Non-thermal fixed points

Jürgen Berges (TU Darmstadt)
Alexander Rothkopf (Tokyo U.)
Jonas Schmidt (DESY)

Phys. Rev. Lett., in print (2008)
arXiv:0803.0131 [hep-ph]

Content

I. Nonequilibrium functional RG/
   2PI effective action

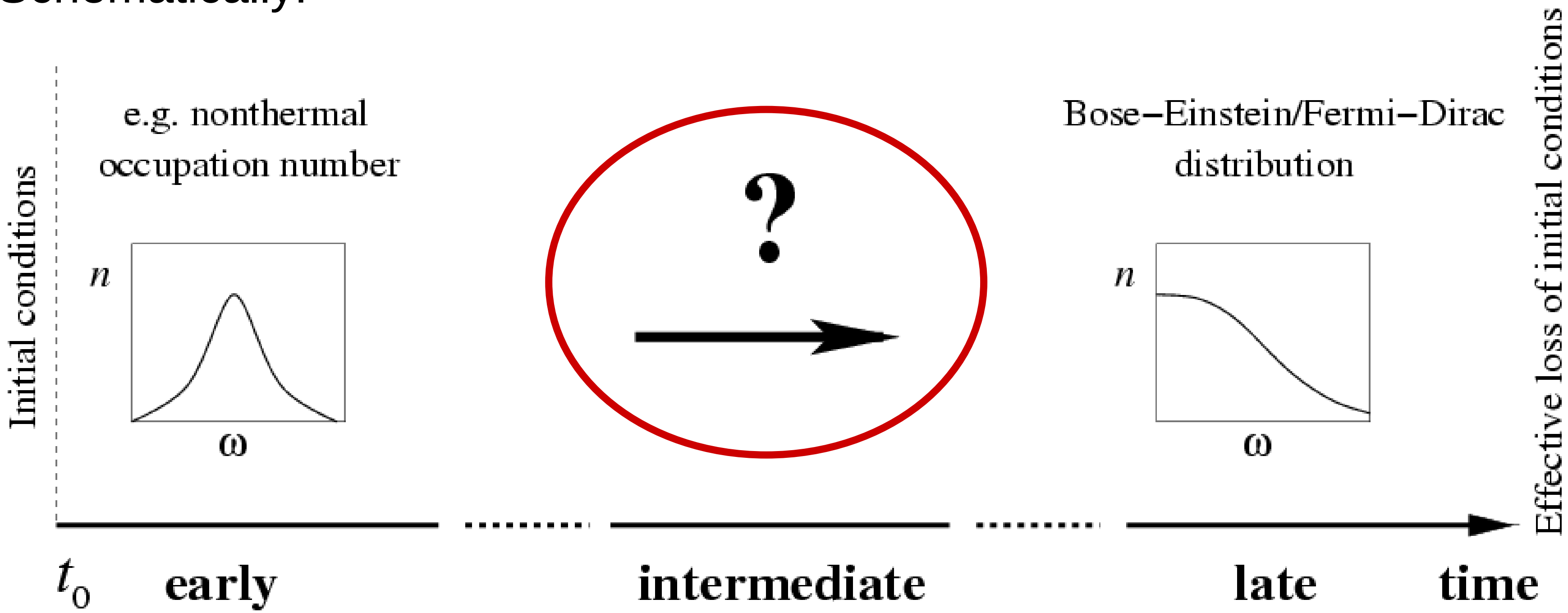
II. Non-thermal fixed points

III. Applications
   - inflaton preheating dynamics
   - fermion instability dynamics
   - non-Abelian plasma instabilities

Nonequilibrium dynamics

Thermalization process?

Schematically:



- Characteristic nonequilibrium time scales?
- *Diverging time scales far from equilibrium, described by scaling solutions?*

→ Non-thermal fixed points

Nonequilibrium vs. equilibrium

E.g. time-ordered 2-point function: $G(x,y) = \langle T_C\, \Phi(x)\Phi(y)\rangle - \langle\Phi(x)\rangle\langle\Phi(y)\rangle$

spectral function $\sim \langle[\Phi,\Phi]\rangle$ i.e. which states are available?

$$\boxed{G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0 - y^0)}$$

statistical propagator $\sim \langle\{\Phi,\Phi\}\rangle$ i.e. how often is a state occupied?

Nonequilibrium vs. equilibrium

E.g. time-ordered 2-point function: $G(x,y) = \langle T_C\, \Phi(x)\Phi(y)\rangle - \langle\Phi(x)\rangle\langle\Phi(y)\rangle$

spectral function $\sim \langle[\Phi,\Phi]\rangle$ i.e. which states are available?

$$\boxed{G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0 - y^0)}$$

statistical propagator $\sim \langle\{\Phi,\Phi\}\rangle$ i.e. how often is a state occupied?

Tremendous simplification in thermal equilibrium $G(x,y)=G^{(eq)}(x-y)$:

$$F^{(\mathrm{eq})}(\omega,\mathbf{p}) = -i\left(\frac{1}{2} + n_{\mathrm{BE}}(\omega)\right)\rho^{(\mathrm{eq})}(\omega,\mathbf{p})$$ *“ fluctuation-dissipation relation“*

Nonequilibrium vs. equilibrium

E.g. time-ordered 2-point function: $G(x,y) = \langle T_C\, \Phi(x)\Phi(y)\rangle - \langle\Phi(x)\rangle\langle\Phi(y)\rangle$

spectral function $\sim \langle[\Phi,\Phi]\rangle$ i.e. which states are available?

$$\boxed{G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0 - y^0)}$$

statistical propagator $\sim \langle\{\Phi,\Phi\}\rangle$ i.e. how often is a state occupied?

Tremendous simplification in thermal equilibrium $G(x,y)=G^{(eq)}(x-y)$:

$$F^{(\mathrm{eq})}(\omega,\mathbf{p}) = -i\left(\frac{1}{2} + n_{\mathrm{BE}}(\omega)\right)\rho^{(\mathrm{eq})}(\omega,\mathbf{p})$$ *"fluctuation-dissipation relation"*

Nonequilibrium:

$$\boxed{F \not\sim \rho}$$

(fermions: $F^{(f)}(x,y) = \frac{1}{2}\langle[\Psi(x),\bar{\Psi}(y)]\rangle$, $\rho^{(f)}(x,y) = i\langle\{\Psi(x),\bar{\Psi}(y)\}\rangle$)

Infrared scaling

Scaling behavior:

$$F(\omega, \mathbf{p}) = s^{2+\alpha} F(s^z \omega, s\mathbf{p}) \qquad , \qquad \rho(\omega, \mathbf{p}) = s^{2-\eta} \rho(s^z \omega, s\mathbf{p})$$

with universal critical exponents η, *z* and α (“occupation number exponent“)

