Wilsonian investigations into the UV properties of gravity

ERG 2008, Heidelberg, July 3, 2008

[A. Codello, R. P. and C. Rahmede, e-Print: arXiv:0805.2909 [hep-th]]

Perturbation theory

$$\Gamma_k = \sum_i g_i \mathcal{O}_i$$

$$= \int d^d x \sqrt{g} \left[2Z\Lambda - ZR + \frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 + \frac{1}{\rho} E + \frac{1}{\tau} \nabla^2 R + \ldots \right]$$

$$Z = \frac{1}{16\pi G} \; ; \qquad \frac{1}{\xi} = -\frac{\omega}{3\lambda} \; ; \qquad \frac{1}{\rho} = \frac{\theta}{\lambda}$$

$$\frac{1}{\epsilon} \int d^4 x \sqrt{g} \left[\frac{7}{10} R_{\mu\nu} R^{\mu\nu} + \frac{1}{60} R^2 + \frac{53}{45} E - \frac{19}{15} \nabla^2 R \right] \qquad \text{1 loop}$$

$$\frac{1}{\epsilon} \int d^4 x \sqrt{g} \, R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} R^{\alpha\beta}{}_{\mu\nu} \qquad \text{2 loops}$$

The Problems of QG

- interaction strength grows like $\tilde{G} = Gk^2$
- lack of predictivity

Possible RG solution

- $\tilde{G} = G(k)k^2 \longrightarrow \tilde{G}_*$ (Fixed Point)
- more generally, $\tilde{g}_i \longrightarrow \tilde{g}_{i*}$ where $\tilde{g}_i = k^{-d_i} g_i$
- predictivity requires that only a finite number of couplings is arbitrary
- The RG trajectories that flow into the FP for $t = \log \frac{k}{k_0} \longrightarrow \infty$ form the UV critical surface.

Asymptotic Safety I

- If (1) a FP exist, and (2) UV critical surface is finite-dimensional, then (a) reaction rates remain finite (in units of k) for $t \to \infty$ and (b) theory is predictive

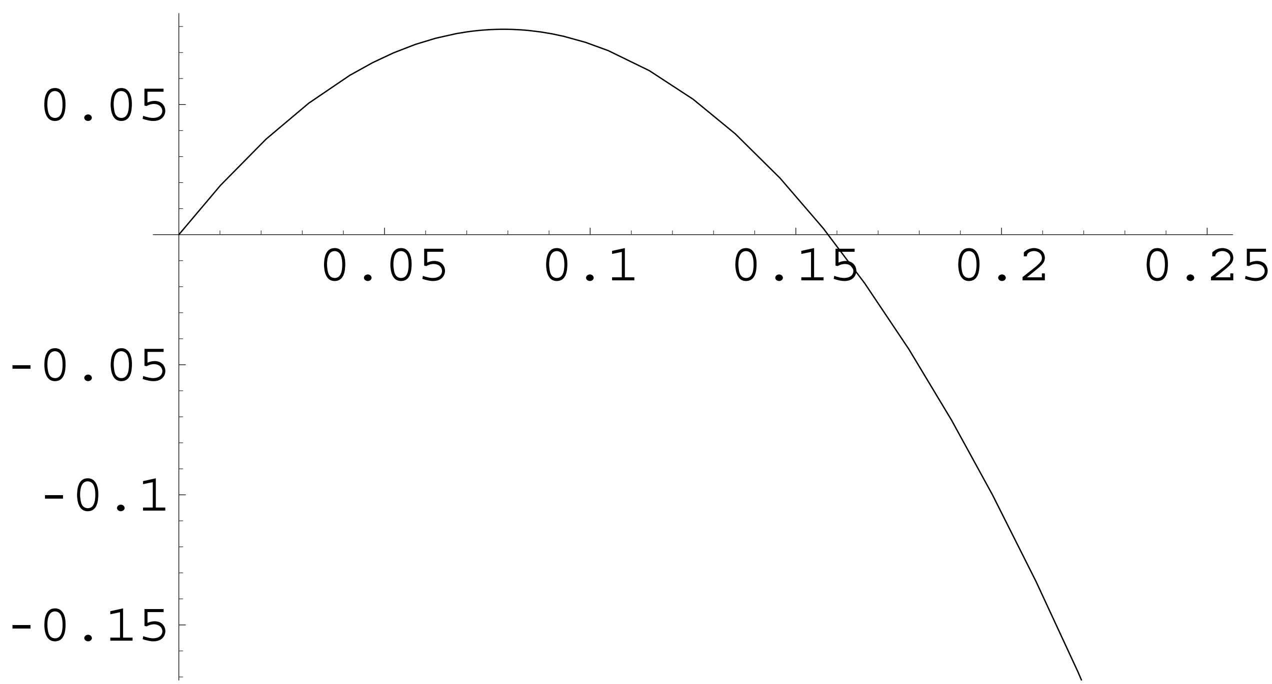
Asymptotic Safety II

- Linearize flow around FP: $M_{ij}\big|_* = \frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j}\big|_*$
- positive critical exponent = negative eigenvalue = UV attractive = relevant
- negative critical exponent = positive eigenvalue = UV repulsive = irrelevant
- Example: QCD. Gaußian Fixed Point at $\tilde{g}_{i*} = 0$.
- $\tilde{\beta}_i = \partial_t \tilde{g}_i = -d_i \tilde{g}_i + k^{-d_i} \beta_i$
- $M_{ij}\big|_* = -d_i \delta_{ij}$
- UV critical surface=span{renormalizable couplings}.

