I. BAKAS

Renormalization group equations and geometric flows

ERG 2008
Heidelberg
3 July '08

Non-linear sigma models

Geometric field theories

$$\Sigma^{(2)} \longrightarrow M^{(n)}$$

with classical action

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2 z \sqrt{\det \gamma}\, \gamma^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu$$

Σ = world-sheet

M = target space (Riemannian)

γ_{ab} = world-sheet metric

$G_{\mu\nu}$ = target space metric

$(z, \bar{z})$ = local coordinates on Σ

X^μ = " " on M

Assume $\partial\Sigma = \phi$ now ; later $\partial\Sigma \neq \phi$

Target space geometry can be constrained by imposing various special conditions on 2-d FT:

- $N=2$ supersymmetry $\Rightarrow$ Kähler and generaligations thereof
- conformal invariance at the quantum level impies

$$R_{\mu\nu} = 0$$

(starting point in σ-model approach to string theory)

- Here, we will provide the geometric interpretation of the RG eqs of 2-d σ-models

These theories are perturbatively renormalizable. Metric $G_{\mu\nu}$ is coupling "constant"

$$G_{\mu\nu}(X;t)$$

that depends on logarithm of length scale on Σ (RG time t). To lowest order

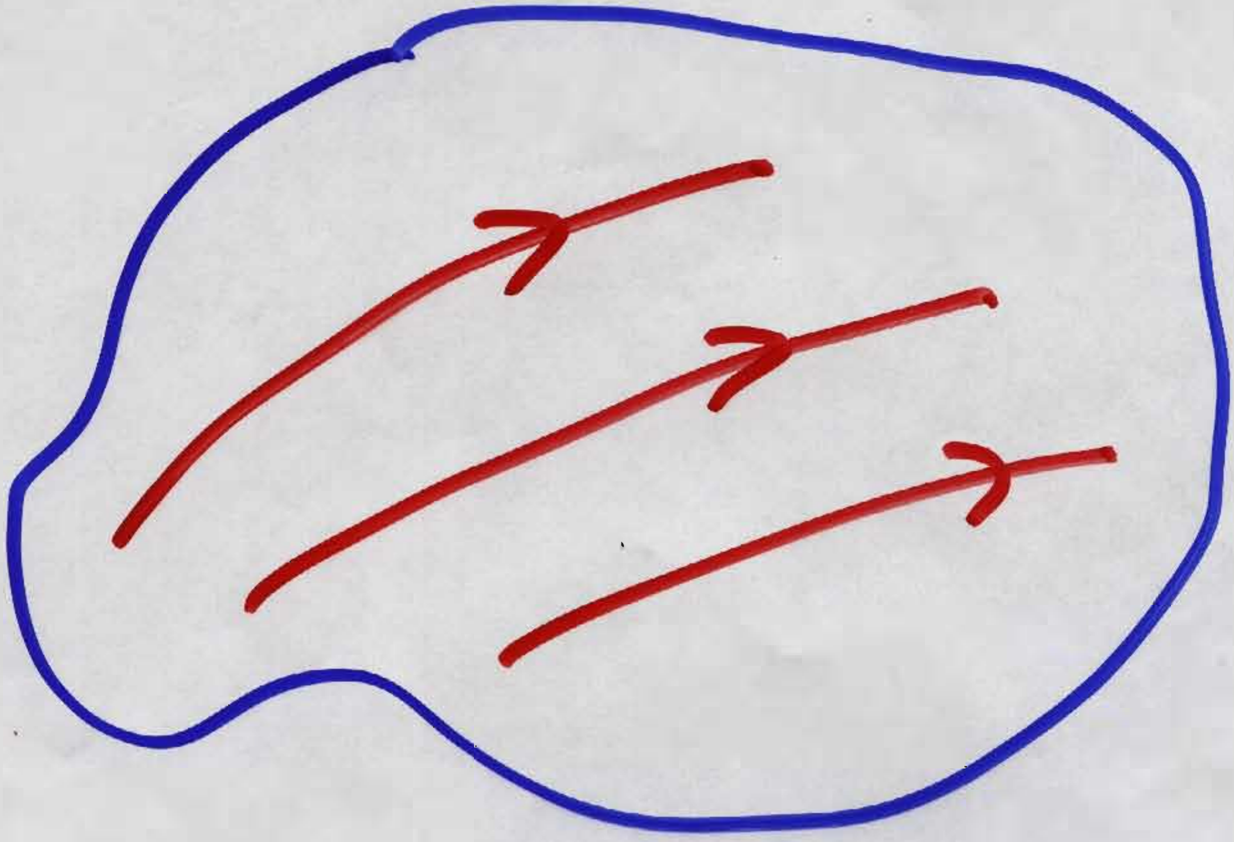
$$\frac{\partial G_{\mu\nu}}{\partial t} \equiv -\beta(G_{\mu\nu}) = -R_{\mu\nu}\,.$$

