Simple recipe for symmetry using ERG

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Plan of the talk

1. A brief review of the ERG differential equations

2. Perturbative renormalizability

3. Recipe for realizing symmetry — "quantum invariance" of the action

   (a) gauge symmetry (not discussed here)
   - QED [Bonini, D'Attanasio, & Marchesini '94, HS '06; Freire & Wetterich '96 for scalar QED; Igarashi, Itoh, & HS '06, Kugo, Higashi, & Itou '07 for BV]
   - YM [Becchi '92, Ellwanger '94, ... ; Morris & Rosten for manifestly gauge invariant formulation]

   (b) supersymmetry — WZ model (discussed if time allows)

Summary

1. No loss of information along the ERG trajectory.

2. Perturbatively renormalized theories are specified by a finite number of parameters that control the action at large cutoff Λ.

3. Symmetry is realized as the "quantum" invariance of the Wilson action under non-linear symmetry transformation of fields.

4. The antifield formalism is useful, if not essential, for showing the consistency of the recipe.

Generalized ERG differential equations

1. Let $S[\phi]$ be the action of a real scalar field theory in D dimensional euclidean space.

2. We generate a one-parameter family of actions S_t equivalent to S:

$$\exp\left[S_t[\phi]\right] = \int [d\phi'] \exp\left[S[\phi']\right]$$
$$\times \exp\left[-\frac{1}{2}\int_p A_t(p)^2 \left\{\phi(p) - Z_t(p)\phi'(p)\right\}\left\{\phi(-p) - Z_t(p)\phi'(-p)\right\}\right]$$

   (a) $\phi(p) \sim Z_t(p)\phi'(p)$ is the **block spin**.
   (b) $\frac{1}{A_t(p)}$ is the width of field diffusion; $S_t[\phi]$ is obtained by an **incomplete integration** of $S[\phi']$. More integration for larger p^2.

3. The t dependence of the action is given by the ERG differential equation of Wilson [Wilson & Kogut '74, sect. 11]:

$$\begin{aligned} \partial_t S_t &= \int_p \Bigg[F_t(p) \cdot \phi(p) \frac{\delta S_t}{\delta \phi(p)} \\ &\quad + G_t(p) \cdot \frac{1}{2} \left\{ \frac{\delta S_t}{\delta \phi(p)} \frac{\delta S_t}{\delta \phi(-p)} + \frac{\delta^2 S_t}{\delta \phi(p) \delta \phi(-p)} \right\} \Bigg] \end{aligned}$$

where

$$\begin{cases} F_t(p) \equiv -\partial_t \ln Z_t(p) \\ G_t(p) \equiv -2 \frac{1}{A_t(p)^2} \partial_t \ln \left(A_t(p) Z_t(p) \right) \end{cases}$$

[Wegner & Houghton '73; Wetterich '93 for 1PI Γ]

4. Relation between S_t and S

(a) Define the generating functionals:

$$\begin{cases} \mathrm{e}^{W[J]} & \equiv \int [d\phi] \exp\left[S[\phi] + i\int_p J(p)\phi(-p)\right] \\ \mathrm{e}^{W_t[J]} & \equiv \int [d\phi] \exp\left[S_t[\phi] + i\int_p J(p)\phi(-p)\right] \end{cases}$$

(b) A simple gaussian integration gives

$$\mathrm{e}^{W_t[J]} = \exp\left[-\frac{1}{2}\int_p \frac{1}{A_t(p)^2} J(p)J(-p) + W\left[Z_t(p)J(p)\right]\right]$$

Hence,

$$\begin{cases} \langle\, \phi(p)\phi(-p)\, \rangle_{S_t} & = \frac{1}{A_t(p)^2} + Z_t(p)^2 \langle\, \phi(p)\phi(-p)\, \rangle_S \\ \langle\, \phi(p_1)\cdots\phi(p_{n>1})\, \rangle^{\mathrm{c}}_{S_t} & = \prod_{i=1}^n Z_t(p_i)\cdot \langle\, \phi(p_1)\cdots\phi(p_n)\, \rangle^c_S \end{cases}$$

(c) Conversely,

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_S &= \frac{1}{Z_t(p)^2}\langle\, \phi(p)\phi(-p)\,\rangle_{S_t} - \frac{1}{A_t(p)^2 Z_t(p)^2} \\ \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_S^c &= \prod_{i=1}^n \frac{1}{Z_t(p_i)}\cdot\langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_{S_t}^c \end{cases}$$

The original correlation functions can be constructed as long as Z_t and $A_t Z_t$ are non-vanishing. Hence,

S_t and S are equivalent.