Gravity in $2+\epsilon$

$$\tilde{G} = G k^{\epsilon}$$

$$\beta_{\tilde{G}} = \epsilon \tilde{G} - \frac{38}{3}\tilde{G}^2$$



Can we take $\epsilon \to 2$?

One Loop Corrections in Einstein's Theory

$$G(r) = G\left[1 - \frac{167}{30\pi}\frac{G\hbar}{r^2}\right]$$

[N.E.J. Bjerrum-Bohr, J.F. Donoghue and B.R. Holstein, Phys. Rev. D 68, 084005 (2003)]

$$\beta_{\tilde{G}} = 2\tilde{G} - \frac{167}{15\pi}\tilde{G}^2$$

ERGE: Beta functions

$$\Gamma_k(\phi) = \sum_i g_i(k)\mathcal{O}_i(\phi)$$

$$\partial_t \Gamma_k = \sum_i \partial_t g_i \mathcal{O}_i = \sum_i \beta_{g_i} \mathcal{O}_i$$

compare with

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} \partial_t R_k$$

read off beta functions. Generally scheme-dependent.

ERGE: Gravity

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$S_{GF}(h,\bar{g}) = \int d^4x \sqrt{\bar{g}}\, \chi_\mu \bar{g}^{\mu\nu} \chi_\nu$$

$$\chi_\nu = \bar{\nabla}^\mu h_{\mu\nu} + \beta \bar{\nabla}_\nu h$$

$$\Delta S_k(h,\bar{g}) = \frac{1}{2} \int d^4x \sqrt{\bar{g}}\; h_{\mu\nu} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} R_k(-\bar{\nabla}^2) h_{\rho\sigma}$$

$$\Gamma_k(\bar{g}_{\mu\nu}, \langle h_{\mu\nu} \rangle)$$

$$\Gamma_k(g_{\mu\nu}) = \Gamma_k(g_{\mu\nu}, 0)$$

[M. Reuter, Phys. Rev. D **57** 971(1998)]

[D. Dou and R. P., Class. and Quantum Grav. **15** 3449 (1998)]

ERGE: CUTOFF TYPES

$$\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} = \Delta = -\nabla^2 + \mathbf{E}_1(R) + \mathbf{E}_2(g_i, R)$$

$$P_k(z) = z + R_k(z)$$

type I

$$\Delta + R_k(-\nabla^2) = P_k(-\nabla^2) + \mathbf{E}_1(R) + \mathbf{E}_2(g_i, R)$$

type II

$$\Delta + R_k(-\nabla^2 + \mathbf{E}_1) = P_k(-\nabla^2 + \mathbf{E}_1) + \mathbf{E}_2$$

type III

$$\Delta + R_k(\Delta) = P_k(\Delta)$$

Heat Kernel Technique I

$$\mathrm{Tr}W(\Delta) = Q_{\frac{d}{2}}(W)B_0(\Delta) + Q_{\frac{d}{2}-1}(W)B_2(\Delta) + \ldots + Q_0(W)B_{2d}(\Delta) + \ldots$$

$$Q_n[W] = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}W(z)\;; \qquad Q_{-n}[W] = (-1)^n W^{(n)}(0)\;. \qquad n > 0$$

[A.H. Chamseddine and A. Connes, Comm. Math. Phys. **186**, 731 (1997)]

[M. Reuter, Phys. Rev. D **57**, 971 (1998)]

Heat Kernel Technique II

$$\begin{aligned}\partial_t \Gamma_k &= \frac{1}{2}\mathrm{Tr}\left(\frac{\partial_t R_k(\Delta)}{P_k(\Delta)}\right)\\ &= \frac{1}{2}\left[Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right)B_0(\Delta) + Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)B_2(\Delta) + \ldots\right]\end{aligned}$$

with optimized cutoff $R_k(z) = (k^2 - z)\theta(k^2 - z)$

[D.Litim, Phys.Rev.D64:105007,2001]

$$Q_n\left(\frac{\partial_t P}{P^\ell}\right) = \frac{2}{n}k^{2(n-\ell+1)}\ ; \qquad Q_0\left(\frac{\partial_t P}{P^\ell}\right) = 2k^{2(-\ell+1)}$$

