (for the HM, brute force works well).

- We consider finite size scaling for isolated blocks of linear size N.

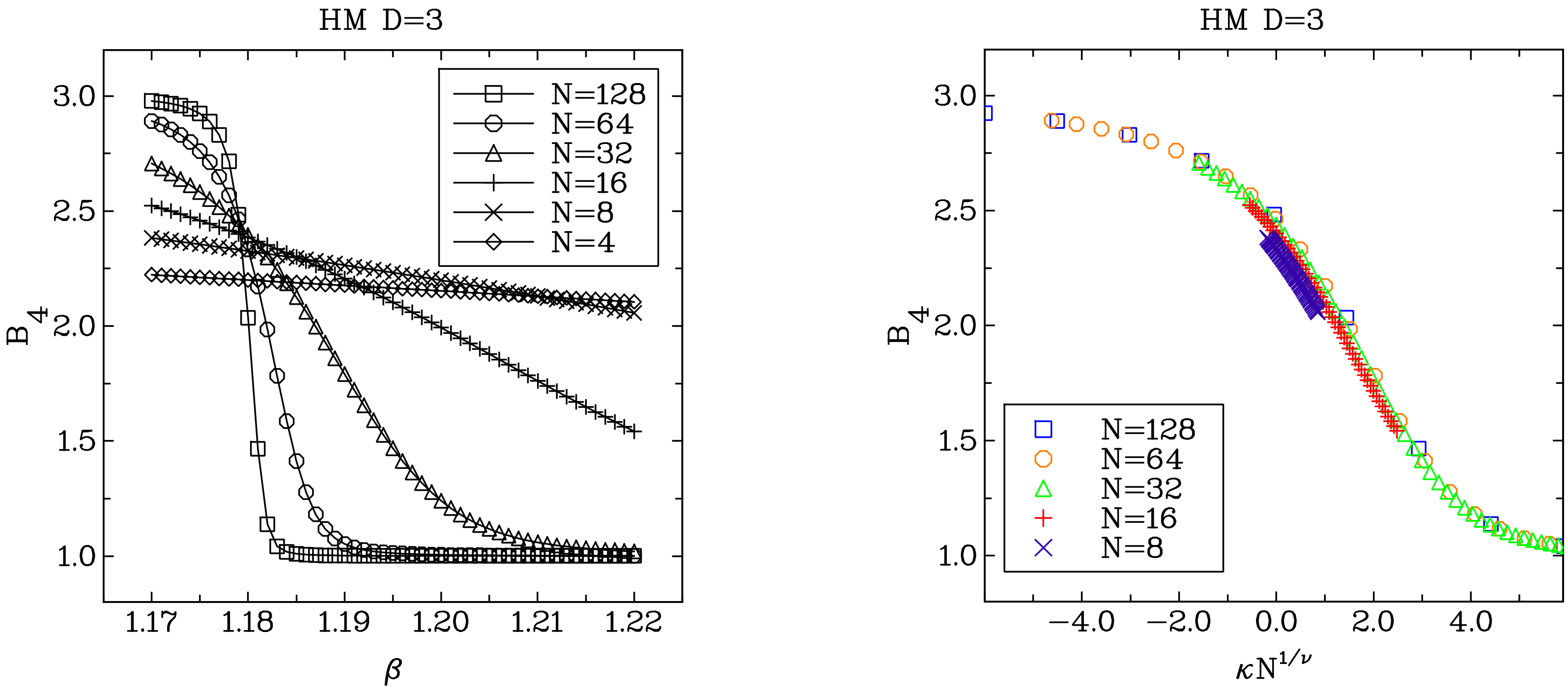

Figure 2: B_4 versus β (left) and $\kappa N^{1/\nu}$ (right), for $N = 8$, 16, 32, 64 and 128 for the Ising hierarchical model.

$$B_4(\beta,N) \simeq B_4(\beta_c,\infty) + f_1\kappa N^{1/\nu} + f_2\kappa^2 N^{2/\nu} + (c_0 + c_1\kappa N^{1/\nu})N^{-\omega} \ .$$

In the linear approximation ($f_2 = c_1 = 0$), we recover the standard linear FSS formula for the point of intersection denoted $(\beta^\star(N,N'), B_4^\star(N,N'))$ between the two curves $B_4(\beta,N)$ and $B_4(\beta,N')$, namely

$$\begin{aligned} \beta^\star(N,N') &= \beta_c + \beta_c(c_0/f_1)L(N,N') \ , \\ B_4^\star(N,N') &= B_4 + c_0 M(N,N') \ , \end{aligned} \tag{1}$$

with

$$\begin{aligned} L(N,N') &= (N^{-\omega} - N'^{-\omega})/(N'^{1/\nu} - N^{1/\nu}) \ , \\ M(N,N') &= (N^{-\omega-1/\nu} - N'^{-\omega-1/\nu})/(N'^{1/\nu} - N^{1/\nu}) \ . \end{aligned} \tag{2}$$