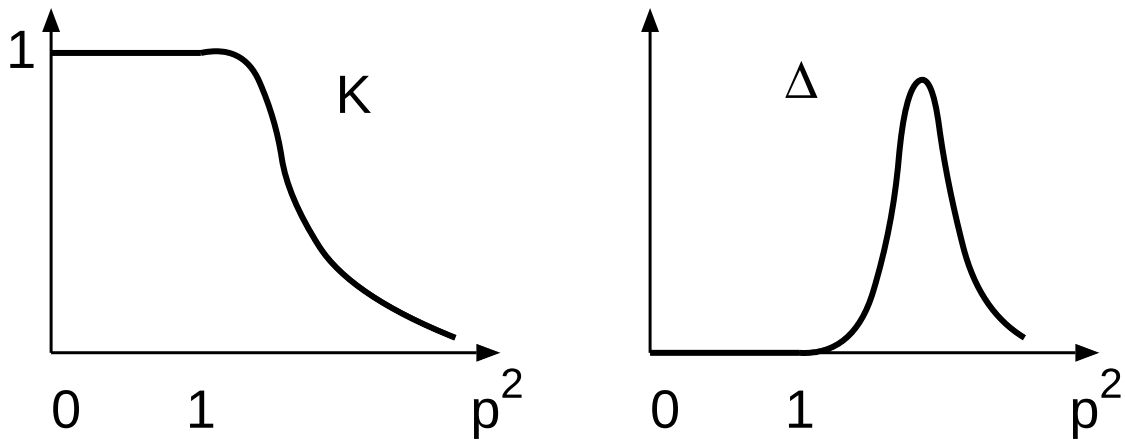
[Wilson & Kogut '74; Rosten's "ERG invariants"]

5. We adopt Polchinski's choice: [Polchinski '83]

$$\begin{cases} Z_t(p) = \frac{K(pe^t)}{K(p)} \\ \frac{1}{A_t(p)^2} = \frac{K(pe^t)}{p^2+m^2}\left(1 - \frac{K(pe^t)}{K(p)}\right) \end{cases} \Longrightarrow \begin{cases} F_t(p) = \frac{\Delta(pe^t)}{K(pe^t)} \\ G_t(p) = \frac{\Delta(pe^t)}{p^2+m^2} \end{cases}$$

where

$$\Delta(p) \equiv -2p^2 \frac{d}{dp^2} K(p)$$



This implies

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_S - \frac{1}{p^2+m^2} K(p)(1-K(p)) \\ \qquad = \frac{K(p)^2}{K(p\mathrm{e}^t)^2} \left\{ \langle\, \phi(p)\phi(-p)\,\rangle_{S_t} - \frac{1}{p^2+m^2} K(p\mathrm{e}^t)\,(1-K(p\mathrm{e}^t)) \right\} \\ \langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_S = \prod_{i=1}^{2n} \frac{K(p_i)}{K(p_i \mathrm{e}^t)} \cdot \langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_{S_t} \end{cases}$$

Perturbative renormalizability

1. We split

$$S(\Lambda) = S_{\mathrm{free}}(\Lambda) + S_{\mathrm{int}}(\Lambda)$$

where $\Lambda \equiv \Lambda_0 \mathrm{e}^{-t}$, and S_{free} is the free action

$$S_{\mathrm{free}}(\Lambda) \equiv -\frac{1}{2} \int_p \phi(-p)\phi(p) \frac{p^2 + m^2}{K\left(\frac{p}{\Lambda}\right)}$$

The Λ dependence of S_{int} given by the Polchinski ERG diff eq.

$$-\Lambda \frac{\partial}{\partial \Lambda} S_{\mathrm{int}}(\Lambda) = \int_p \frac{\Delta(p/\Lambda)}{p^2 + m^2} \frac{1}{2} \left\{ \frac{\delta S_{\mathrm{int}}}{\delta \phi(p)} \frac{\delta S_{\mathrm{int}}}{\delta \phi(-p)} + \frac{\delta^2 S_{\mathrm{int}}}{\delta \phi(p) \delta \phi(-p)} \right\}$$