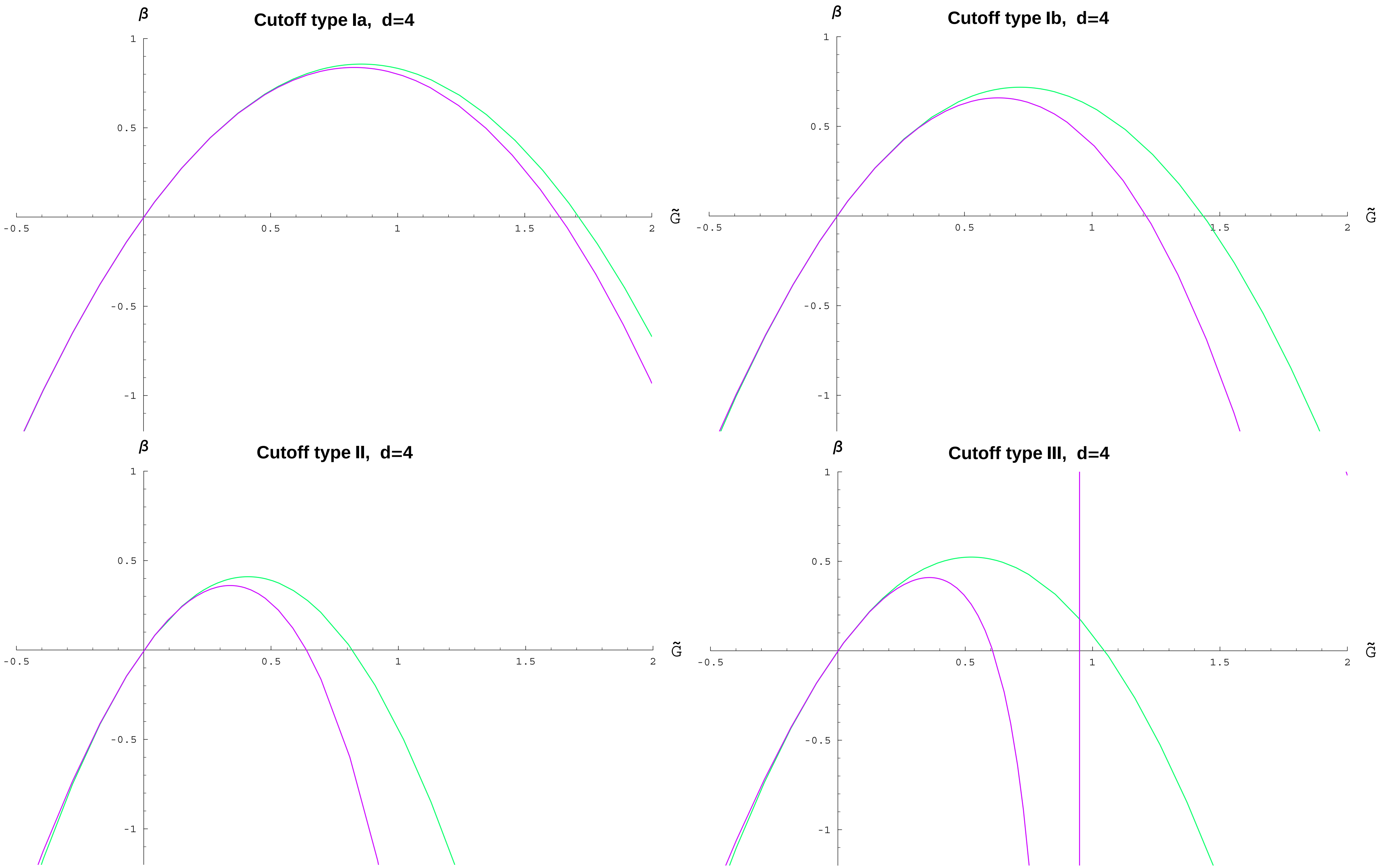
Einstein–Hilbert truncation I

$$\Gamma_k(g) = \frac{1}{16\pi G}\int dx\sqrt{g}(2\Lambda - R)$$

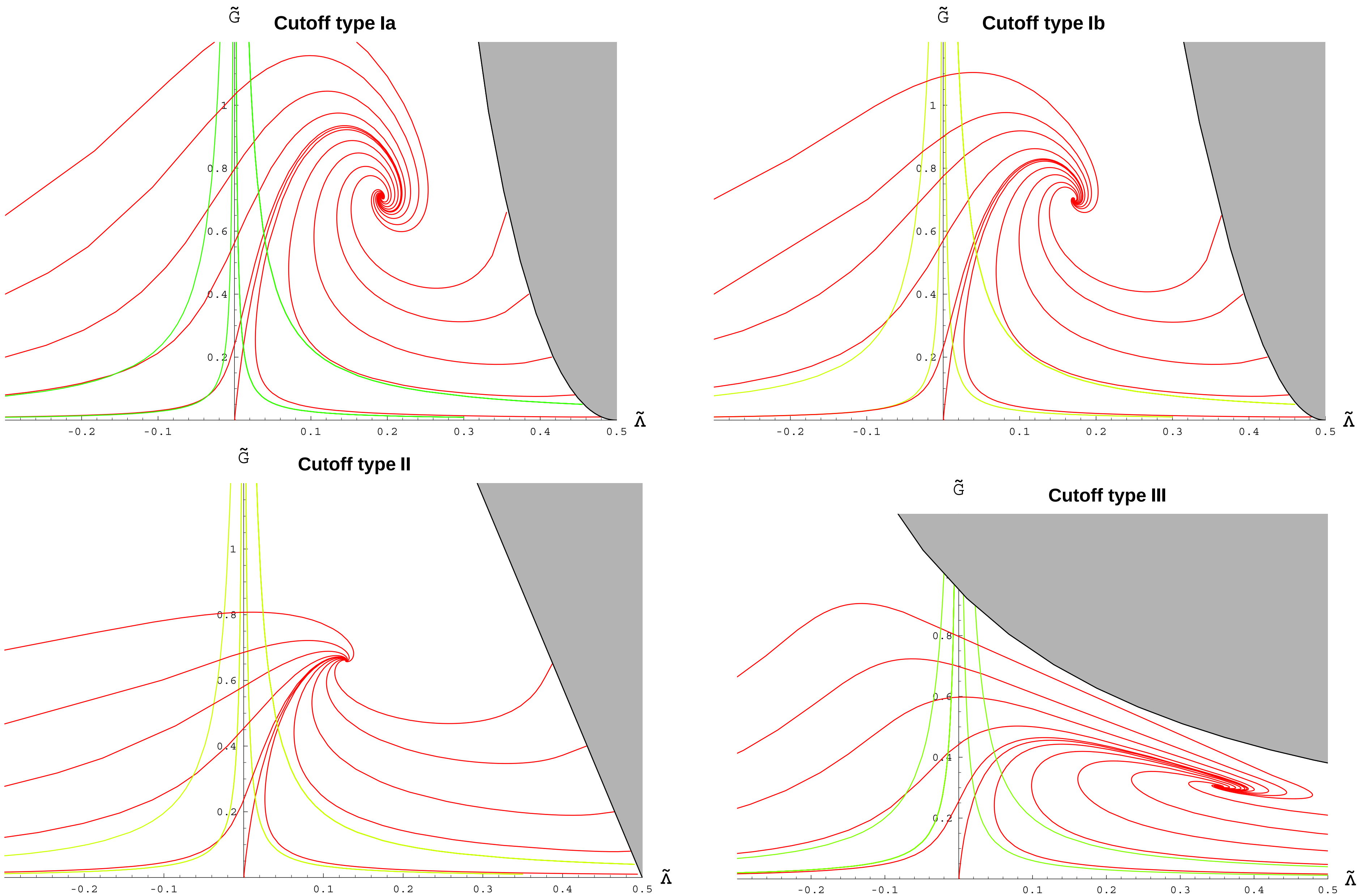
$$\beta_{\tilde{\Lambda}} = \frac{-2(1-2\tilde{\Lambda})^2\tilde{\Lambda} + \frac{36-41\tilde{\Lambda}+42\tilde{\Lambda}^2-600\tilde{\Lambda}^3}{72\pi}\tilde{G} + \frac{467-572\tilde{\Lambda}}{288\pi^2}\tilde{G}^2}{(1-2\tilde{\Lambda})^2 - \frac{29-9\tilde{\Lambda}}{72\pi}\tilde{G}}$$