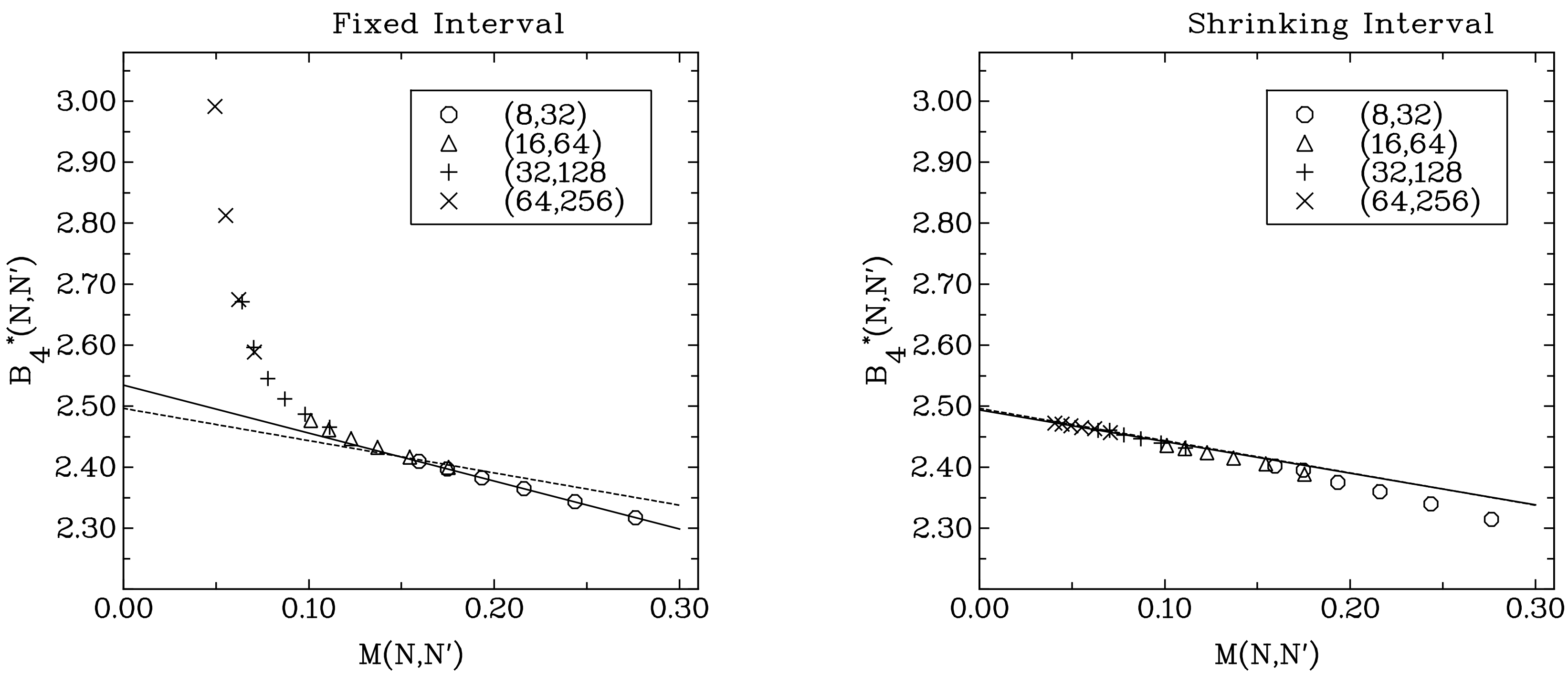

Figure 3: Empirical values of $B_4^\star(N,N')$ versus $M(N,N')$ obtained with the fixed interval procedure (left) and with the shrinking interval procedure (right) for 4 sets of 6 pairs of values. The solid line are linear fits. The dash line is the behavior expected from independent accurate calculations.

The shrinking interval procedure

In the literature, B_4 is often plotted for different volumes but at fixed values of β. It is better to shrink the interval as the volume increases. Given N and an estimate $\bar{\beta}_c$ of β_c from smaller volumes, we should restrict

$$|\beta - \bar{\beta}_c| < \epsilon (f_1/f_2) \bar{\beta}_c N^{-1/\nu} \ .$$

The value of ϵ needs to be chosen carefully. On one hand, we need ϵ small enough in order to control the nonlinear effects. On the other hand, if ϵ is very small, we need a correspondingly good estimate of β_c. In addition, when ϵ is too small, the intersections may be far away from the interval.

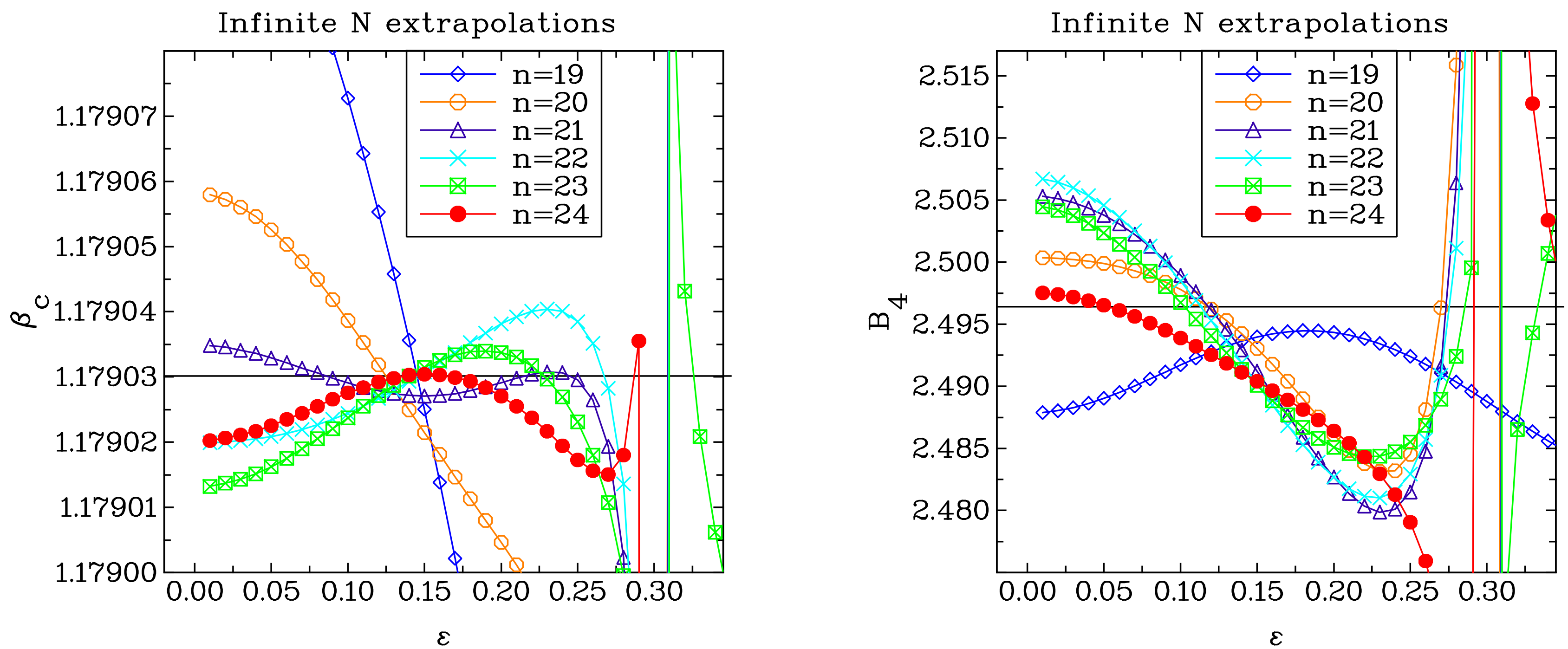

Figure 4: Infinite volume extrapolations of β_c and B_4 based on 15 point linear fits from the intersections among the B_4 curves at $N = 2^{n/3}$ and the 5 values of N immediately below, for n between 19 and 24.