[Polchinski '83]

2. Need for an **"initial condition"** (ϕ^4 theory in $D = 4$ here)

(a) **bare action**

$$S_{\rm int}(\Lambda_0) = -\int d^4x \left[\left\{ \Lambda_0^2 A_2(\ln \Lambda_0/\mu) + m^2 B_2(\ln \Lambda_0/\mu) \right\} \frac{1}{2}\phi^2 \right.$$

$$\left. + C_2(\ln \Lambda_0/\mu) \frac{1}{2} \left(\partial_\mu \phi\right)^2 + A_4(\ln \Lambda_0/\mu) \frac{\lambda}{4!}\phi^4 \right]$$

Choose the coefficients as power series of λ so that for a finite Λ

$$\lim_{\Lambda_0 \to \infty} S_{\rm int}(\Lambda)$$

is well-defined. [Polchinski '83]

(b) **asymptotic condition** [HS '03]

$$S_{\rm int}(\Lambda) \stackrel{\Lambda\to\infty}{\longrightarrow} \int d^4x \left[\left(\Lambda^2 a_2(\ln \Lambda/\mu) + m^2 b_2(\ln \Lambda/\mu) \right) \frac{1}{2}\phi^2 \right.$$
$$\left. + c_2(\ln \Lambda/\mu)\frac{1}{2}\left(\partial_\mu \phi\right)^2 + a_4(\ln \Lambda/\mu)\frac{1}{4!}\phi^4 \right]$$

i. μ is an arbitrary renormalization scale.

ii. The theory has three parameters: ($a_2(0)$ cannot be controlled.)

A. $b_2(0)$ normalizes $m^2 \Longrightarrow b_2(0) = 0$

B. $c_2(0)$ normalizes $\phi \Longrightarrow c_2(0) = 0$

C. $a_4(0)$ defines the coupling constant $\lambda \Longrightarrow a_4(0) = -\lambda$

iii. m^2 and λ parameterize the whole ERG trajectory $S(\Lambda)$, not just for $\Lambda = \mu$.

iv. 1-loop calculations

$$\begin{cases} a_2 &= \frac{\lambda}{4} \int_q \frac{\Delta(q)}{q^2} \\ b_2 &= -\frac{\lambda}{(4\pi)^2} \ln \frac{\Lambda}{\mu} \\ c_2 &= 0 \\ a_4 &= -\lambda - \frac{3\lambda^2}{(4\pi)^2} \ln \frac{\Lambda}{\mu} \end{cases}$$

a_2 depends on the choice of K.

v. μ dependence given by the **"ordinary" RG equations**

$$-\mu\frac{\partial}{\partial\mu}S(\Lambda) = \beta_m(\lambda)m^2\mathcal{O}_m + \beta_\lambda(\lambda)\mathcal{O}_\lambda + \gamma(\lambda)\mathcal{N}$$

where

$$\begin{cases} \mathcal{O}_m = -\partial_{m^2}S - \int_p \frac{K(1-K)}{(p^2+m^2)^2}\frac{1}{2}\left\{\frac{\delta S}{\delta\phi(p)}\frac{\delta S}{\delta\phi(-p)} + \frac{\delta^2 S}{\delta\phi(p)\delta\phi(-p)}\right\} \\ \mathcal{O}_\lambda = -\partial_\lambda S \\ \mathcal{N} = -\int_p \phi(p)\frac{\delta S}{\delta\phi(p)} - \int_p \frac{K(1-K)}{p^2+m^2}\left\{\frac{\delta S}{\delta\phi(p)}\frac{\delta S}{\delta\phi(-p)} + \frac{\delta^2 S}{\delta\phi(p)\delta\phi(-p)}\right\} \end{cases}$$

so that

$$\left(-\mu\frac{\partial}{\partial\mu} + \beta_m m^2\partial_{m^2} + \beta_\lambda\partial_\lambda - n\gamma\right)\langle\,\phi(p_1)\cdots\phi(p_n)\,\rangle_\infty = 0$$

[HS '06]

Realization of symmetry

1. Observation: $S(\Lambda)$ with a finite Λ gives the correlation functions in the continuum limit.

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_\infty &= \frac{1}{K\left(\frac{p}{\Lambda}\right)^2} \langle\, \phi(p)\phi(-p)\,\rangle_{S(\Lambda)} + \frac{1-1/K\left(\frac{p}{\Lambda}\right)}{p^2+m^2} \\ \langle\, \phi(p_1)\cdots\phi(p_{n>2})\,\rangle_\infty &= \prod_{i=1}^{n} \frac{1}{K\left(\frac{p_i}{\Lambda}\right)} \cdot \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_{S(\Lambda)} \end{cases}$$

2. Whatever symmetry of the continuum limit must be realized in $S(\Lambda)$.

Note ϕ is a generic field in the following.