$$\beta_{\tilde{G}} = \frac{2(1-2\tilde{\Lambda})^2\tilde{G} - \frac{373-654\tilde{\Lambda}+600\tilde{\Lambda}^2}{72\pi}\tilde{G}^2}{(1-2\tilde{\Lambda})^2 - \frac{29-9\tilde{\Lambda}}{72\pi}\tilde{G}}$$

Einstein–Hilbert truncation II



Einstein–Hilbert truncation III



EINSTEIN–HILBERT TRUNCATION IV

- Non–Gaussian FP with $\tilde{G}_* = \frac{1}{16\pi\tilde{Z}} = 0.701$ and $\tilde{\Lambda}_* = 0.171$, UV attractive in both directions.

  [W. Souma, arXiv:gr-qc/0006008.]
  [O. Lauscher and M. Reuter, Phys. Rev. D **65** (2002) 025013]

- Several tests prove robustness.

  [O. Lauscher and M. Reuter, Class. Quant. Grav. **19** (2002) 483]
  [O. Lauscher and M. Reuter, Int. J. Mod. Phys. A **17** (2002) 993]

- Adding term R^2 brings about relatively small modifications.

  [O. Lauscher and M. Reuter, Phys. Rev. D **66** (2002) 025026.]

- Other dimensions

  [P, Fischer and D.Litim, Phys.Lett.B638:497-502,2006]

- Need further tests

Divergences in Einstein theory

$$k\frac{d\Gamma_k}{dk}\bigg|_{\sim R^2} = \frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\left[\frac{7}{10}R_{\mu\nu}R^{\mu\nu} + \frac{1}{60}R^2 + \frac{53}{45}E - \frac{19}{15}\nabla^2 R\right]$$

$$g(k) \approx \log k$$

$$k\frac{d\Gamma_k}{dk}\bigg|_{\sim R^3} = \frac{c}{k^2}\frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\,R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}$$

$$g(k) \approx ck^{-2}$$

Higher derivative gravity

$$\Gamma_k = \int d^4x \sqrt{g} \left[2Z\Lambda - ZR + \frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 + \frac{1}{\rho} E \right]$$

$$Z = \frac{1}{16\pi G} \; ; \qquad \frac{1}{\xi} = -\frac{\omega}{3\lambda} \; ; \qquad \frac{1}{\rho} = \frac{\theta}{\lambda}$$

K.S. Stelle, Phys. Rev. **D16**, 953 (1977).

J. Julve, M. Tonin, Nuovo Cim. **46B**, 137 (1978).

E.S. Fradkin, A.A. Tseytlin, Phys. Lett. **104 B**, 377 (1981).

I.G. Avramidi, A.O. Barvinski, Phys. Lett. **159 B**, 269 (1985).

G. de Berredo–Peixoto and I. Shapiro, Phys.Rev. **D71** 064005 (2005).

A. Codello and R. P., Phys.Rev.Lett. **97** 22 (2006)

Gauge fixing

$$S_{GF} = \int d^4x\sqrt{g}\,\chi_\mu Y^{\mu\nu}\chi_\nu$$

$$Y^{\mu\nu} = \frac{1}{\alpha}\left[g^{\mu\nu}\nabla^2 + \gamma\nabla^\mu\nabla^\nu - \delta\nabla^\nu\nabla^\mu\right]$$

$$\chi_\nu = \nabla^\mu h_{\mu\nu} + \beta\nabla_\nu h$$

$$S_c = \int d^4x\sqrt{g}\,\bar{c}_\nu(\Delta_{gh})_\mu^\nu c^\mu$$

$$(\Delta_{gh})_\mu^\nu = -\delta_\mu^\nu\Box - (1+2\beta)\nabla_\mu\nabla^\nu + R_\mu^\nu$$

$$S_b = \frac{1}{2}\int d^4x\sqrt{g}\,b_\mu Y^{\mu\nu}b_\nu$$

$$\left(\Gamma_k + S_{GF}\right)^{(2)} = \frac{1}{2}\int d^4x\sqrt{g}\;\delta g\,\mathbf{K}\boldsymbol{\Delta}^{(4)}\delta g$$

$$\boldsymbol{\Delta}^{(4)} = \mathbf{1}\Box^2 + \mathbf{V}^{\rho\lambda}\nabla_\rho\nabla_\lambda + \mathbf{U}$$