Special case: $R_{\mu\nu} = a\,G_{\mu\nu}$

$$G_{\mu\nu}(t) = (1-at)\,G_{\mu\nu}(0)$$

- if $a>0$ space shrinks uniformly to zero size. Asymptotically free as $t\to-\infty$
- if $a<0$ space will expand

Evolution equation in superspace ($\equiv$ all Riemannian metrics on M)



- non-linear generalization of heat flow equation for metric (tends to dissipate curvature perturbations)
- RG equation $\equiv$ Ricci flow.

Ancient solutions exhibit asymptotic freedom.

In mathematics one often uses the normalized Ricci flow

$$\frac{\partial}{\partial t} G_{\mu\nu} = -R_{\mu\nu} + \frac{1}{\dim M} \bar{R} G_{\mu\nu}$$

where $\bar{R}$ is mean scalar curvature

$$\bar{R} = \frac{1}{\mathrm{Vol}(M)} \int_M d^n x \sqrt{\det G}\, R[G]$$

- follows from Ricci flow by appropriate rescaling of metric and time reparametrization
- preserves $\mathrm{Vol}(M)$
- fixed points are canonical metrics on M (useful to study uniformization problems)

Another useful variant includes reparametrizations along the flow wrt vector field ξ_μ, as

$$\frac{\partial}{\partial t} G_{\mu\nu} = -R_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

For gradient vector fields $\xi_\mu = \nabla_\mu \Phi$

$$\frac{\partial}{\partial t} G_{\mu\nu} = -R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \Phi$$

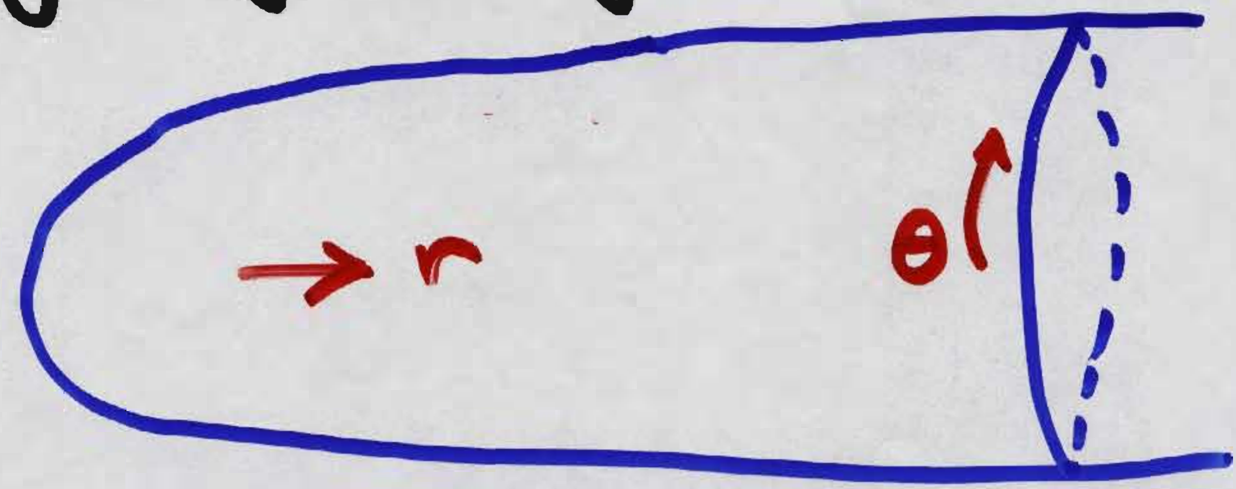
where Φ plays role of dilaton.