3. **Universal form of invariance** [Becchi '92, Igarashi, So, Ukita '02 with AF]

   (a) The action $S(\Lambda)$ is "invariant" under a symmetry transformation "$\delta\phi(p) = \mathcal{O}(p)$":

$$\Sigma(\Lambda) \equiv e^{-S} \int_p K\left(\frac{p}{\Lambda}\right) \frac{\delta}{\delta\phi(p)} \left(\mathcal{O}(p) e^{S}\right)$$
$$= \int_p K\left(\frac{p}{\Lambda}\right) \left(\mathcal{O}(p) \frac{\delta S}{\delta\phi(p)} + \frac{\delta\mathcal{O}(p)}{\delta\phi(p)}\right) = 0$$

   (b) $\int_p K\left(\frac{p}{\Lambda}\right) \mathcal{O}(p) \frac{\delta S}{\delta\phi(p)}$ is the **change of the action** under an infinitesimal change of fields:

$$\delta\phi(p) = K\left(\frac{p}{\Lambda}\right) \mathcal{O}(p)$$

   (c) $\int_p K\left(\frac{p}{\Lambda}\right) \frac{\delta\mathcal{O}(p)}{\delta\phi(p)}$ is the **jacobian** of the above change.

(d) $\Sigma = 0$ gives the Ward identities in the continuum limit:

$$\sum_{i=1}^{n} \langle\, \phi(p_1) \cdots \mathcal{O}(p_i) \cdots \phi(p_n)\, \rangle_\infty = 0$$

(e) $\Sigma(\Lambda)$ is a **composite operator** satisfying

$$-\Lambda \frac{\partial}{\partial \Lambda} \Sigma(\Lambda) = \int_p \frac{\Delta(p/\Lambda)}{p^2 + m^2} \left\{ \frac{\delta S_{\mathrm{int}}}{\delta \phi(-p)} \frac{\delta}{\delta \phi(p)} + \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right\} \Sigma(\Lambda)$$

[Becchi '92]

This type of diff eq. is satisfied by any infinitesimal deformation of S.

(f) If $\Sigma \to 0$ as $\Lambda \to \infty$, then $\Sigma(\Lambda) = 0$ for any Λ.

4. **Perturbative solution of** $\Sigma = 0$

(a) Loop expansions:

$$S_{\rm int}(\Lambda) = \sum_{l=0}^{\infty} S_{{\rm int};l}(\Lambda), \quad \Sigma(\Lambda) = \sum_{l=0}^{\infty} \Sigma_l(\Lambda)$$

(b) **induction hypothesis**: $S_{{\rm int};0,\cdots,l-1}$ constructed so that $\Sigma_{0,\cdots,l-1} = 0$

(c) $\Sigma_{0,\cdots,l-1} = 0$ implies Σ_l has no Λ dependence for large Λ [Becchi '92]:

$$-\Lambda \frac{\partial}{\partial \Lambda} \Sigma_l(\Lambda) = \int_p \frac{\Delta(p/\Lambda)}{p^2+m^2} \frac{\delta S_{{\rm int},l}}{\delta \phi(-p)} \frac{\delta \Sigma_l(\Lambda)}{\delta \phi(p)} \stackrel{\Lambda \to \infty}{\longrightarrow} 0$$

(d) We fine-tune the parameters of $S_{{\rm int};l}$ so that $\Sigma_l = 0$.

But is it possible?

(e) To show the possibility of such fine-tuning, we can resort to the **antifield formalism** in which the antifields ϕ^* generate the symmetry transformation. [Batalin-Vilkovisky '81]

$$\bar{\Sigma}(\Lambda) \equiv \mathrm{e}^{-S} \int_p K(p/\Lambda) \frac{\delta}{\delta\phi(p)} \frac{\overrightarrow{\delta}}{\delta\phi^*(-p)} \mathrm{e}^{\bar{S}} = 0$$

(f) ϕ^* has the opposite statistics to ϕ. Hence, $\bar{\Sigma}$ is a fermionic scalar composite operator.