Beta functions I

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$$

$$\beta_\xi = -\frac{1}{(4\pi)^2}\left(10\lambda^2 - 5\lambda\xi + \frac{5}{36}\right)$$

$$\beta_\rho = \frac{1}{(4\pi)^2}\frac{196}{45}\rho^2\lambda$$

$$\lambda(k) = \frac{\lambda_0}{1 + \lambda_0 \frac{1}{(4\pi)^2}\frac{133}{10}\log\left(\frac{k}{k_0}\right)}$$

$$\omega(k) \to \omega_* \approx -0.0228$$

$$\theta(k) \to \theta_* \approx 0.327$$

Beta functions II

$$\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{1}{(4\pi)^2}\left[\frac{1+20\omega^2}{256\pi\tilde{G}\omega^2}\lambda^2 + \frac{1+86\omega+40\omega^2}{12\omega}\lambda\tilde{\Lambda}\right]$$

$$-\frac{1+10\omega^2}{64\pi^2\omega}\lambda + \frac{2\tilde{G}}{\pi} - q(\omega)\tilde{G}\tilde{\Lambda}$$

$$\beta_{\tilde{G}} = 2\tilde{G} - \frac{1}{(4\pi)^2}\frac{3+26\omega-40\omega^2}{12\omega}\lambda\tilde{G} - q(\omega)\tilde{G}^2$$

where $q(\omega) = (83 + 70\omega + 8\omega^2)/18\pi$

Flow in $\tilde{\Lambda}$–$\tilde{G}$ plane I

$$\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{2\tilde{G}}{\pi} - q_* \tilde{G}\tilde{\Lambda}$$

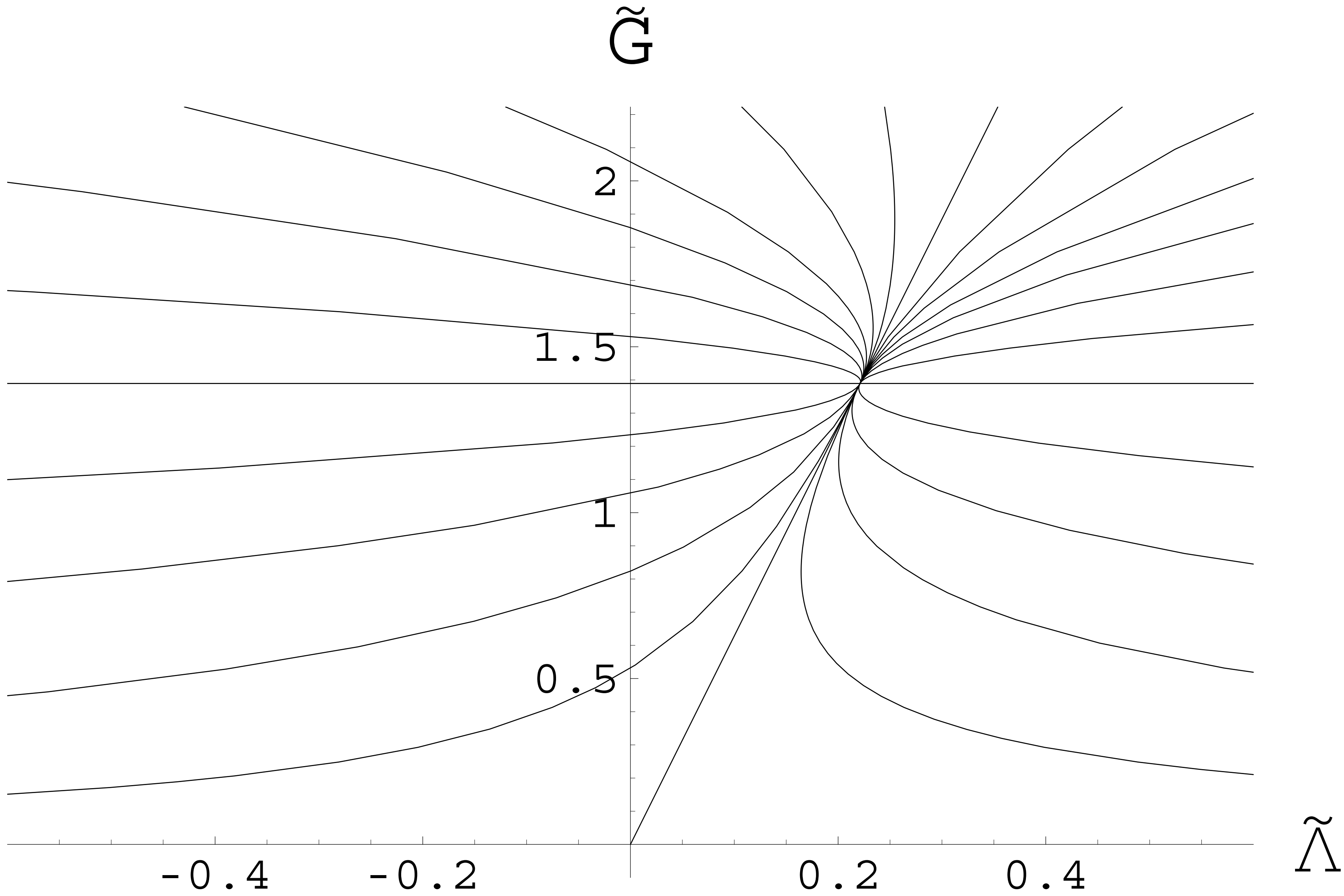
$$\beta_{\tilde{G}} = 2\tilde{G} - q_* \tilde{G}^2$$