Gradient Ricci solitons satisfy

$$R_{\mu\nu} = 2 \nabla_\mu \nabla_\nu \Phi$$

and give rise to non-trivial CFT as generalized fixed points

Prime example is semi-infinite cigar geometry



associated to solution

$$ds^2 = dr^2 + \tanh^2 r \, d\theta^2$$

$$\Phi(r) = \log(\cosh r)$$

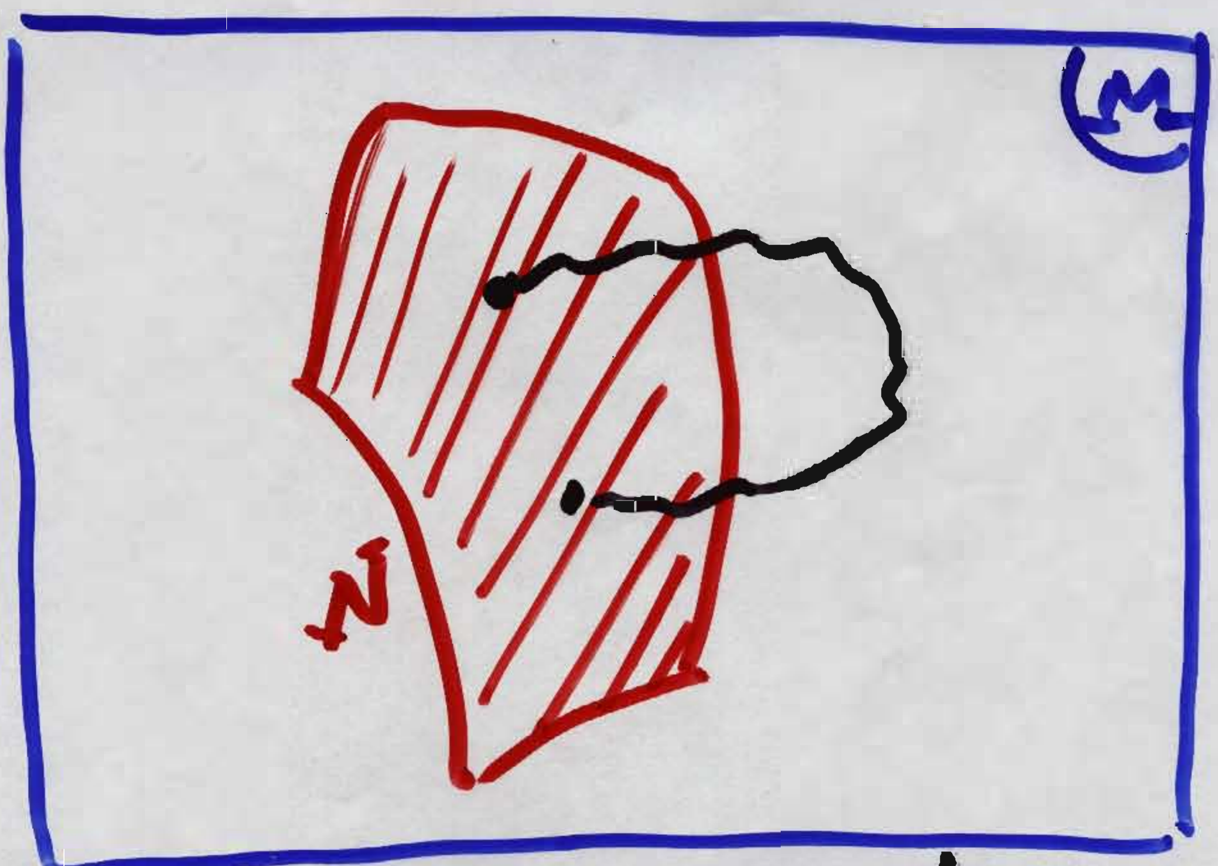
- describes the semi-classical geometry of Euclidean 2-d black-hole solution in string theory.

Dirichlet sigma models

Consider world-sheets with boundary eg disc so that $\partial\Sigma = S^1$. Then

$$X^\mu{}_{|\partial\Sigma} = f^\mu(y^A)$$

amounts to introducing branes in M as embedded submanifolds with local coordinates y^A. Thus



D-branes are minimal submanifolds with zero extrinsic curvature that correspond to conformal boundary conditions. In general, however, the embedding functions

$$f^\mu(y^A; t)$$

depend on RG time t, so that

$$\frac{\partial f^\mu}{\partial t} = \sum_{\sigma=1}^{\mathrm{codim}} H^\sigma \hat{n}^\mu_\sigma + \xi^\mu$$