(g) Given a composite operator $\mathcal{O}$, we define its **BRST transformation** by

$$\delta_Q \mathcal{O} \equiv \mathrm{e}^{-\bar{S}} \int_p K(p/\Lambda) \frac{\delta}{\delta\phi(p)} \frac{\overrightarrow{\delta}}{\delta\phi^*(-p)} \left(\mathrm{e}^{\bar{S}} \mathcal{O}\right)$$

(h) By construction, $\bar{\Sigma}$ is nilpotent:

$$\delta_Q \bar{\Sigma} = 0$$

This constrains the asymptotic behavior of $\bar{\Sigma}$.

(i) **We only need to show** $\bar{\Sigma} = 0$ **for large** Λ where both $\bar{S}$ and $\bar{\Sigma}$ are local (polynomials of derivatives of fields).

$\Longrightarrow$ Only **classical** BRST cohomology is required.

$$\delta_0 \bar{\Sigma}_l = 0 \stackrel{?}{\Longrightarrow} \bar{\Sigma}_l = \delta_0 \bar{S}_l$$

where δ_0 is defined with $\bar{S}_0$, satisfying $\delta_0 \delta_0 = 0$.

Perturbative construction of the WZ model

1. We construct supersymmetric theories without superfields or auxiliary fields. [Bonini & Vian '98 with superfields]

2. The classical action

$$S_{cl} \equiv \int d^4x \left[\bar\chi_L \sigma \cdot \partial \chi_R + \frac{1}{2}\left(m \bar\chi_R \chi_R + \bar m \bar\chi_L \chi_L\right) + \partial_\mu \bar\phi \partial_\mu \phi + |m|^2 \bar\phi \phi \right.$$
$$\left. + g\phi \frac{1}{2} \bar\chi_R \chi_R + \bar g \bar\phi \frac{1}{2} \bar\chi_L \chi_L + m\phi \frac{\bar g}{2} \bar\phi^2 + \bar m \bar\phi \frac{g}{2} \phi^2 + \frac{|g|^2}{4} |\phi|^4 \right]$$

is invariant under ($\xi_{R,L}$ are anticommuting constant spinors)

$$\begin{cases} \delta\phi = \bar\xi_R \chi_R, & \delta\chi_R = \bar\sigma_\mu \xi_L \partial_\mu \phi - \left(\bar m \bar\phi + \frac{\bar g}{2}\bar\phi^2\right)\xi_R \\ \delta\bar\phi = \bar\xi_L \chi_L, & \delta\chi_L = \sigma_\mu \xi_R \partial_\mu \bar\phi - \left(m\phi + \frac{g}{2}\phi^2\right)\xi_L \end{cases}$$

3. To quantize the Wess-Zumino model perturbatively, we split

$$S = S_{\rm free} + S_{\rm int}$$

Using the two-component spinor notation ($\bar{\chi} \equiv \chi^T \sigma_y$)

$$S_{\rm free} \equiv - \int_p \frac{1}{K_b(p/\Lambda)} \bar{\phi}(-p)\phi(p)(p^2 + |m|^2)$$
$$- \int_p \frac{1}{K_f(p/\Lambda)} \left[\bar{\chi}_L(-p)\sigma_\mu i p_\mu \chi_R(p) + \frac{m}{2}\bar{\chi}_R(-p)\chi_R(p) + \frac{\bar{m}}{2}\bar{\chi}_L(-p)\chi_L(p) \right]$$