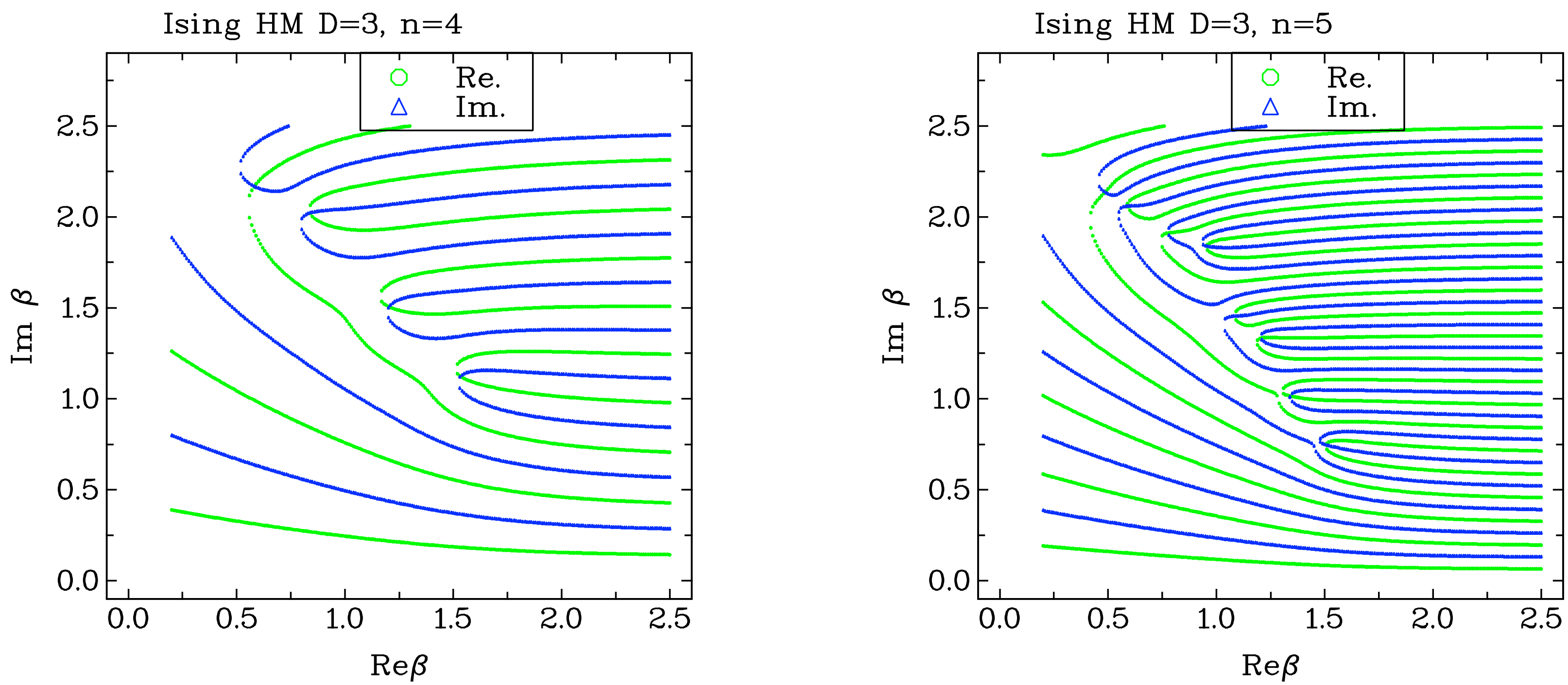

Figure 16: Zeros of the real (green) and imaginary (blue) part of Z for the $D = 3$ Ising HM for $n = 4$ and 5 ($V = 2^n$).