is driven by extrinsic mean curvature vector with

$\hat{n}^\mu_\sigma$: unit normal vectors

$$H^\sigma = g^{AB} K^\sigma_{AB}$$

induced metric

second fundamental form

In this case, boundary RG flow equation $\equiv$ mean curvature flow

- it derives as gradient flow from area functional

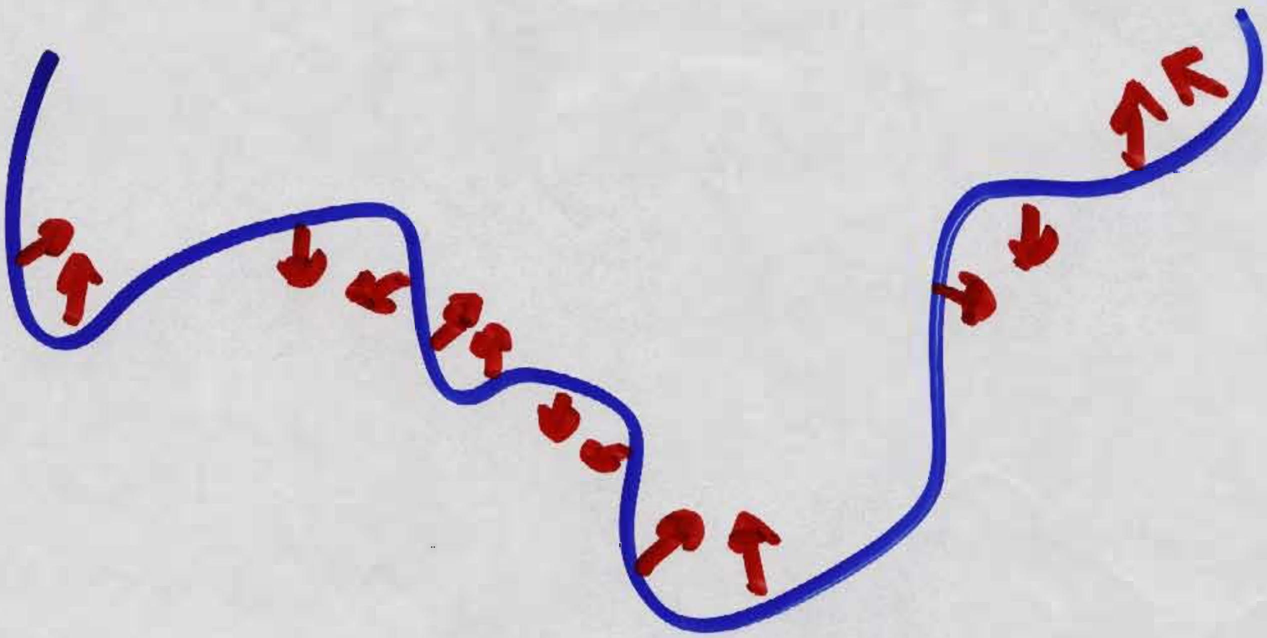
$$S = \int_N d^n y \, e^{-\Phi} \sqrt{\det g}$$

for $\xi^\mu = \nabla^\mu \Phi$

It arises as special case of Dirac-Born-Infeld action

- the simplest (yet non-trivial) case is the curve shortening problem on $\mathbb{R}^2$ (boundary effects in QFT of 2 free bosons)

Thus, for planar curves have



$$\frac{\partial x}{\partial t} = -\frac{\varphi'\varphi''}{(1+\varphi'^2)^2} + \xi^x$$