where $q_* = q(\omega_*) \approx 1.440$

$$\tilde{\Lambda}(t) = \frac{(2\pi\tilde{\Lambda}_0 - \tilde{G}_0(1 - e^{4t}))e^{-2t}}{\pi(2 - q_*\tilde{G}_0(1 - e^{2t}))} \; ; \qquad \tilde{G}(t) = \frac{2\tilde{G}_0 e^{2t}}{2 - q_*\tilde{G}_0(1 - e^{2t})}$$

$$\tilde{\Lambda}_* = \frac{1}{\pi q_*} \approx 0.221 \; , \qquad \tilde{G}_* = \frac{2}{q_*} \approx 1.389 \; .$$

Flow in $\tilde{\Lambda}$–$\tilde{G}$ plane II



With matter

n_S scalars, n_D Dirac, n_M Maxwell fields, minimally coupled

$$\Delta\beta_\lambda = -2a_\lambda^{(4)}\lambda^2$$

$$\Delta\beta_\xi = -a_\xi^{(4)}\xi^2$$

$$\Delta\beta_\rho = -a_\rho^{(4)}\rho^2$$

$$\Delta\beta_{\tilde G} = 32\pi a^{(4)}\tilde G^2$$

$$\Delta\beta_{\tilde\Lambda} = 8\pi a^{(0)}\tilde G + 32\pi a^{(2)}\tilde\Lambda\tilde G$$

FP still exists

Coefficients

$$a^{(0)} = \frac{1}{32\pi^2}\left(n_S - 4n_D + 2n_M\right)$$

$$a^{(2)} = \frac{1}{96\pi^2}\left(n_S - 2n_D - 4n_M\right)$$

$$a_\lambda^{(4)} = \frac{1}{2880\pi^2}\left(\frac{3}{2}n_S + 9n_D + 18n_M\right)$$

$$a_\xi^{(4)} = \frac{1}{2880\pi^2}\left(-\frac{1}{2}n_S - \frac{11}{4}n_D - 31n_M\right)$$

$$a_\rho^{(4)} = \frac{1}{2880\pi^2}\frac{5}{2}n_S$$

$f(R)$ GRAVITY

$$\Gamma_k(g_{\mu\nu}) = \int d^4x \sqrt{g} f(R)$$

$$f(R) = \sum_{i=0}^{n} g_i(k) R^i$$

- workable for $n \leq 8$

  A. Codello, R.P. and C. Rahmede Int.J.Mod.Phys.A23:143-150 e-Print:0705.1769 [hep-th]; P.F. Machado, F. Saueressig, Phys. Rev. **D** e-Print: arXiv:0712.0445 [hep-th])