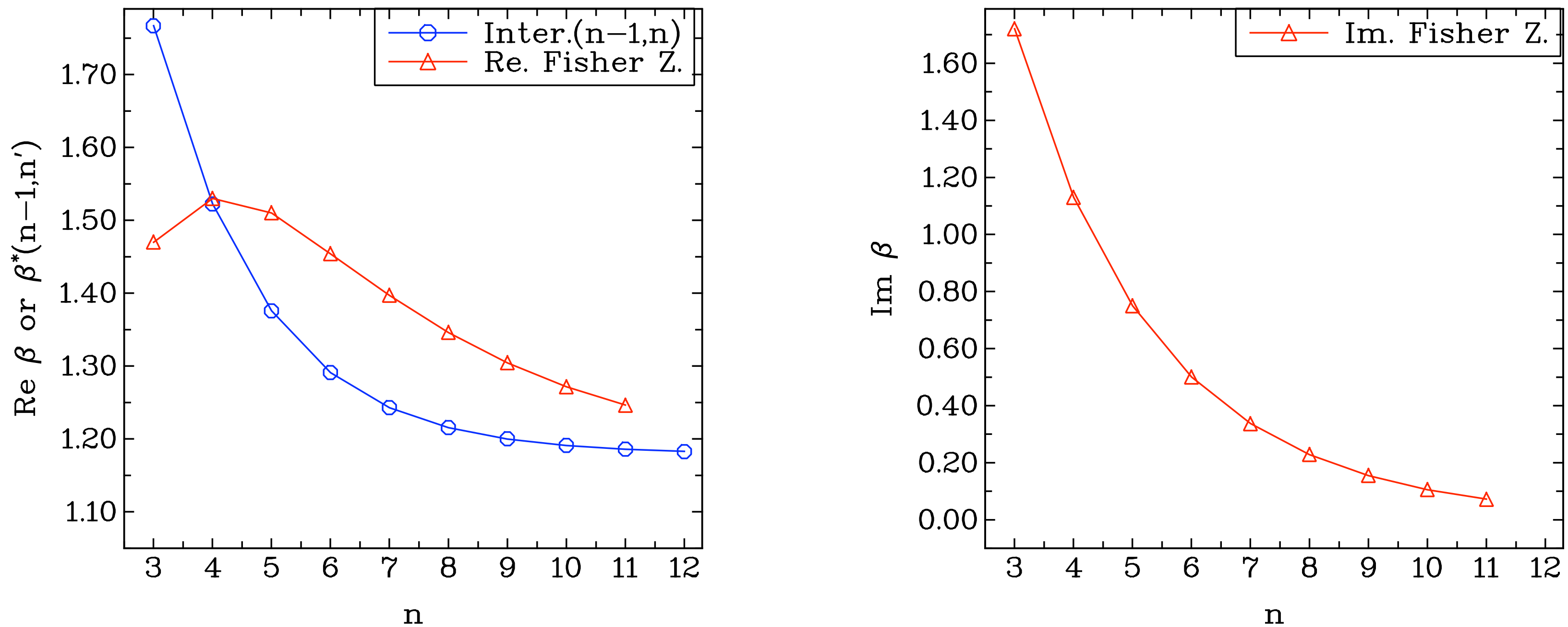

Figure 17: Re and Im part of the zero closest to the real axis compared to the intersections of Binder curves $\beta^{\star}(n-1,n)$ for V up to 2^{11}.

3. The zero volume limit

$B_4 = f(u_\kappa N^{1/\nu}, u_1 N^{-\omega_1}, u_2 N^{-\omega_2}, \dots)$

The ω_i are widely spaced for the HM

$$\begin{aligned} \omega_1 &= 0.655736 \\ \omega_2 &= 3.17995 \\ \omega_3 &= 5.91212 \end{aligned}$$

A strategy to get accurate estimates at not too large volume is to try to fine tune u_κ and u_1 to the smallest possible values. Fine tuning u_1 can be done by looking for the crossing of the first and second irrelevant directions at very small volume. This was done for a LG measure.

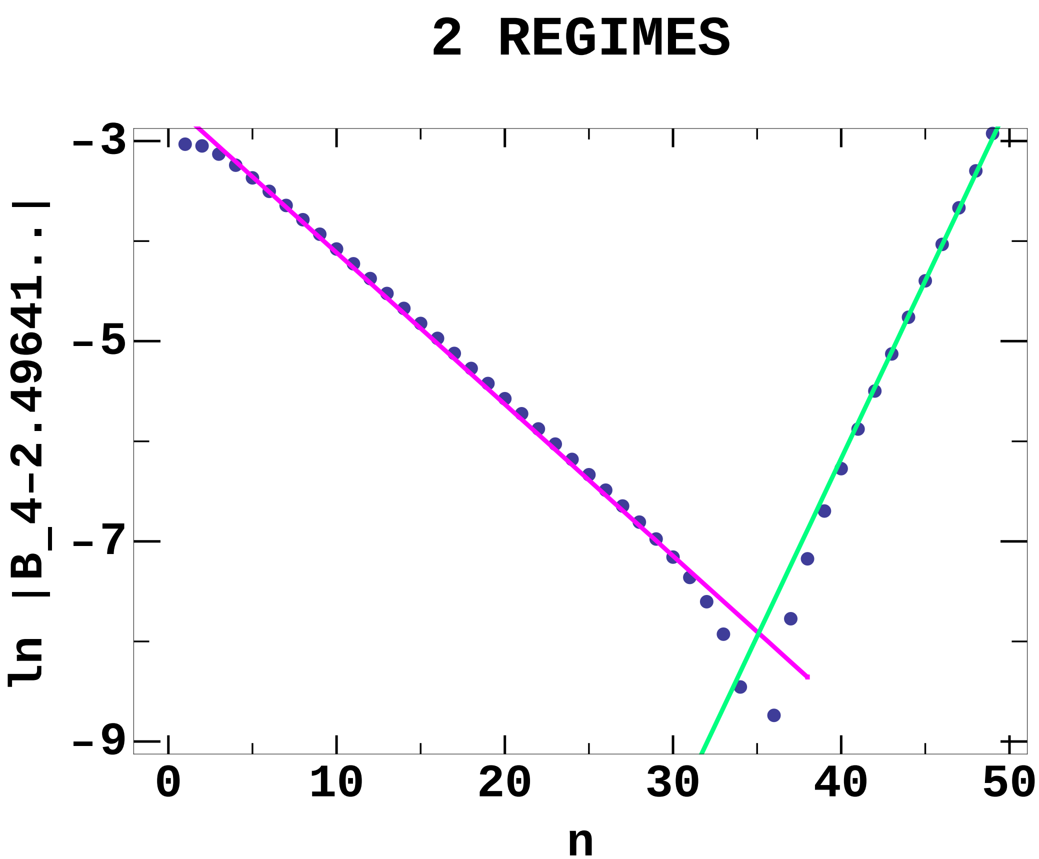

Figure 5: $\ln|B_4 - 2.49641845|$, versus $n =$Log$_2 V$. The two lines have slopes corresponding to the first irrelevant direction and the relevant direction (from left to right). β was fine tuned with 8 digits.