Infrared scaling

Scaling behavior:

$$F(\omega,\mathbf{p}) = s^{2+\alpha} F(s^z\omega, s\mathbf{p}) \quad , \quad \rho(\omega,\mathbf{p}) = s^{2-\eta} \rho(s^z\omega, s\mathbf{p})$$

with universal critical exponents η, *z* and α (“occupation number exponent“)

- Thermal equilibrium: $n_{BE}(\omega) \overset{\omega \ll T}{\sim} \frac{T}{\omega}$

  i.e. from the fluctuation-dissipation relation $\Rightarrow \quad \alpha - z + \eta = 0$

  For relativistic 3+1d scalar theories in *equilibrium* typically:

$$z \simeq 1\,,\ \eta \simeq 0 \quad \text{i.e.} \quad \alpha \simeq 1$$

Infrared scaling

Scaling behavior:

$$F(\omega, \mathbf{p}) = s^{2+\alpha} F(s^z \omega, s\mathbf{p}) \quad , \quad \rho(\omega, \mathbf{p}) = s^{2-\eta} \rho(s^z \omega, s\mathbf{p})$$

with universal critical exponents η, *z* and α ("occupation number exponent")

- Thermal equilibrium: $n_{BE}(\omega) \overset{\omega \ll T}{\sim} \dfrac{T}{\omega}$

  i.e. from the fluctuation-dissipation relation $\Rightarrow \quad \alpha - z + \eta = 0$

  For relativistic 3+1d scalar theories in *equilibrium* typically:

$$z \simeq 1\,,\; \eta \simeq 0 \quad \text{i.e.} \quad \alpha \simeq 1$$

- *Non-thermal* infrared fixed point: e.g.

$$\alpha \simeq 4 \; (!)$$

  $\rightarrow$ *strongly enhanced infrared fluctuations*

Nonperturbative descriptions

$$\Gamma$$

Two-particle irreducible expansions

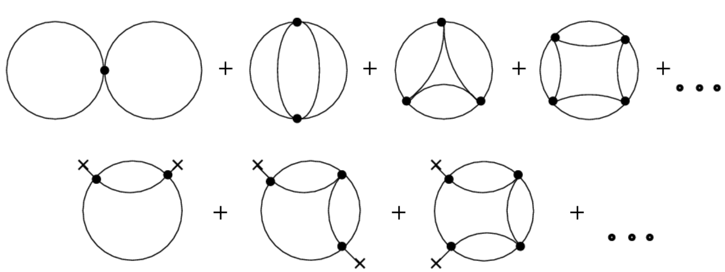
$$\Gamma[\phi,G] = S[\phi] + \frac{1}{2}\mathrm{Tr}\ln G^{-1} + \frac{1}{2}\mathrm{Tr}\, G_0^{-1}(\phi)G + \Gamma_2[\phi,G]$$

$$\delta\Gamma/\delta G = 0$$

Luttinger, Ward ´60; Baym ´62;
Cornwall, Jackiw, Tomboulis ´74,…

2PI-1/N to NLO: ($\lambda(\phi_a\phi_a)^2$)

$$\Gamma_2[\phi,G] \; =$$



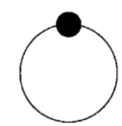
Berges ´02 ; Aarts, Ahrensmeier, Baier, Berges, Serreau ´02

Functional renormalization group

$$\Gamma_k[\phi] = \Gamma[\phi,G] + \frac{1}{2}\mathrm{Tr}\, GR$$

$R = R(k)$: $\quad k\frac{\partial}{\partial k} \equiv 2k^2\frac{\partial}{\partial k^2}$

$$\frac{\partial \Gamma_R[\phi]}{\partial k} = \frac{1}{2}\mathrm{Tr}\, G\frac{\partial R}{\partial k}$$

$$\frac{\partial \Gamma_k}{\partial k} = \frac{1}{2}$$ 

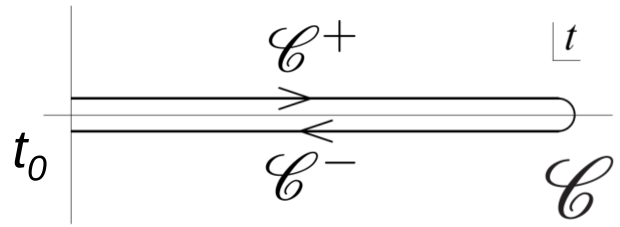
Wetterich ‘93

$$G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0-y^0)$$

Nonequilibrium proper vertices

Closed *real-time* contour (no imaginary part; $t_0 \to -\infty$ for fixed points):

$$\int_x \equiv \int_{\mathscr{C}} \mathrm{d}x^0 \int \mathrm{d}^d x$$



with $\phi^{\pm}(x): \quad x^0 \in \mathcal{C}^{\pm}$

$$\phi(x) = \frac{1}{2}\left(\phi^+ + \phi^-\right) \quad , \quad \tilde{\phi}(x) = \phi^+ - \phi^-$$

Nonequilibrium average action $\Gamma_k[\phi, \tilde{\phi}]$, correlators and proper vertices:

$$\Gamma_{\phi\tilde{\phi}}(x,y) \equiv \frac{\delta^2 \Gamma_k[\phi,\tilde{\phi}]}{\delta\phi(x)\delta\tilde{\phi}(y)} \quad , \quad \Gamma_{\phi\tilde{\phi}\phi\phi}(x,y,z,w) \equiv \frac{\delta^4 \Gamma_k[\phi,\tilde{\phi}]}{\delta\phi(x)\delta\tilde{\phi}(y)\delta\phi(z)\delta\phi(w)} \quad , \ldots$$

Retarded/advanced $G^R(x,y) = G^A(y,x)$ with $\rho(x,y) = G_R(x,y) - G_A(x,y)$ and

$$G_R = -\left(\Gamma_{\tilde{\phi}\phi} + R_R\right)^{-1},\ G_A = -\left(\Gamma_{\phi\tilde{\phi}} + R_A\right)^{-1},\ F = -i\left(\Gamma_{\tilde{\phi}\phi} + R_R\right)^{-1}\Gamma_{\tilde{\phi}\tilde{\phi}}\left(\Gamma_{\phi\tilde{\phi}} + R_A\right)^{-1}$$

$$(R_R(x,y) = R_A(y,x))$$

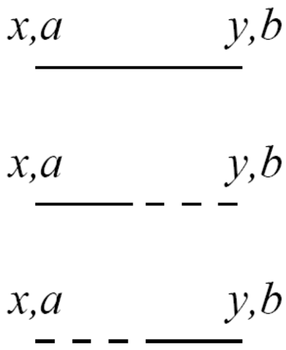
Diagrammatics

Nonequilibrium propagators:

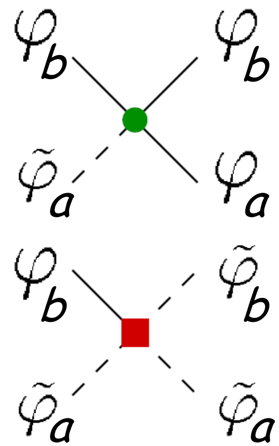
$F_{ab}(x,y) =$

$G^R_{ab}(x,y) =$

$G^A_{ab}(x,y) =$



Classical vertices, here quartic:

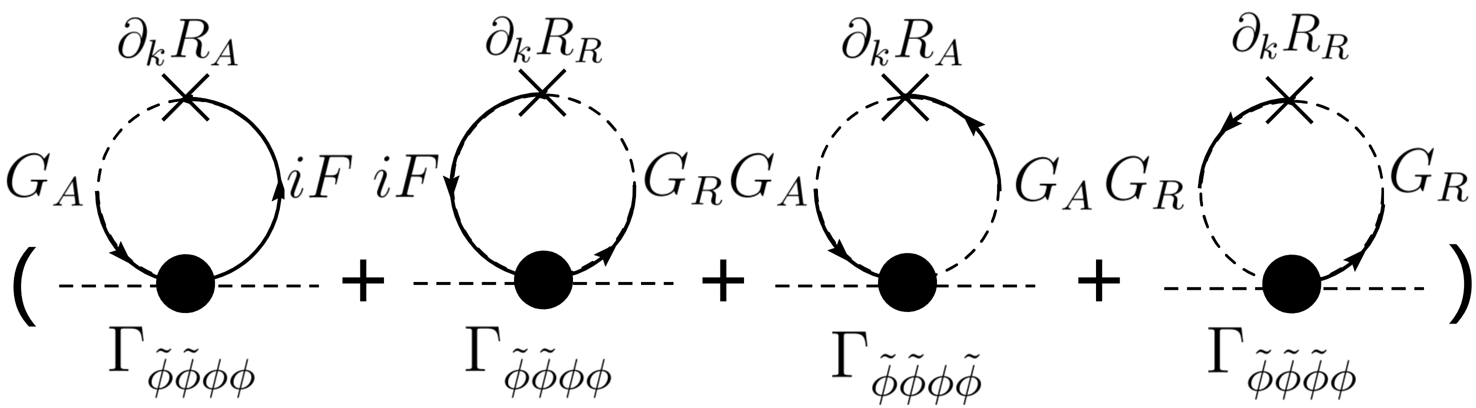


absent in classical-statistical limit ($F^2 \gg \rho^2$)

Exact RG: $\phi = \tilde\phi = 0$

e.g.

$\partial_k \Gamma_{\tilde\phi\tilde\phi} = -\frac{i}{2}\Big($



$\Big)$

accordingly for $\partial_k \Gamma_{\tilde\phi\tilde\phi\phi\phi}$ etc., all in *Minkowski space*

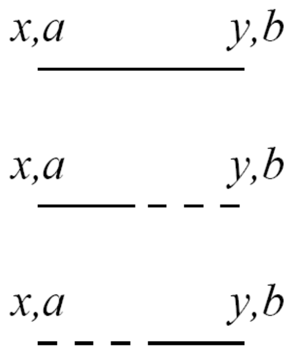
Diagrammatics

Nonequilibrium propagators:

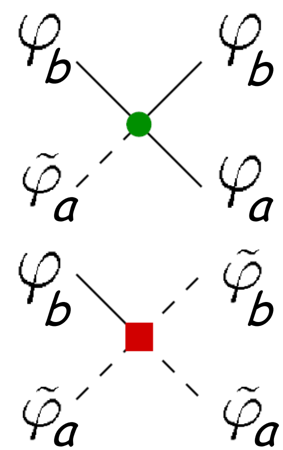
$F_{ab}(x,y) =$

$G^R_{ab}(x,y) =$

$G^A_{ab}(x,y) =$



Classical vertices, here quartic:

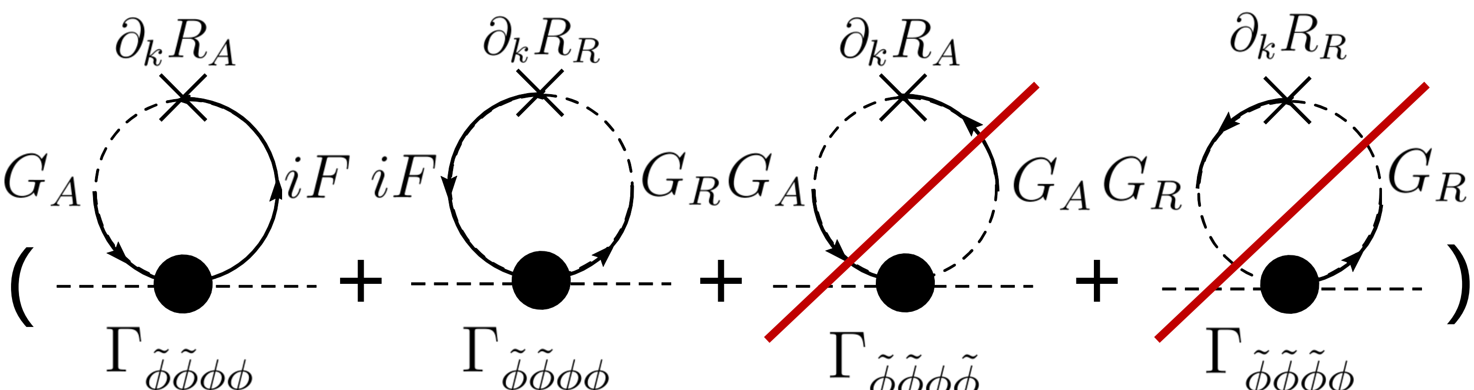


absent in classical-statistical limit ($F^2 \gg \rho^2$)

Exact RG: $\phi = \tilde\phi = 0$

e.g.

$$\partial_k \Gamma_{\tilde\phi\tilde\phi} = -\frac{i}{2}\Big(\quad\Big)$$



subleading for fixed point

accordingly for $\partial_k \Gamma_{\tilde\phi\tilde\phi\phi\phi}$ etc., all in *Minkowski space*

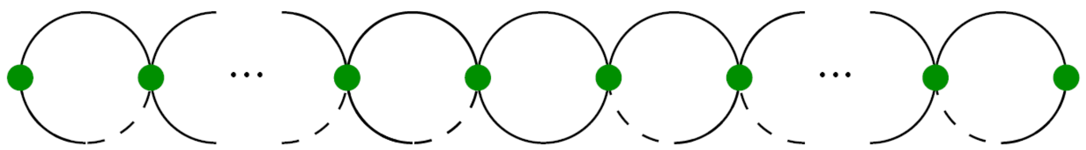
Approximations

N-component scalar theory with quartic interactions:

- classical-statistical limit
- estimate four-vertices from 2PI *1/N* expansion to NLO:

$$\Gamma_{\tilde{\phi}\tilde{\phi}\phi\phi}(p) = -i\,\lambda_{\text{eff}}(p)\,\frac{\lambda}{9}\int_q F(p-q)F(q) \quad \text{, etc.}$$

$$\lambda_{\text{eff}}(p) = \frac{\lambda}{\left|1 + \frac{\lambda}{3}\int_q G_R(p-q)F(q)\right|^2}$$