$y(t) = \varphi(x(t), t)$

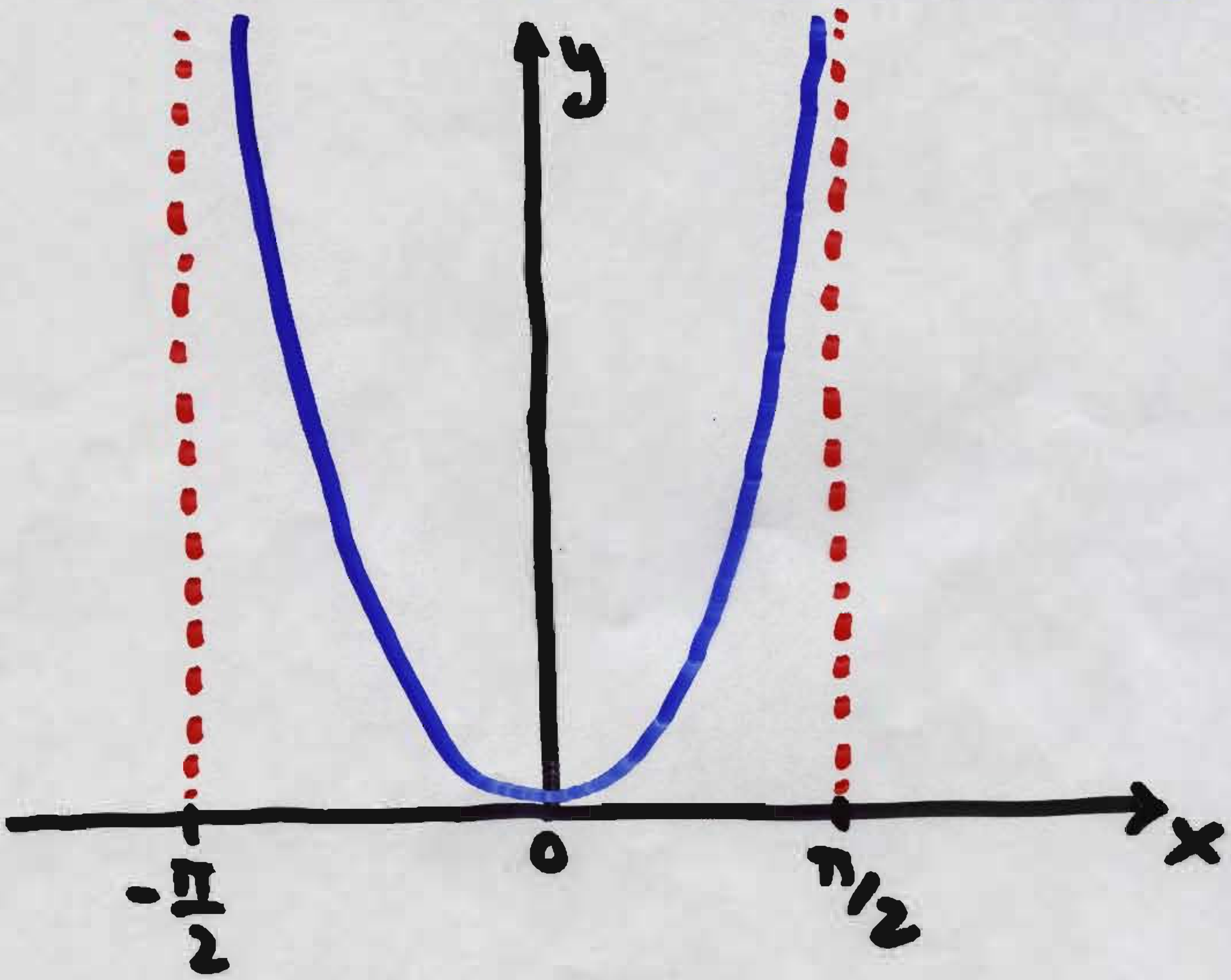
$$\frac{\partial y}{\partial t} = \frac{\varphi''}{(1+\varphi'^2)^2} + \xi^y$$

ie/

$$\boxed{\frac{\partial \varphi}{\partial t} = \frac{\varphi''}{1+\varphi'^2} + \xi^y - \varphi'\xi^x}$$

Simple solitonic solution is hair-pin (or grim-reaper) associated to translational KVF

$\xi^\mu = \nabla^\mu \Phi$ with $\Phi(y) = y$



with

$$y(x) = -\log(\cos x)$$

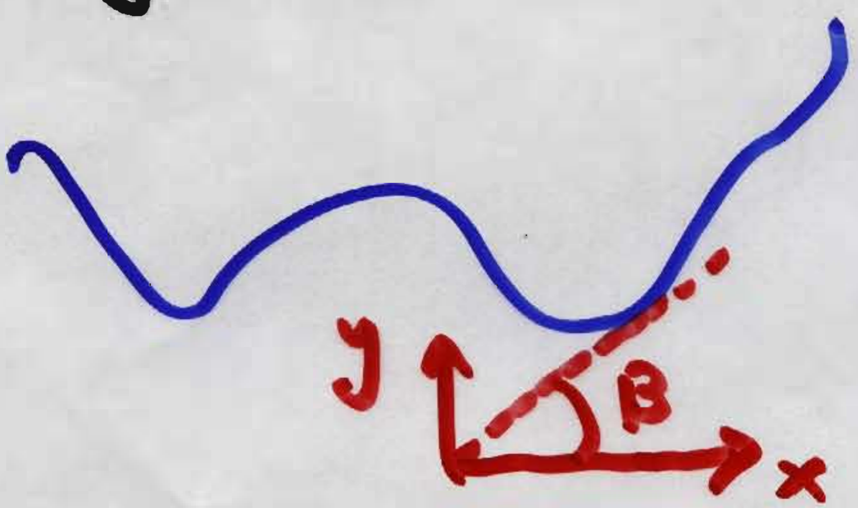
Homothetic solutions

$$y = \varphi(x(t), t) = \sqrt{2ct}\, \varphi\left(\frac{x}{\sqrt{2ct}}, 1\right)$$

or equivalently

$$H\hat{n} = c\vec{r}$$

Using the slope of the curve



$$\tan\beta = \varphi'(x)$$

the boundary flow assumes the form

$$\frac{\partial H}{\partial t} = H^2 \frac{\partial^2 H}{\partial \beta^2} + H^3$$