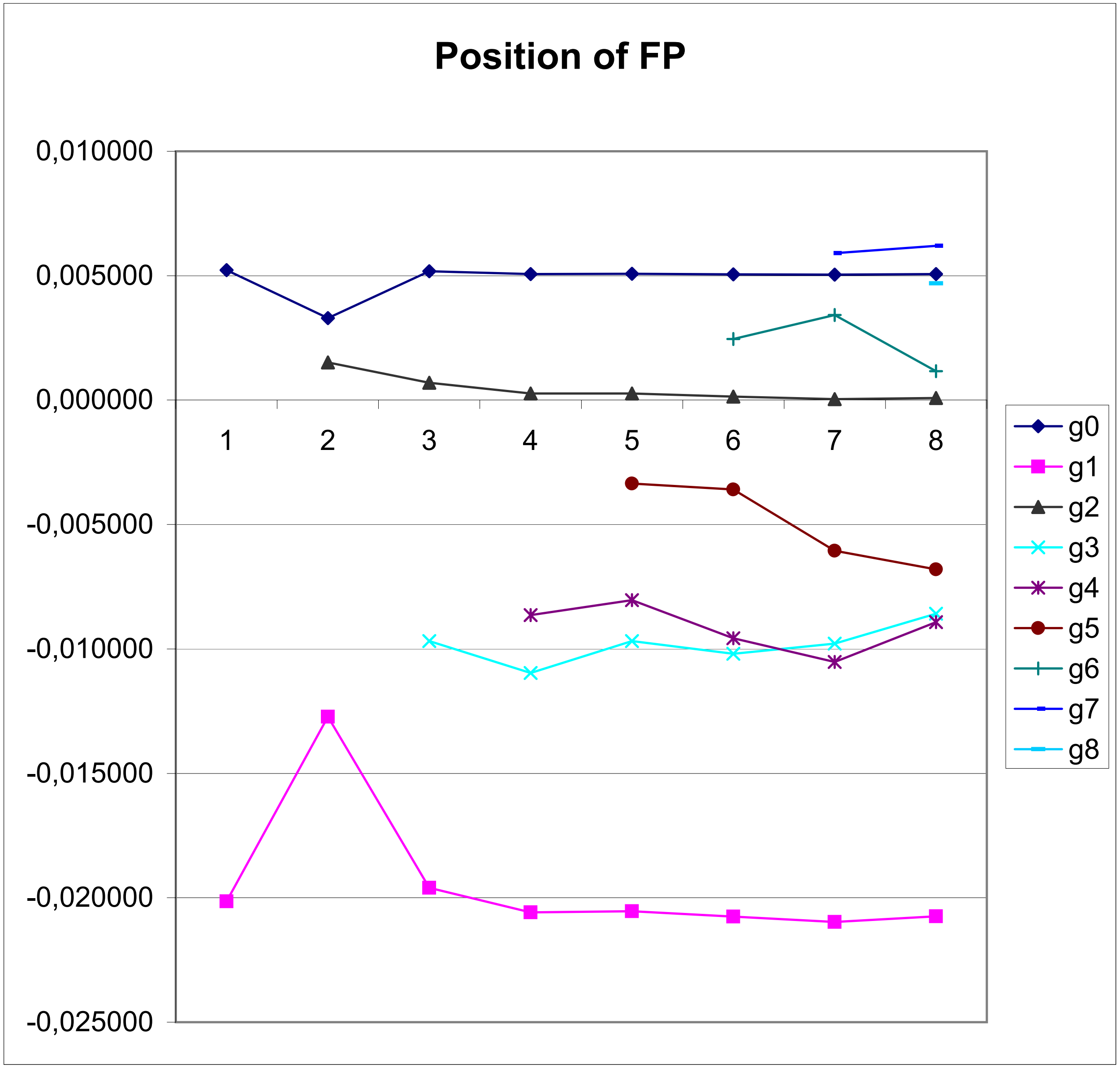
$f(R)$ GRAVITY II

Position of FixedPoint $(\times 10^{-3})$

n	$\tilde{g}_{0*}$	$\tilde{g}_{1*}$	$\tilde{g}_{2*}$	$\tilde{g}_{3*}$	$\tilde{g}_{4*}$	$\tilde{g}_{5*}$	$\tilde{g}_{6*}$	$\tilde{g}_{7*}$	$\tilde{g}_{8*}$
1	5.23	-20.1							
2	3.29	-12.7	1.51						
3	5.18	-19.6	0.70	-9.7					
4	5.06	-20.6	0.27	-11.0	-8.65				
5	5.07	-20.5	0.27	-9.7	-8.03	-3.35			
6	5.05	-20.8	0.14	-10.2	-9.57	-3.59	2.46		
7	5.04	-20.8	0.03	-9.78	-10.5	-6.05	3.42	5.91	
8	5.07	-20.7	0.09	-8.58	-8.93	-6.81	1.17	6.20	4.70

Critical exponents

n	$Re\vartheta_1$	$Im\vartheta_1$	ϑ_2	ϑ_3	$Re\vartheta_4$	$Im\vartheta_4$	ϑ_6	ϑ_7	ϑ_8
1	2.38	2.17							
2	1.38	2.32	26.9						
3	2.71	2.27	2.07	-4.23					
4	2.86	2.45	1.55	-3.91	-5.22				
5	2.53	2.69	1.78	-4.36	-3.76	-4.88			
6	2.41	2.42	1.50	-4.11	-4.42	-5.98	-8.58		
7	2.51	2.44	1.24	-3.97	-4.57	-4.93	-7.57	-11.1	
8	2.41	2.54	1.40	-4.17	-3.52	-5.15	-7.46	-10.2	-12.3


Position of FP
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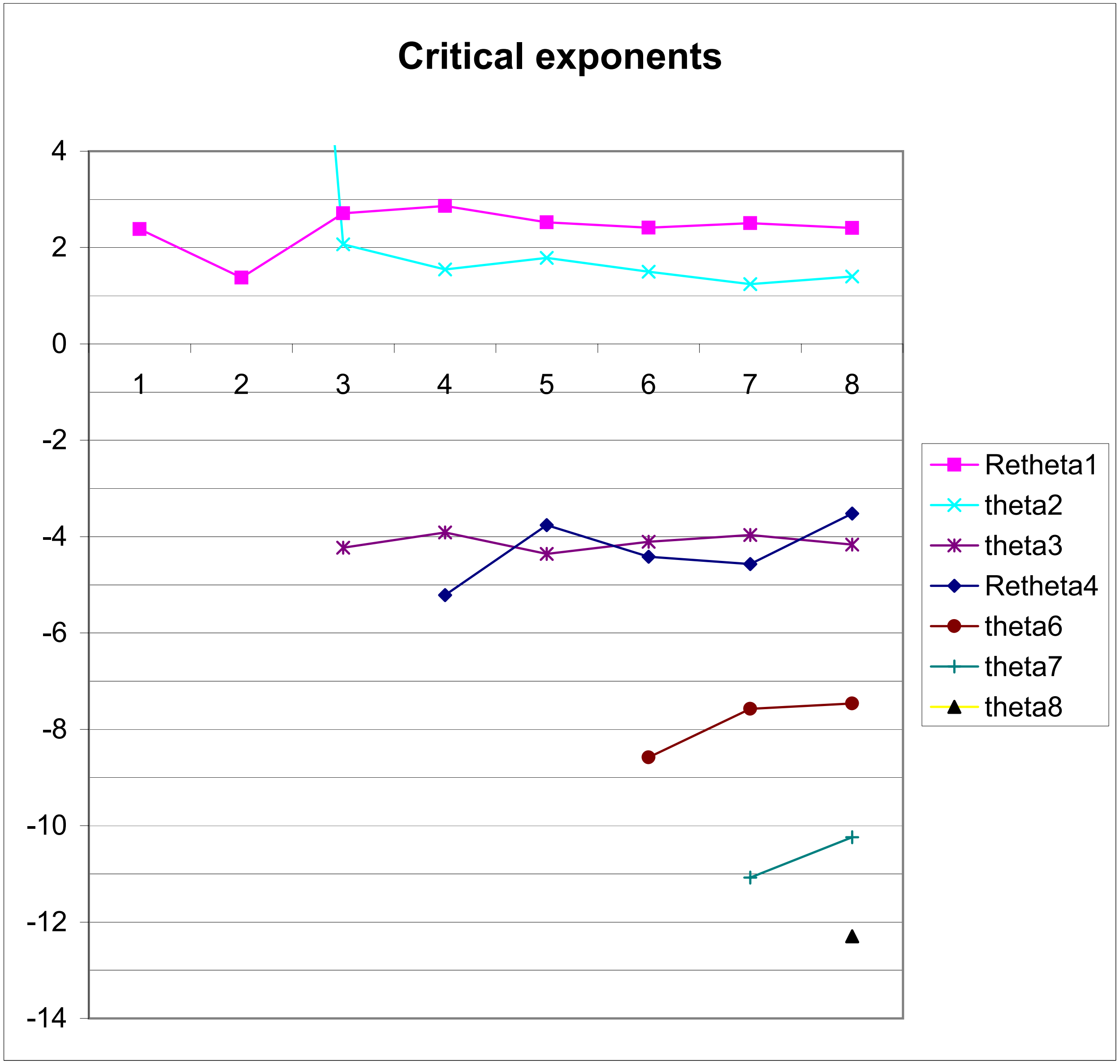
0,005000
0,000000
-0,005000
-0,010000
-0,015000
-0,020000
-0,025000
1
2
3
4
5
6
7
8
g0
g1
g2
g3
g4
g5
g6
g7
g8