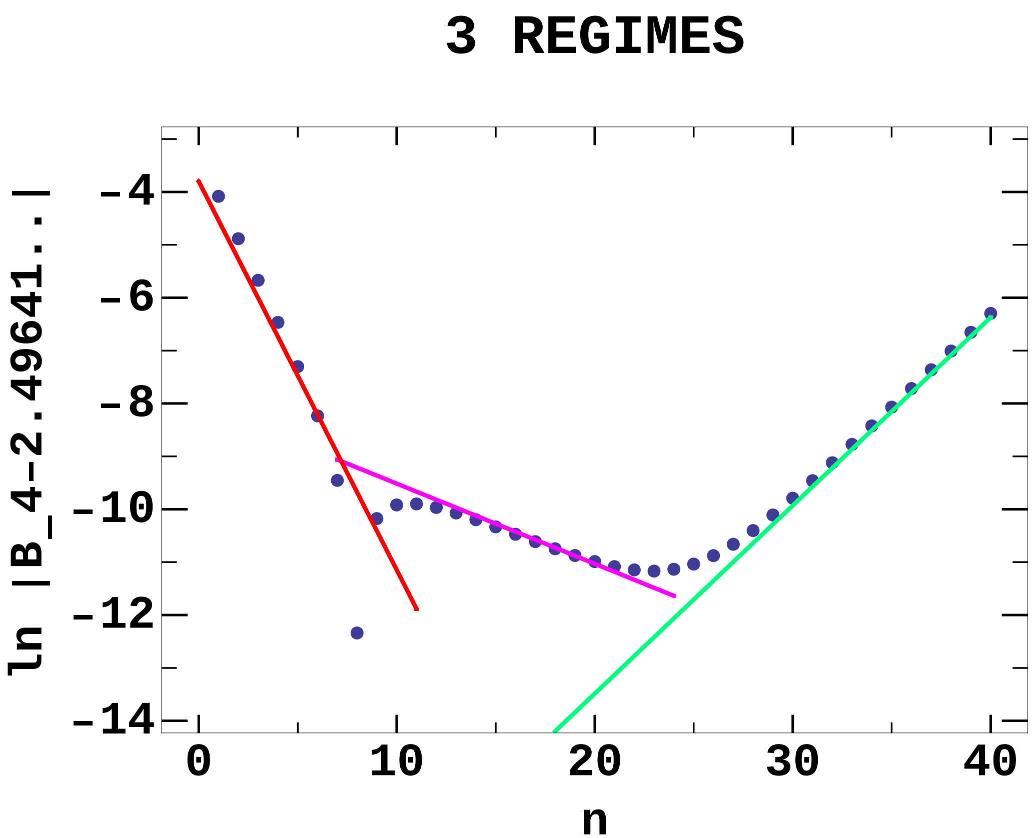

Figure 6: $\ln|B_4 - 2.49641845|$, versus $n =$Log$_2 V$. The three lines have slopes corresponding to the second and first irrelevant directions and the relevant direction (from left to right). β was fine tuned with 8 digits and λ_4 with 3 digits.

4. Continuum limit of discrete RG

The recursion formula can be extended for arbitrary scale. The number of sites integrated for the HM, namely 2, appears as the exponent. With the replacements $2 \to \ell^D$ and $\frac{c}{4} \to \ell^{-2-D}$ the recursion formula becomes

$$R_{n+1}(k) = C_{n+1}\ \mathrm{e}^{-\frac{1}{2}\beta\frac{\partial^2}{\partial k^2}} \left(R_n(\sqrt{c/4}\ k) \right)^2 ,$$

becomes

$$R_{n+1}(k) = C_{n+1}\ \mathrm{e}^{-\frac{1}{2}\beta\frac{\partial^2}{\partial k^2}} \left(R_n(\ell^{-(D+2)/2}\ k) \right)^{\ell^D} ,$$

The usual equation is obtained for $\ell = 2^{1/3}$.

We are interested in the limit $\ell \to 1$. Working with the integral form, we get for V (essentially the log of R, see review)

$$\frac{\partial V}{\partial t} = DV + (1 - \frac{D}{2})\phi\frac{\partial V}{\partial \phi} - (\frac{\partial V}{\partial \phi})^2 + \frac{\partial^2 V}{\partial \phi^2} \tag{3}$$

which implies the so-called Wilson-Polchinski equation

$\nu_{HM} = 0.649570365$

$\nu_{WP} = 0.649561773$ (Litim; Bervillier, Juttner and Litim)

$\nu_{optimal} = \nu_{WP}$ (Litim)

Numerical issues

R^2 is a very simple multiplication of polynomials (when we use the polynomial truncation)

When ℓ^D is not integer, R^{ℓ^D} needs to be defined by some approximation.

We can use

$$\begin{aligned} R^{\ell^D} &= (1 + (R-1))^{\ell^D} \\ &\simeq 1 + \ell^D (R-1) + (1/2!)\ell^D(\ell^D - 1)(R-1)^2 + \ldots \end{aligned}$$

As $R - 1$ is of order k^2, it is consistent to truncate the sum at order $(R-1)^{l_{max}}$. Numerically, this is very slow if we use $Series$ in Mathematica.