For homothetic solutions,

$$H(\beta,t) = \frac{H(\beta)}{\sqrt{2ct}}$$

and the equation specializes to

$$\frac{d^2 H(\beta)}{d\beta^2} + H(\beta) + \frac{c}{H(\beta)} = 0$$

$$\Downarrow$$

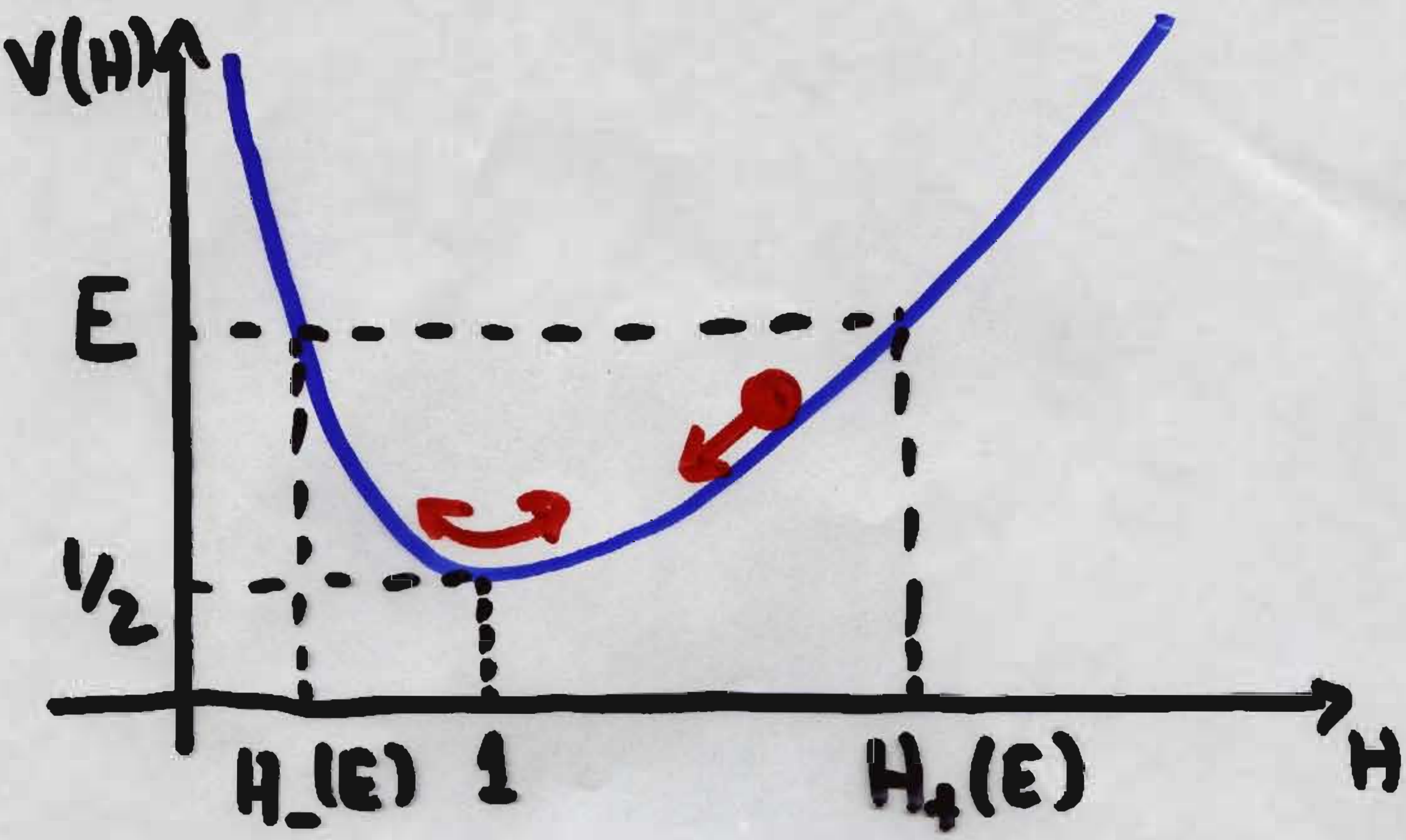
$$\frac{1}{2}\left(\frac{dH}{d\beta}\right)^2 + V(H) = E$$

where

$$V(H) = \frac{1}{2}\left(H^2 + c \log H^2\right)$$

- homothetically shrinking solutions have $c<0$, say $c=-1$

Effective particle motion



Closed curves arise when

$$T(E) = 2\int_{H_-(E)}^{H_+(E)} \frac{dH}{\sqrt{2(E-V(H))}}$$

Closed curves arise when

$$T(E) = 2\int_{H_-(E)}^{H_+(E)} \frac{dH}{\sqrt{2(E - V(H))}}$$

satisfies

$$T(E) = 2\pi \frac{p}{q}$$

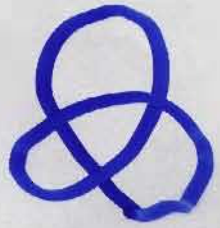
with $\quad \frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2}.$