- Sharp cutoff with $R_R(p) = R_A(p) = \begin{cases} 0 & \text{for } p^2 \geq k^2 \\ \infty & \text{for } p^2 < k^2 \end{cases}$

$$\Rightarrow \quad iF(p)\left[G_A(p)k\partial_k R_A(p) + G_R(p)k\partial_k R_R(p)\right] = -iF(p)2k^2\delta(p^2-k^2)$$

Fixed points

Identity: ($\Sigma_F(p)\;\rho(p) \;=\; F(p)\;\Sigma_\rho(p)$ i.e. "gain = loss")

$$i\Gamma_{\tilde\phi\tilde\phi}(p)\left[\Gamma^{-1}_{\phi\tilde\phi}(p)-\Gamma^{-1}_{\tilde\phi\phi}(p)\right] = -i\Gamma^{-1}_{\tilde\phi\phi}(p)\Gamma_{\tilde\phi\tilde\phi}(p)\Gamma^{-1}_{\phi\tilde\phi}(p)\left[\Gamma_{\phi\tilde\phi}(p)-\Gamma_{\tilde\phi\phi}(p)\right]$$

Integrating the RG equations with above approximations yields:

$$i\Gamma_{\tilde\phi\tilde\phi}(p) \;\equiv\; \Sigma_F(p) \;=\; -\frac{\lambda}{18}\int_{q,l}\lambda_{\mathrm{eff}}(p-q)\,F(p-q-l)F(q)F(l)$$

$$\begin{aligned}\Gamma_{\phi\tilde\phi}(p)-\Gamma_{\tilde\phi\phi}(p) \;\equiv\; \Sigma_\rho(p) \;=\; -\frac{\lambda}{18}\int_{q,l}\lambda_{\mathrm{eff}}(p-q)\;\Big[\;&\rho(p-q-l)F(q)F(l)\\ &+F(p-q-l)\rho(q)F(l)\\ &+F(p-q-l)F(q)\rho(l)\Big]\end{aligned}$$

⇒ infrared scaling solutions: (z = 1, η = 0)

$$\begin{aligned}F(p) &= s^{2+\alpha}\,F(sp)\\ \rho(p) &= s^{2}\,\rho(sp)\\ \lambda_{\mathrm{eff}}(p) &= s^{\gamma}\,\lambda_{\mathrm{eff}}(sp)\end{aligned}$$

$$\gamma = -2\alpha$$

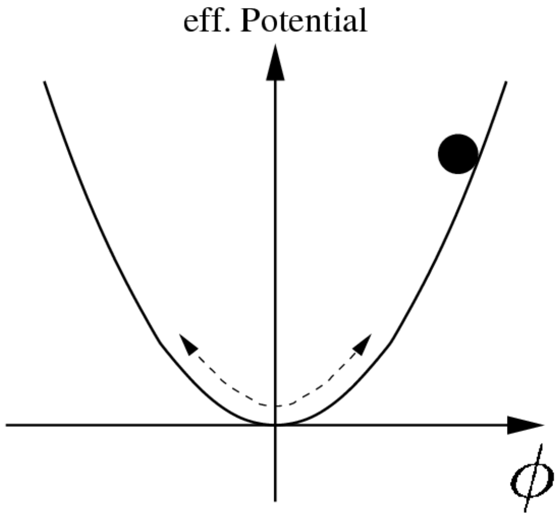
$$\boxed{\alpha = 0\,,\;\alpha = 1\,,\;\alpha = 4\,,\;\alpha = 5}$$

vacuum thermal ***non-thermal***

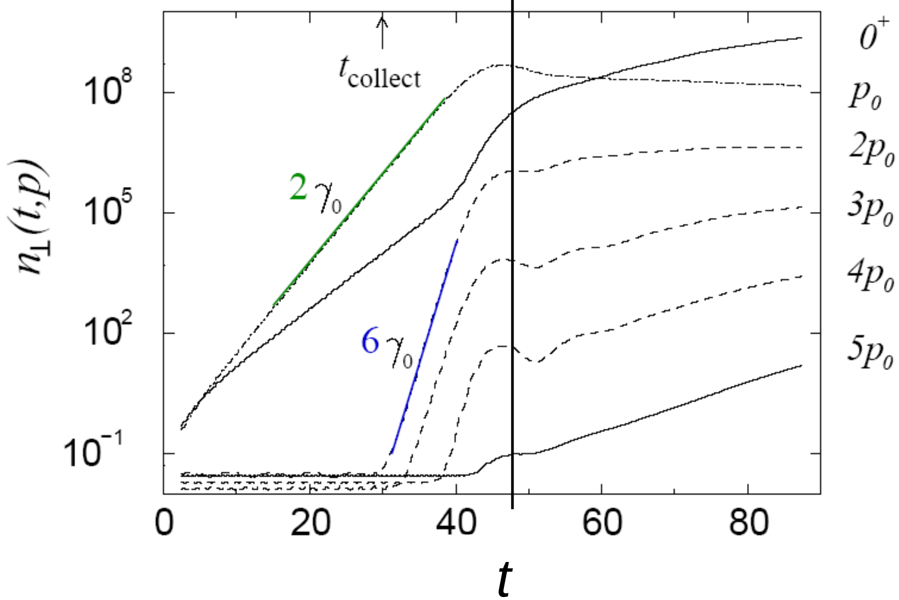
Applications

I) Early universe inflaton preheating dynamics

a) Parametric resonance instability:



fast slow



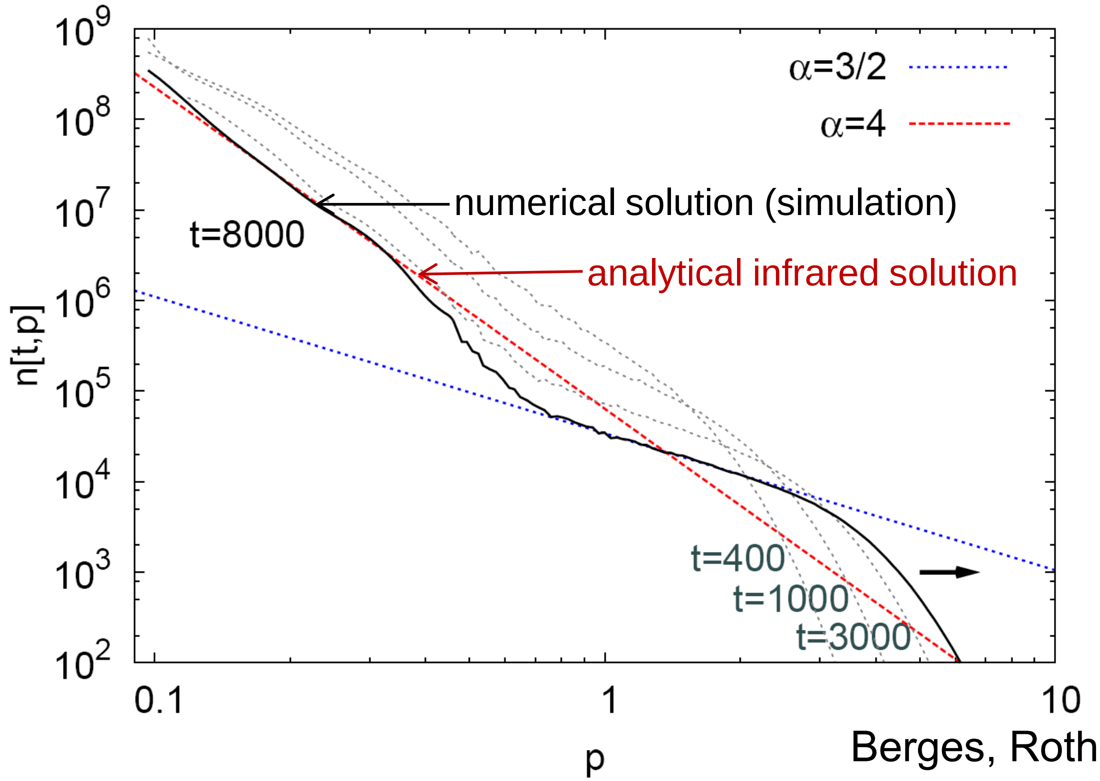
$$n(t,\mathbf{p}) + 1/2 = [F(t,t';\mathbf{p})\partial_t\partial_{t'}F(t,t';\mathbf{p})]^{1/2}_{t=t'}$$

fast: *Nonlinear:* occupation numbers < $1/\lambda$

slow: *Nonperturbative:* saturated occupation numbers $\sim 1/\lambda$
$\rightarrow$ *all processes* $\mathcal{O}(1)$

governed by non-thermal fixed point!

Intermediate-time behavior



Simulation:
460^3 lattice

Berges, Rothkopf, Schmidt '08

Well characterized by non-thermal fixed point!

UV: α = 3/2 coincides with perturbative (Boltzmann) analysis exponent

Micha, Tkachev '04

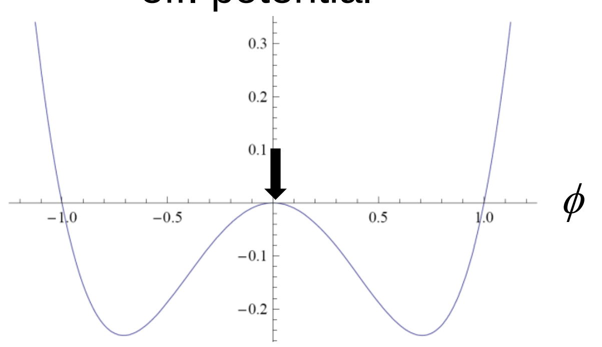
Critical slowing down delays *thermalization*:

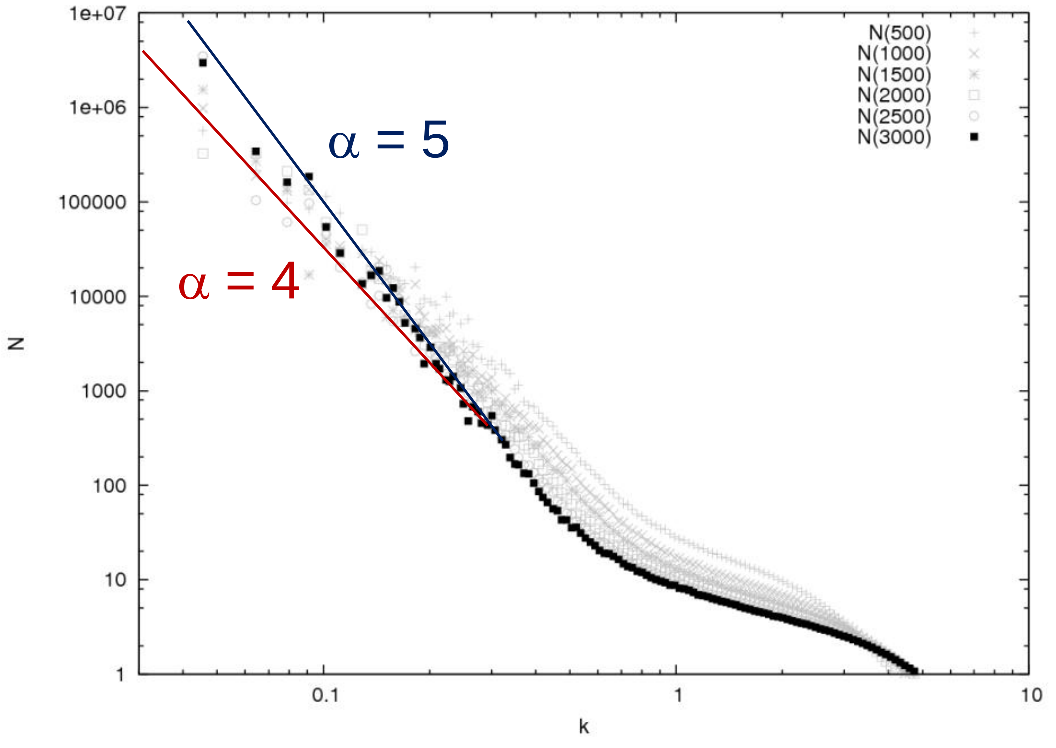
T_{reheat} > 10MeV (BBN!) difficult to achieve in simple inflaton models

Non-thermal fixed points from large class of nonequilibrium instabilities

b) Spinodal/tachyonic instability:

Classical-statistical simulation
(*quench*)