In general

$$K_f \neq K_b$$

4. the asymptotic behavior of $S_{\rm int}$: (R-symmetry & dimensions)

$$S_{\rm int}(\Lambda) \quad \stackrel{\Lambda\to\infty}{\longrightarrow} \quad \int \Bigg[z_1 \bar{\chi}_L \sigma_\mu \partial_\mu \chi_R + z_2 \left(\frac{m}{2} \bar{\chi}_R \chi_R + \frac{\bar{m}}{2} \bar{\chi}_L \chi_L \right)$$

$$+ z_3 \partial_\mu \bar{\phi} \partial_\mu \phi + (a_4 \Lambda^2 + z_4 |m|^2) |\phi|^2$$

$$+ (-1 + z_5) \left(g\phi \frac{1}{2} \bar{\chi}_R \chi_R + \bar{g}\bar{\phi} \frac{1}{2} \bar{\chi}_L \chi_L \right)$$

$$+ (-1 + z_6) \left(m\phi \frac{\bar{g}}{2} \bar{\phi}^2 + \bar{m}\bar{\phi} \frac{g}{2} \phi^2 \right) + (-1 + z_7) \frac{|g|^2}{4} |\phi|^4$$

$$+ z_8 \left(g^2 \bar{m}^2 \phi^2 + \bar{g}^2 m^2 \bar{\phi}^2 \right)$$

$$+ (\Lambda^2 a_9 + |m|^2 z_9) \left(g\bar{m}\phi + \bar{g} m \bar{\phi} \right) \Bigg]$$

where $a_{4,9}$ and $z_i \, (i = 1, \cdots, 9)$ are all functions of $|g|^2$ and $\ln \Lambda/\mu$.

5. The supersymmetry transformation has the same form as the classical one:

$$\begin{cases} \delta\phi(p) & \equiv & \bar{\xi}_R[\chi_R](p) \\ \delta\bar{\phi}(p) & \equiv & \bar{\xi}_L[\chi_L](p) \\ \delta\chi_R(p) & \equiv & \bar{\sigma}_\mu \xi_L i p_\mu [\phi](p) - \left(\bar{m}[\bar{\phi}](p) + \bar{g}\cdot \left[\frac{\bar{\phi}^2}{2}\right](p)\right)\xi_R \\ \delta\chi_L(p) & \equiv & \sigma_\mu \xi_R i p_\mu [\bar{\phi}](p) - \left(m[\phi](p) + g\cdot \left[\frac{\phi^2}{2}\right](p)\right)\xi_L \end{cases}$$

6. The composite operators $[\phi^2/2], [\bar{\phi}^2/2]$ are defined by

$$\begin{cases} \left[\frac{\phi^2}{2}\right](p) \stackrel{\Lambda\to\infty}{\longrightarrow} (1+z_{10})\frac{\phi^2}{2}(p) + z_{11}\bar{g}m\phi(p) + z_{12}\bar{g}^2 m^2 \cdot (2\pi)^4 \delta^{(4)}(p) \\ \left[\frac{\bar{\phi}^2}{2}\right](p) \stackrel{\Lambda\to\infty}{\longrightarrow} (1+z_{10})\frac{\bar{\phi}^2}{2}(p) + z_{11}g\bar{m}\bar{\phi}(p) + z_{12}g^2 \bar{m}^2 \cdot (2\pi)^4 \delta^{(4)}(p) \end{cases}$$

where $z_{10,11,12}$ are functions of $|g|^2$ and $\ln\Lambda/\mu$.

7. Altogether, the theory has **twelve** parameters: $\boxed{z_1(0), \cdots, z_{12}(0)}$
(9 for the action, 3 for the transformation)

8. The parameters are constrained by the invariance:

$$\Sigma(\Lambda) \equiv \int_p K_b(p/\Lambda) \left[\delta\phi(p) \frac{\delta S}{\delta\phi(p)} + \delta\bar{\phi}(p) \frac{\delta S}{\delta\bar{\phi}(p)} + \frac{\delta}{\delta\phi(p)} \delta\phi(p) + \frac{\delta}{\delta\bar{\phi}(p)} \delta\bar{\phi}(p) \right]$$
$$+ \int_p K_f(p/\Lambda) \left[S \frac{\overleftarrow{\delta}}{\delta\chi_R(p)} \delta\chi_R(p) - \mathrm{Tr}\ \delta\chi_R(p) \frac{\overleftarrow{\delta}}{\delta\chi_R(p)} + (R \to L) \right] = 0$$

(a) z_1, z_2, z_5, z_9 are left arbitrary.
(b) The remaining nine are fixed.

9. Proof requires the AF formalism [K. Ülker & HS `arXiv:0804.1072`]

Conclusions

1. The continuum limit can be described by a cutoff action.

2. Whatever symmetry of the continuum limit must be realized in the cutoff action.

3. Using the parameterization of a theory by its asymptotic behavior, only classical analysis is needed to show the possibility of realizing symmetry.