They correspond to so called Abresch-Langer curves with

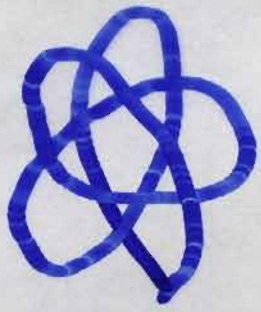
p = winding number

q = number of petals

eg



$p=2,\ q=3$



$p=3,\ q=5$

etc

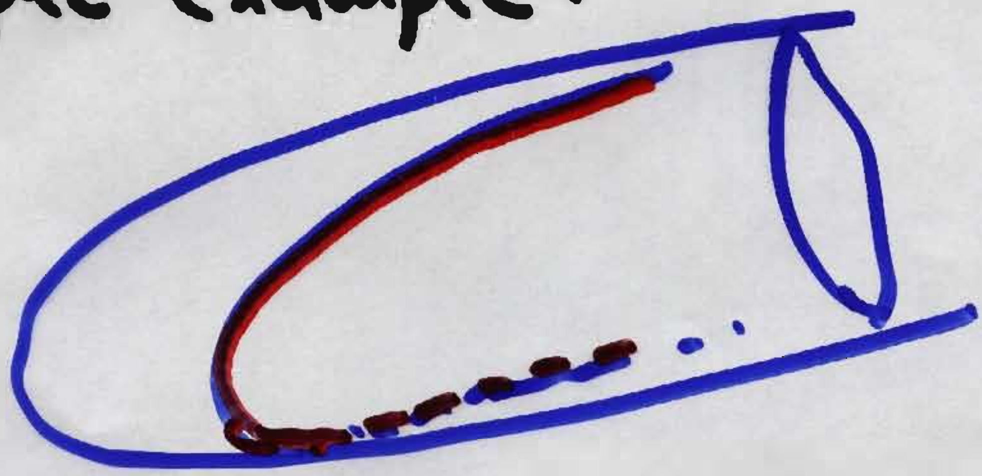
The homothetically shrinking simple curve corresponds to particle sitting at bottom of $V(H)$

Combined bulk and boundary RG flow equations

Have deforming branes in deforming backgrounds

- bulk eq is inert to the branes
- boundary eq depends on the bulk

Simple example:



Hair-pin (D-brane) on cigar described by

$r = \rho(\theta)$ with $\frac{\sinh \rho_0}{\sinh \rho} = \cos(\theta - \theta_0)$

Further generalizations

- **Adding fluxes**: add $B_{\mu\nu}$

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2z \sqrt{\det\gamma} \left[\gamma^{ab} G_{\mu\nu} - i \varepsilon^{ab} B_{\mu\nu} \right] \partial_a X^\mu \partial_b X^\nu$$

anti-symmetric tensor

Then, Ricci flow generalizes to

$$\frac{\partial}{\partial t} G_{\mu\nu} = -\beta(G_{\mu\nu}) = \frac{1}{4} H_{\mu\kappa\lambda} H_\nu{}^{\kappa\lambda} - R_{\mu\nu}$$

$$\frac{\partial}{\partial t} B_{\mu\nu} = -\beta(B_{\mu\nu}) = \frac{1}{2} \nabla_\lambda H^\lambda{}_{\mu\nu}$$

using 3-form strength $H = dB$.

Fixed points

$$R_{\mu\nu} = \frac{1}{4} H_{\mu\kappa\lambda} H_\nu{}^{\kappa\lambda} \; ; \; \nabla_\lambda H^\lambda{}_{\mu\nu} = 0$$