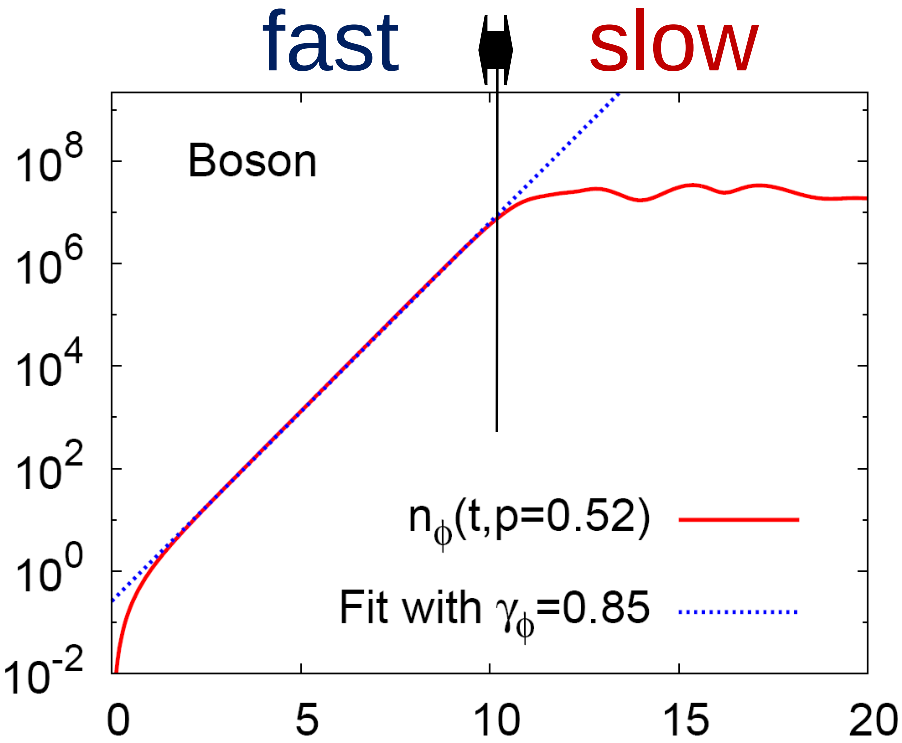


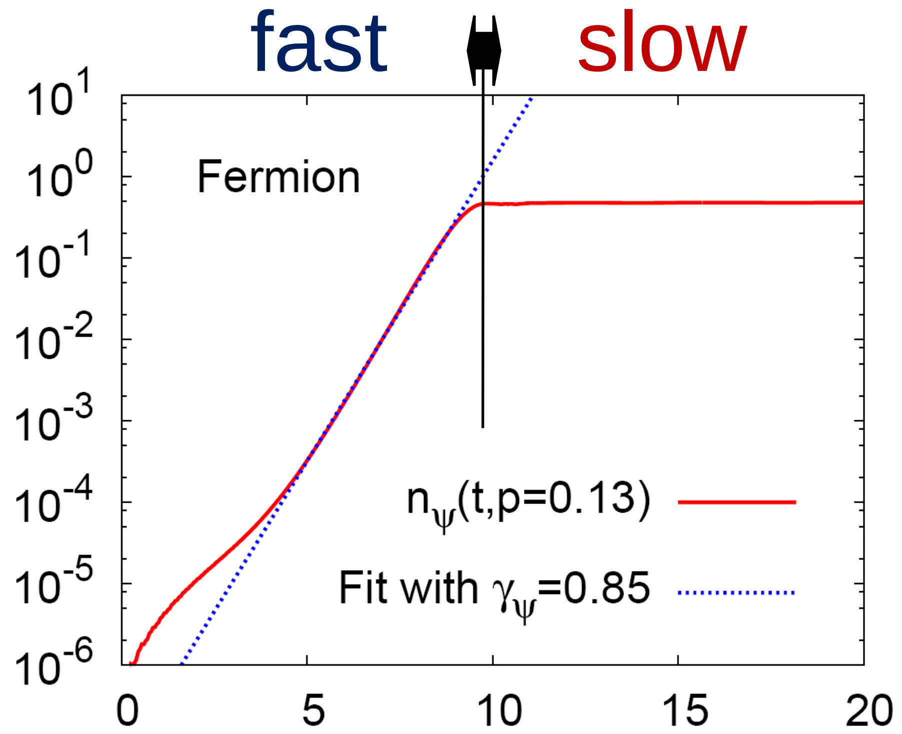


II) Fermion instability dynamics

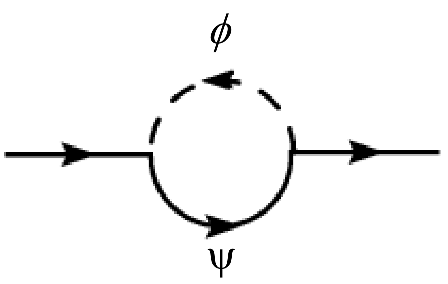
SU(2)×SU(2) Yukawa theory after a *quench* (2PI *1/N* to NLO):

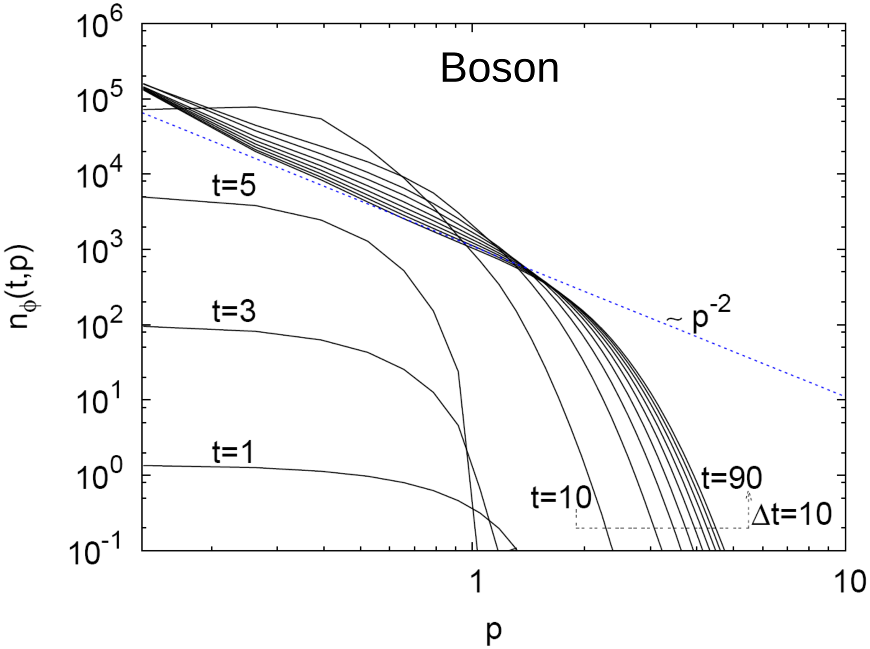
with Pruschke, Rothkopf

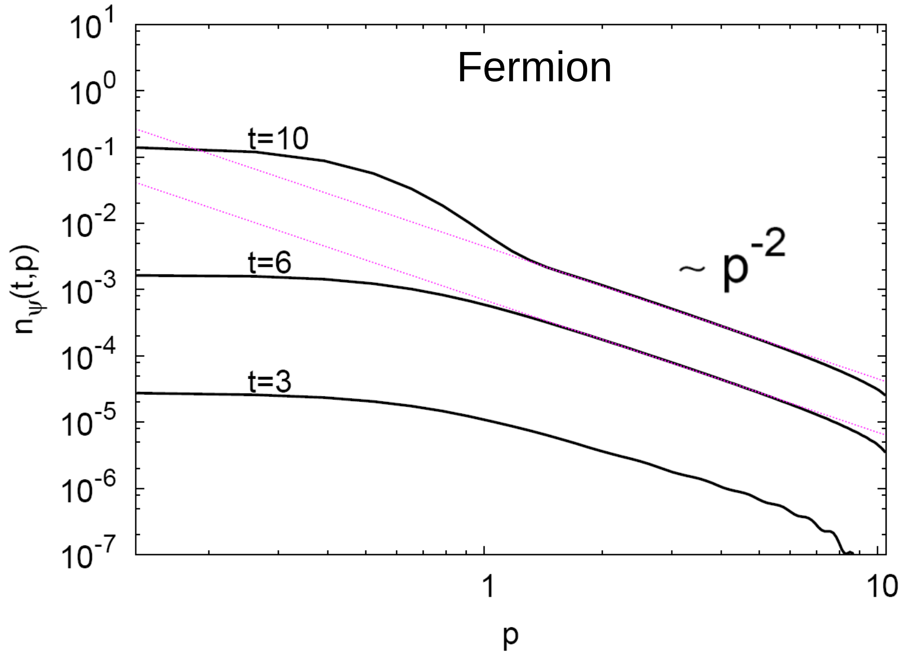




- fermionic instability induced via boson-fermion loop:

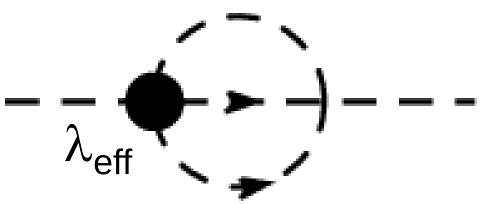


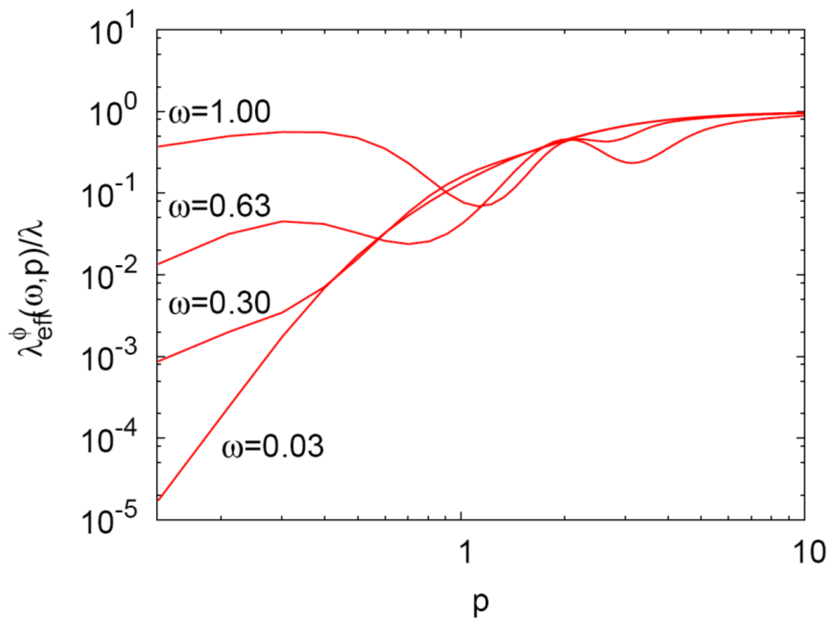




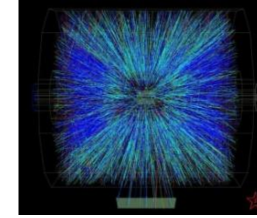
- no IR fixed point for fermions (Pauli principle); IR bosons unaffected

- UV scaling regime, where $\lambda_{\mathrm{eff}} \sim \lambda$:



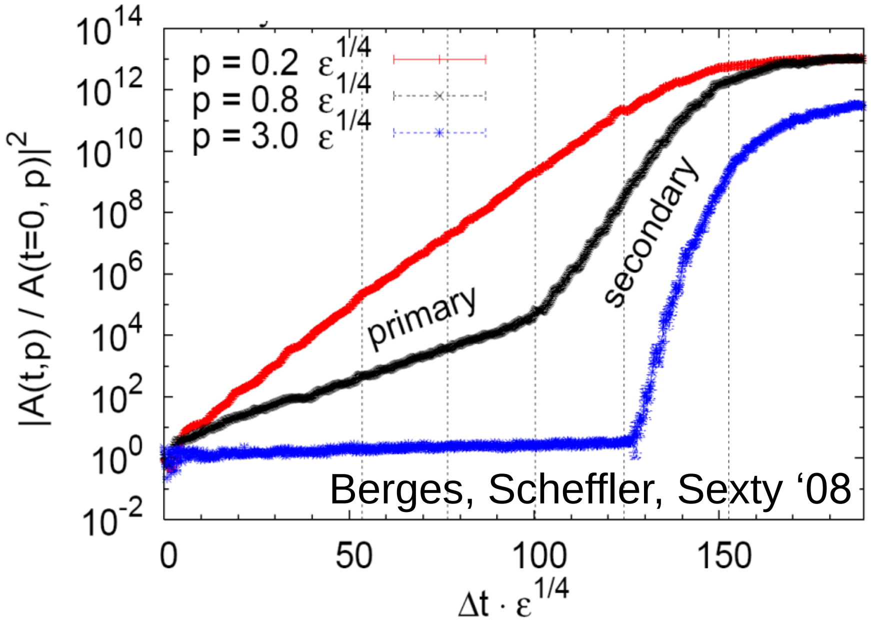


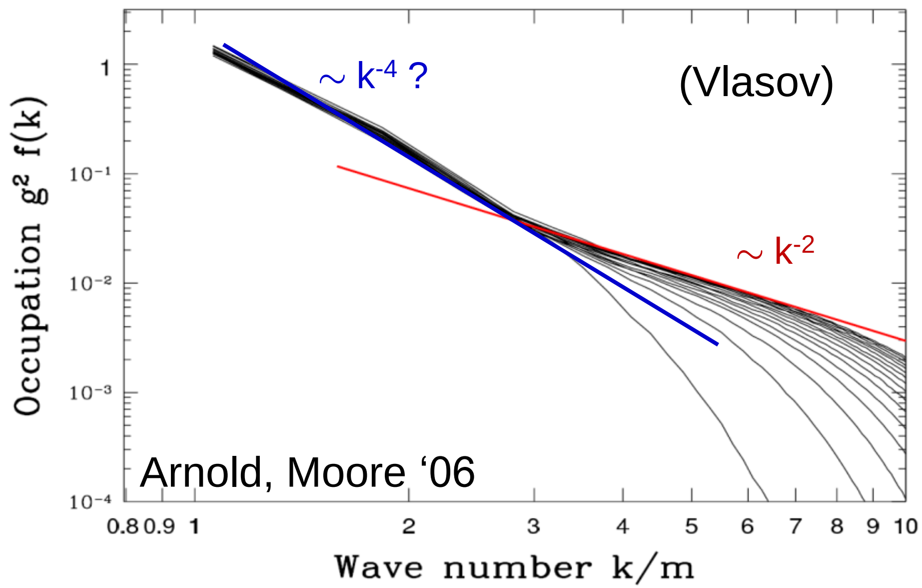
III) Non-Abelian gauge theory dynamics



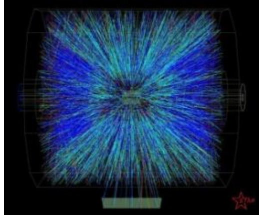
c) Plasma (Chromo-Weibel) instability:

classical-statistical SU(2) pure gauge theory in 3+1d (initial anisotropy)