Critical exponents
4
2
0
-2
-4
-6
-8
-10
-12
-14
1
2
3
4
5
6
7
8
Retheta1
theta2
theta3
Retheta4
theta6
theta7
theta8

$f(R)$ GRAVITY III

Critical surface:

$$\tilde{g}_3 = 0.00061243 + 0.06817374\,\tilde{g}_0 + 0.46351960\,\tilde{g}_1 + 0.89500872\,\tilde{g}_2$$

$$\tilde{g}_4 = -0.00916502 - 0.83651466\,\tilde{g}_0 - 0.20894019\,\tilde{g}_1 + 1.62075130\,\tilde{g}_2$$

$$\tilde{g}_5 = -0.01569175 - 1.23487788\,\tilde{g}_0 - 0.72544946\,\tilde{g}_1 + 1.01749695\,\tilde{g}_2$$

$$\tilde{g}_6 = -0.01271954 - 0.62264827\,\tilde{g}_0 - 0.82401181\,\tilde{g}_1 - 0.64680416\,\tilde{g}_2$$

$$\tilde{g}_7 = -0.00083040 + 0.81387198\,\tilde{g}_0 - 0.14843134\,\tilde{g}_1 - 2.01811163\,\tilde{g}_2$$

$$\tilde{g}_8 = 0.00905830 + 1.25429854\,\tilde{g}_0 + 0.50854002\,\tilde{g}_1 - 1.90116584\,\tilde{g}_2$$

Large N expansion I

$$n_S \approx n_D \approx n_M \approx N \to \infty$$

$$\Delta^{(S)} = -\nabla^2$$

$$\Delta^{(D)} = -\nabla^2 + \frac{R}{4}$$

$$\Delta^{(M)} = -\nabla^2 \delta^\mu_\nu + R^\mu{}_\nu$$

$$\partial_t \Gamma_k = \frac{n_S}{2} \mathrm{Tr}_{(S)} \left(\frac{\partial_t P_k(\Delta^{(S)})}{P_k(\Delta^{(S)})} \right) - \frac{n_D}{2} \mathrm{Tr}_{(D)} \left(\frac{\partial_t P_k(\Delta^{(D)})}{P_k(\Delta^{(D)})} \right) + \frac{n_M}{2} \mathrm{Tr}_{(M)} \left(\frac{\partial_t P_k(\Delta^{(M)})}{P_k(\Delta^{(M)})} \right) - n_M \mathrm{Tr}_{(gh)} \left(\frac{\partial_t P_k(\Delta^{(gh)})}{P_k(\Delta^{(gh)})} \right)$$

Large N expansion II

$$\Gamma_k = \sum_{n=0}^{\infty} \sum_i g_i^{(n)} \mathcal{O}_i^{(n)}$$

$$\partial_t \tilde{g}_i^{(n)} = (2n-4)\tilde{g}_i^{(n)} + a_i^{(n)}$$

For all $n \neq 2$: $\tilde{g}_{i*}^{(n)} = \frac{1}{4-2n} a_i^{(n)}$

For $n = 2$: $g_i^{(2)}(k) = g_i^{(2)}(k_0) + a_i^{(2)} \ln(k/k_0)$

If $R_k(z) = (k^2 - z)\theta(k^2 - z)$, $\tilde{g}_{i*}^{(n)} = 0$ for $n \geq 3$.

[R. P., Phys. Rev. D **73**, 041501 (2006),]

Summary and Conclusions

- Asymptotic Safety of gravity plausible
- Bottom up approach
- Not exclusive
- Implications

FAQs and Bibliography

http://www.percacci.it/roberto/physics/as/