eg SU(2) WZW model

- Geometry stabilized by fluxes

Likewise for the boundary RG flow obtain generalized MCF from DBI action

$$S_{DBI} = \int_N d^n y \, e^{-\Phi} \sqrt{\det(g + b + F)}$$

induced metric

induced anti-sym. tensor

$U(1)$

- **Adding higher curvature terms:**

They become more and more relevant close to singularities

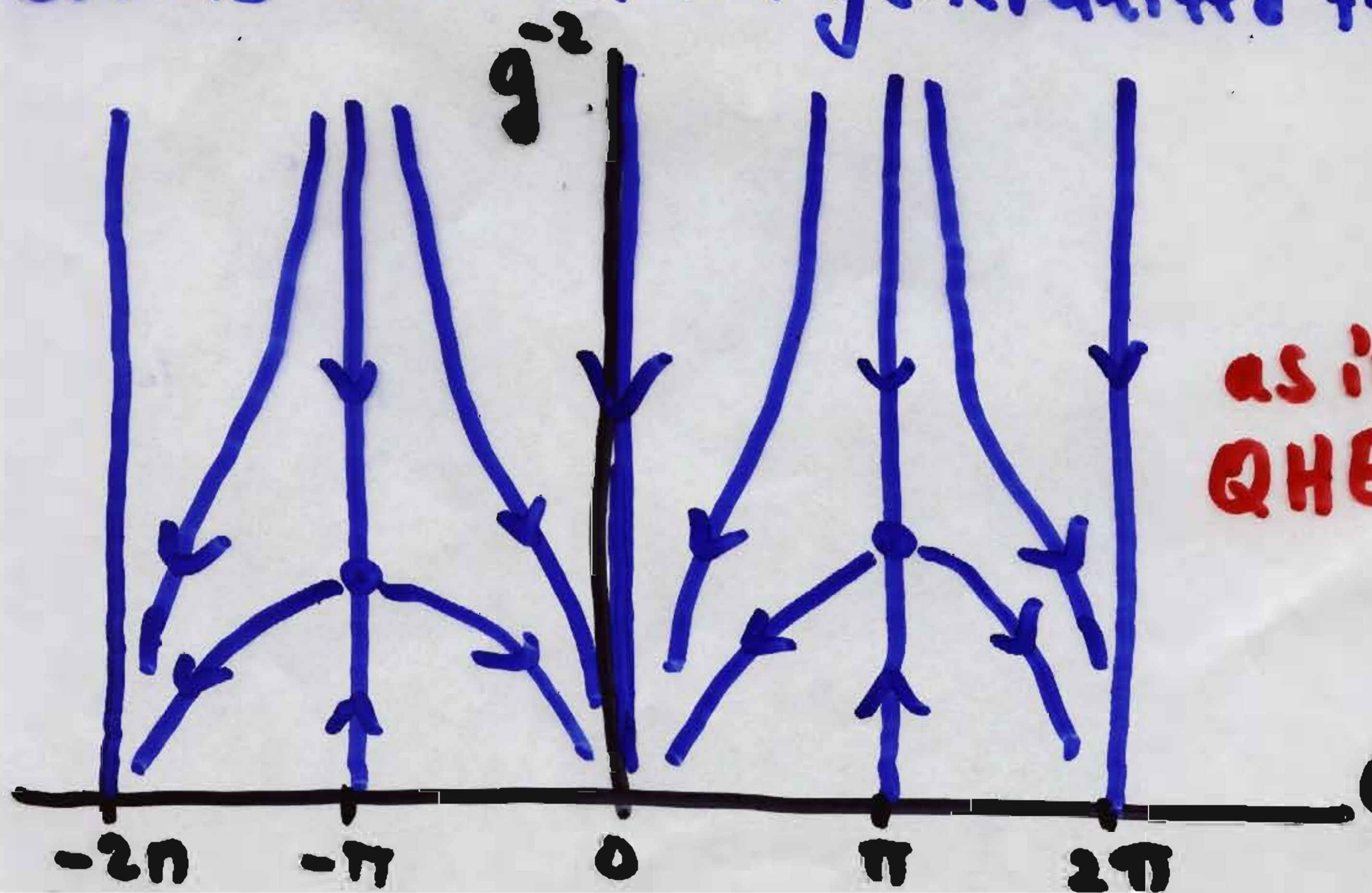
Thus, expect fluxes and perturbative correction to smooth out singularities of Ricci / MCF flow equations.

Instanton effects

Consider O(3) σ-model with θ-term (topological torsion):

$$S = \frac{1}{2g^2}\int_\Sigma (\partial\vec{n})^2 \, dx\, d\tau + i\frac{\theta}{4\pi}\int \vec{n}\cdot\partial_\tau\vec{n}\times\partial_x\vec{n} \, dx\, d\tau$$

Add 1-instanton effect to β-functions and obtain generalized flow



as in QHE

O(3) σ-model + $\theta = \pi$ term
yield $SU(2)_1$ WZW at IR
(Gaussian @ self-dual radius)

Thus instanton effects alter the formation of singularities under the geometric flows. Similar considerations apply to MCF.

Open question: Can all these be understood systematically, eg within the ERG framework

? ?