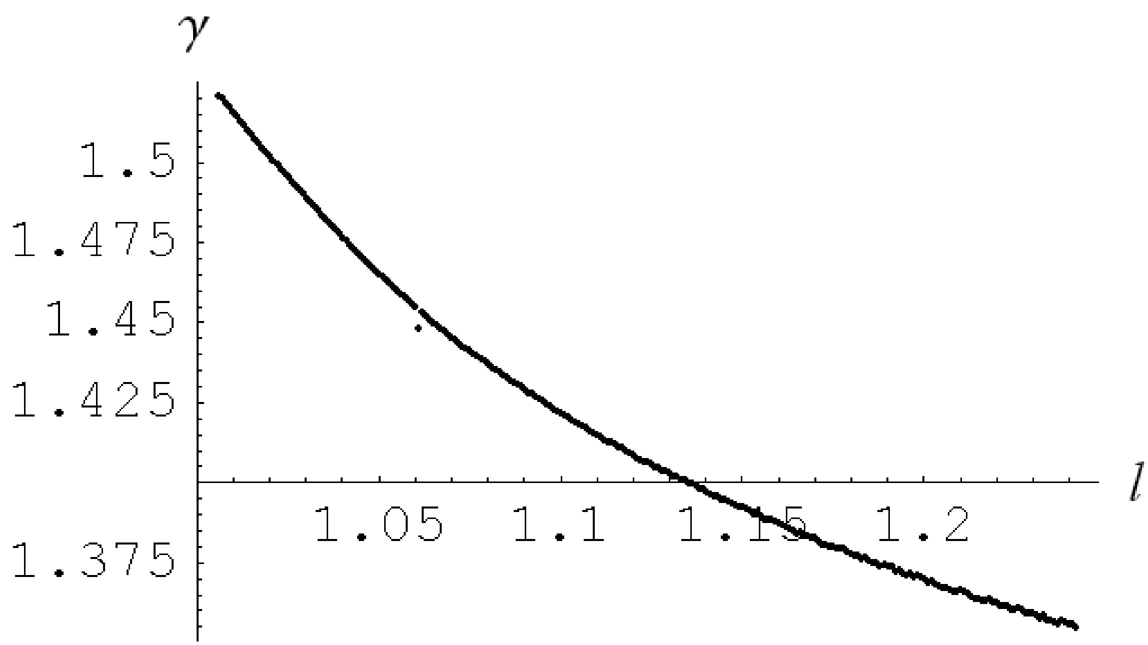

Figure 7: $\gamma = 2\nu$ for $l_{max} = 6$, as $\ell \to 1$, it's going up not down. We also lose the phase bifurcations when $l_{max} = 12$

5. $n(S)$ in Wilson's $SU(2)$ (0807.0185)

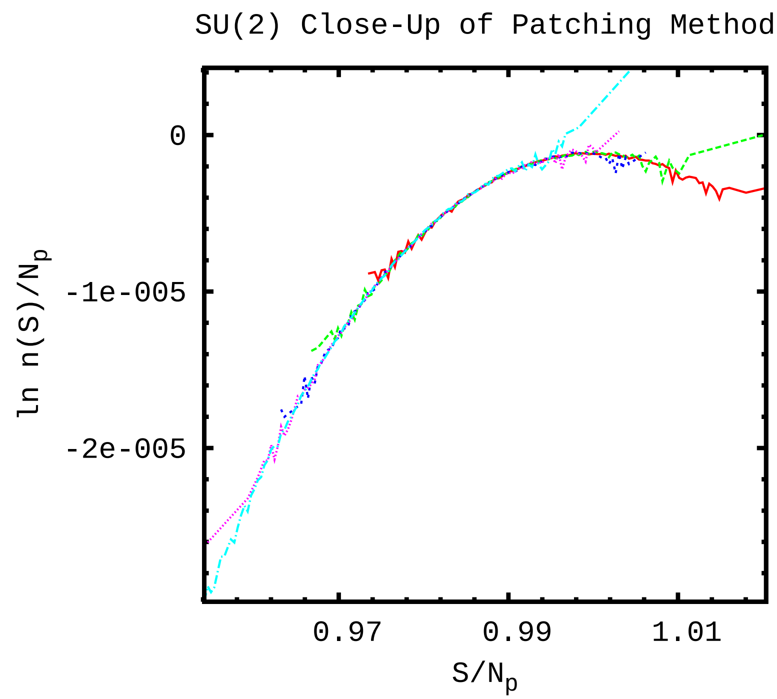

Figure 8: Close-up of the patching process for 6^4 ($n(S) \propto P_\beta(S)\mathrm{e}^{\beta \mathrm{S}}$).

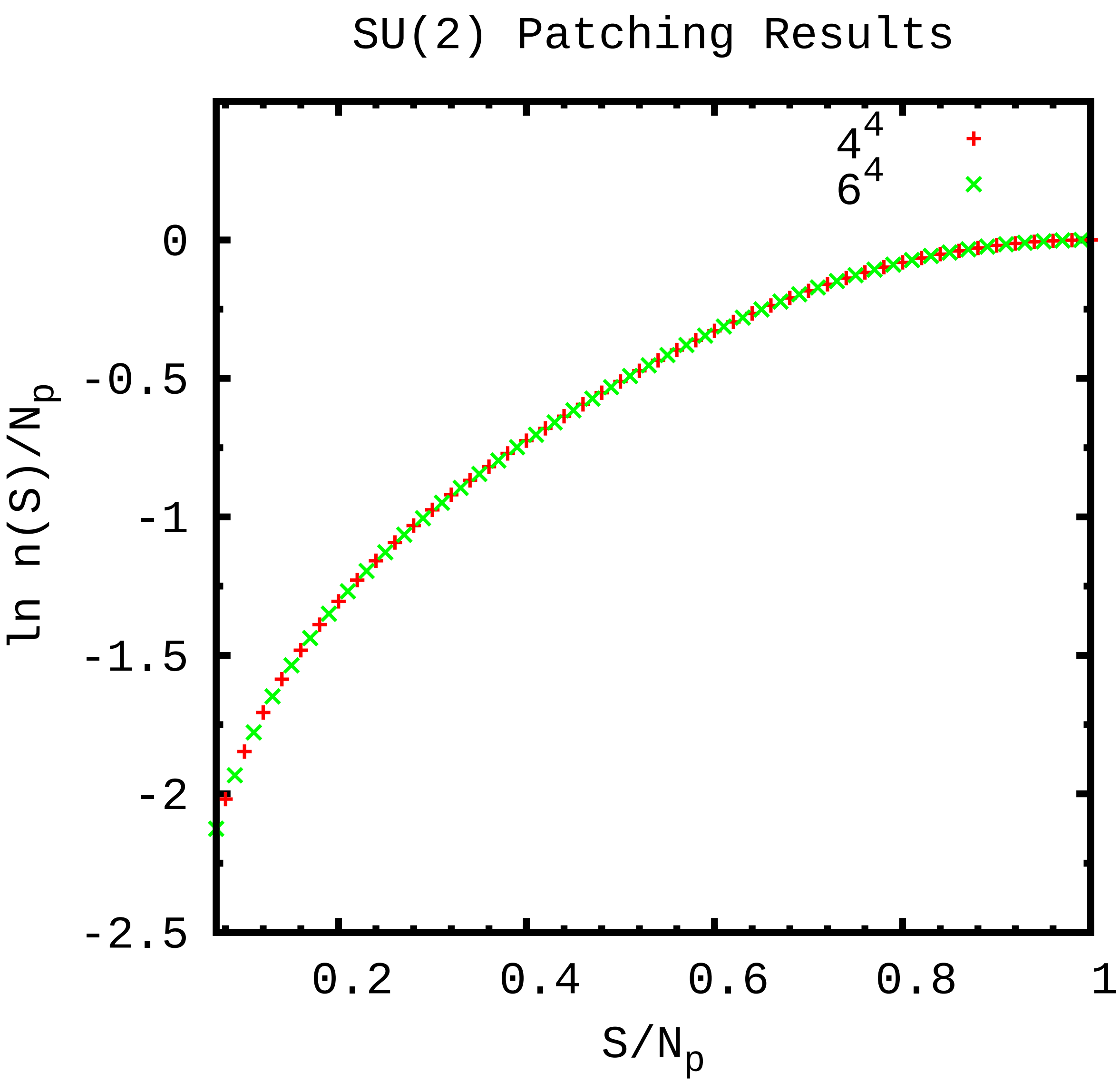

Figure 9: Results of patching for 4^4 and 6^4.

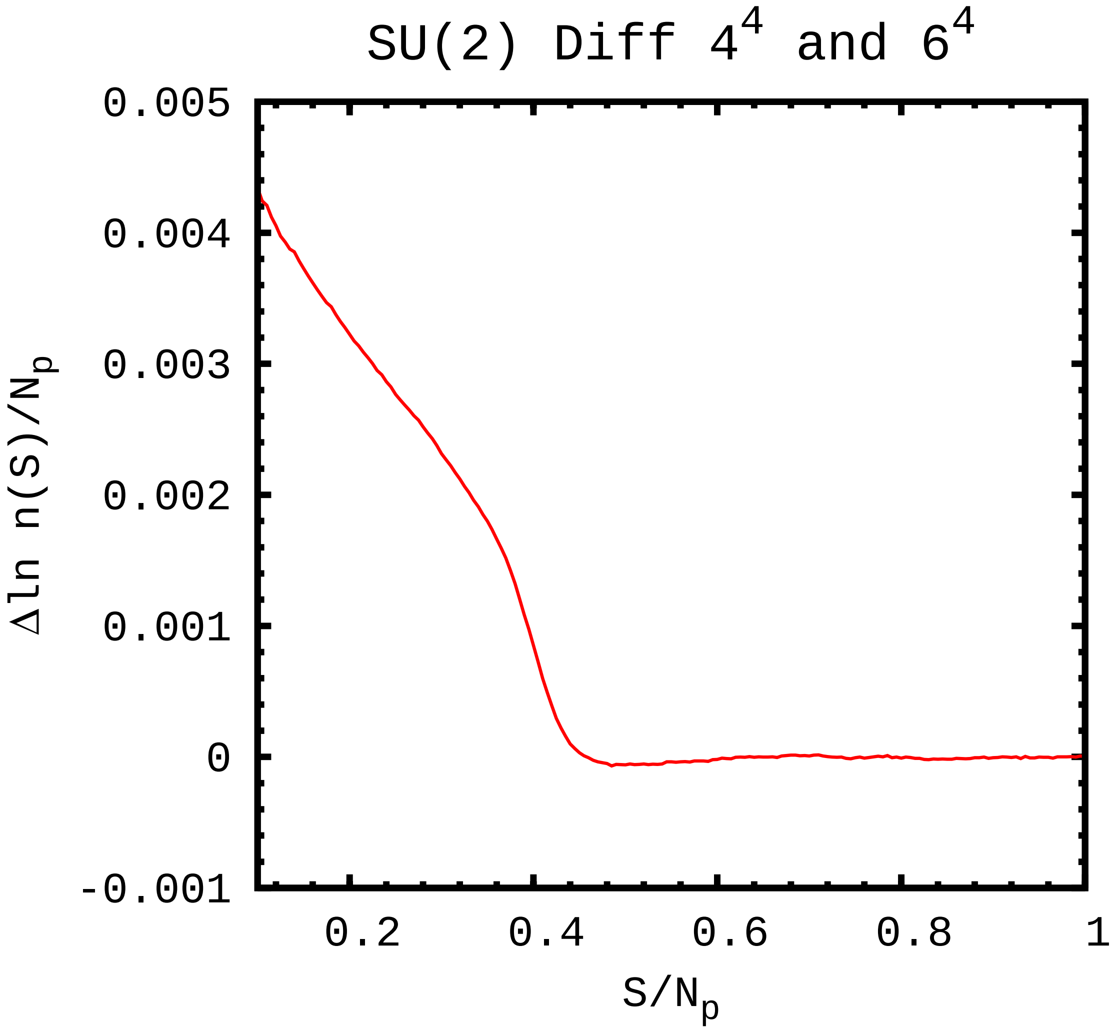

Figure 10: $\Delta \ln n(S)/\mathcal{N}_p$ for 4^4 and 6^4.

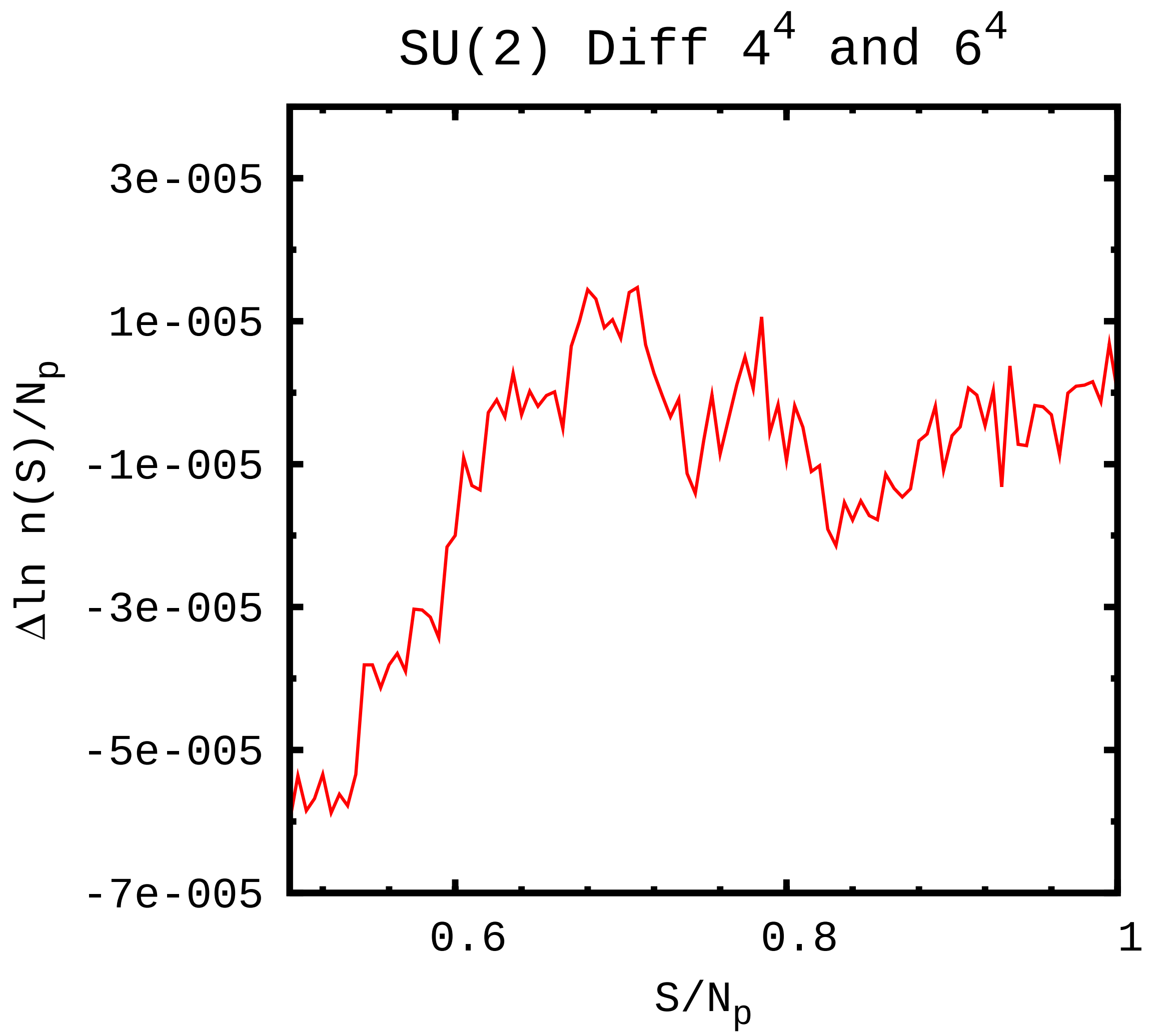

Figure 11: The noise in the tail of $\Delta \ln n(S)/\mathcal{N}_p$for 4^4 and 6^4.

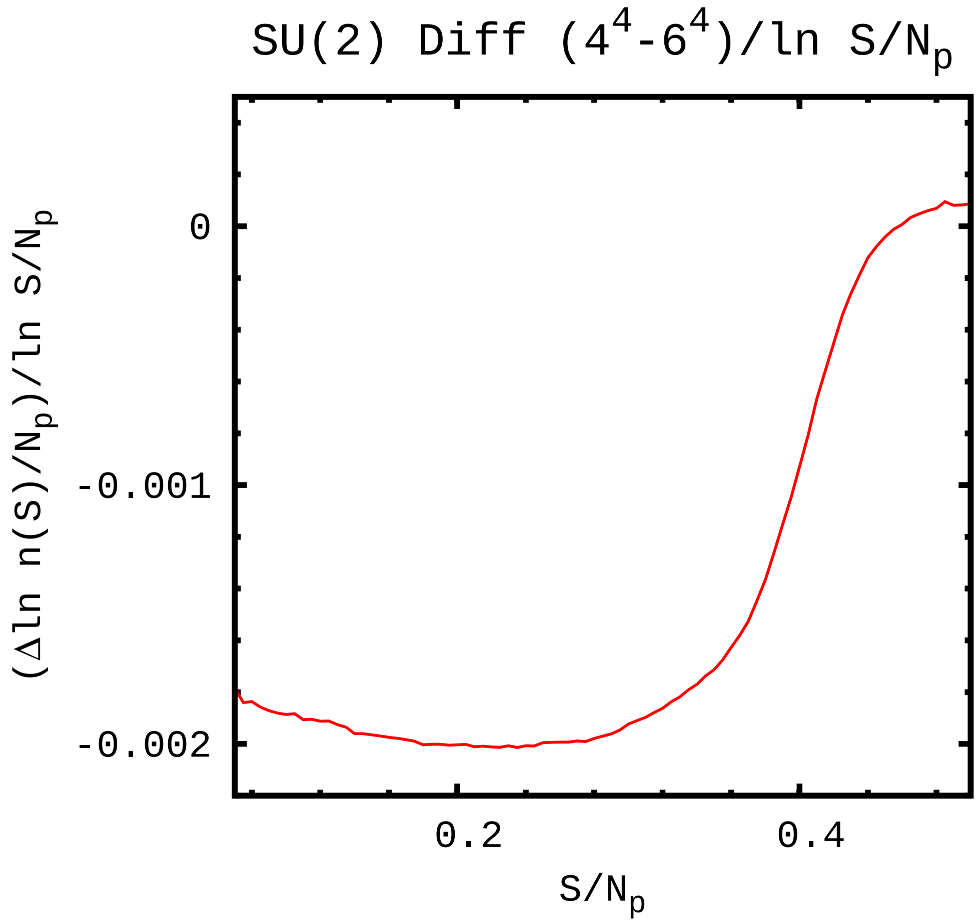

Figure 12: $\Delta \ln n(S)/\mathcal{N}_p$ for 4^4 and 6^4 divided by $ln(S/\mathcal{N}_p)$. Predicted constant is -0.0013.

Conclusions

- Nonlinear effects should be carefully estimated before trying to do accurate calculations of Binder cumulants.

- Irrelevant directions should be studied at small volume.

- Stable numerical methods remain to be developed to perform the $\ell \to 1$ limit for the hierarchical limit.

- The connection between the improvement of the hierarchical approximation and the derivative expansion in an important problem.

- Volume effects under control for the density of states of Wilson action.

- Finite size scaling formalism can be constructed from generalized density of states

- Thanks to the organizers!