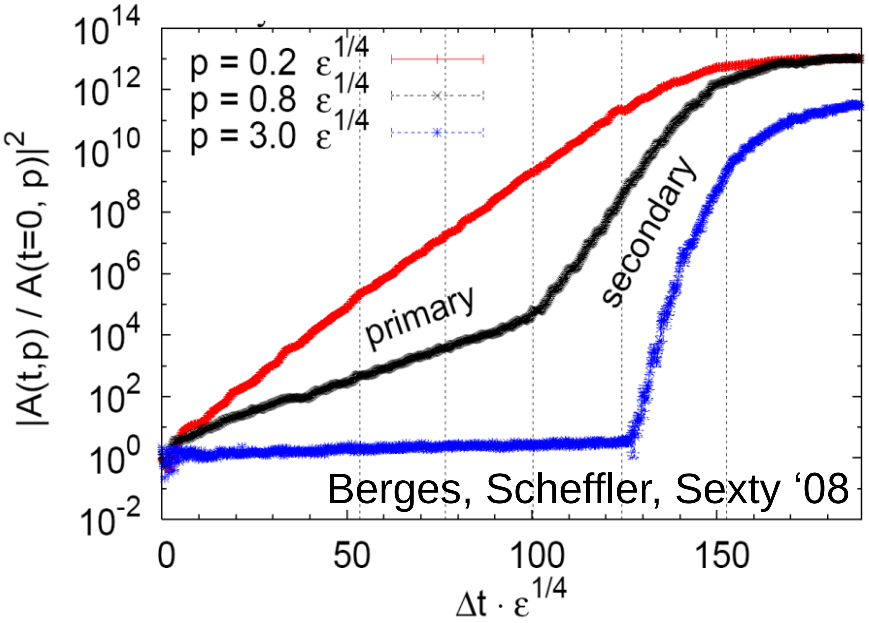


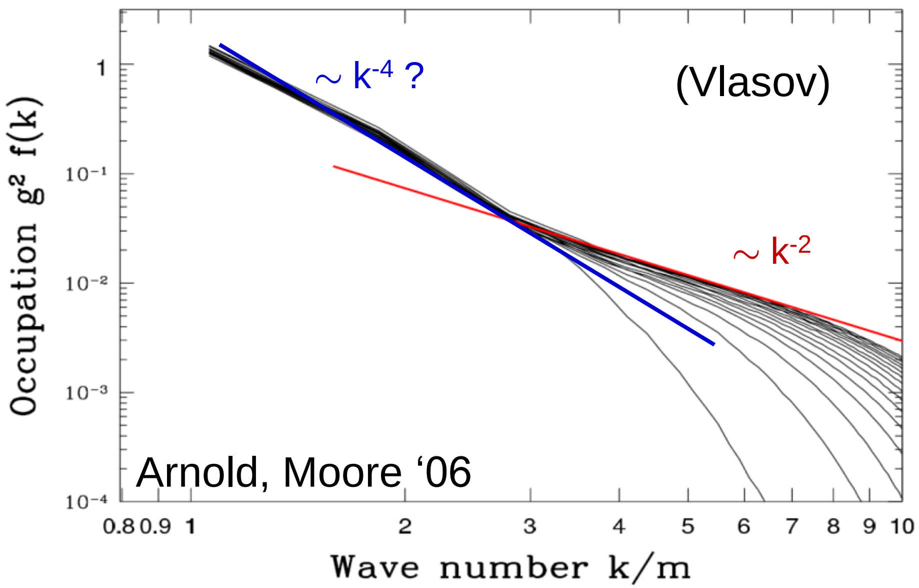
III) Non-Abelian gauge theory dynamics



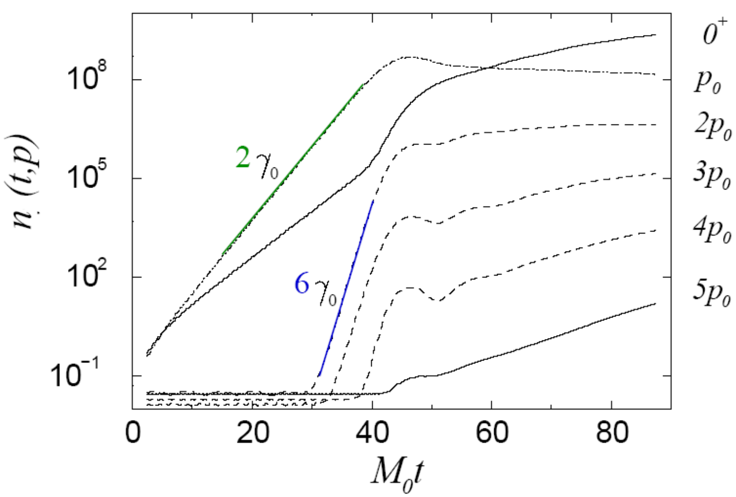
c) Plasma (Chromo-Weibel) instability:

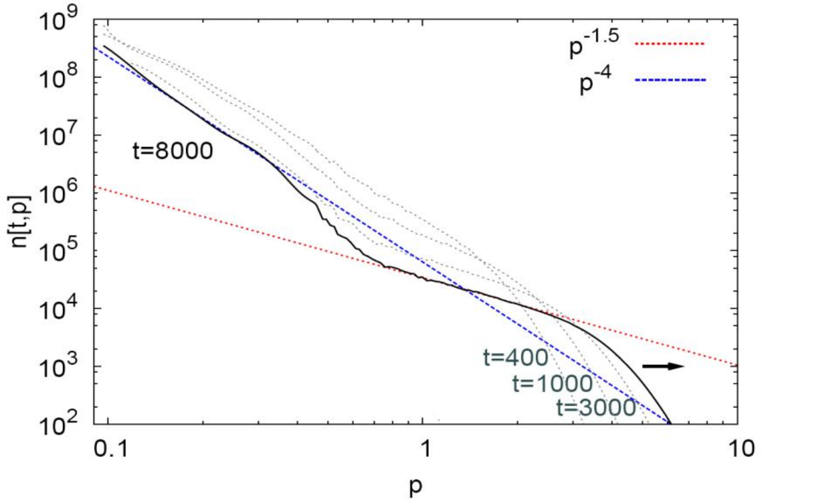
classical-statistical SU(2) pure gauge theory in 3+1d (initial anisotropy)





Compare inflaton:

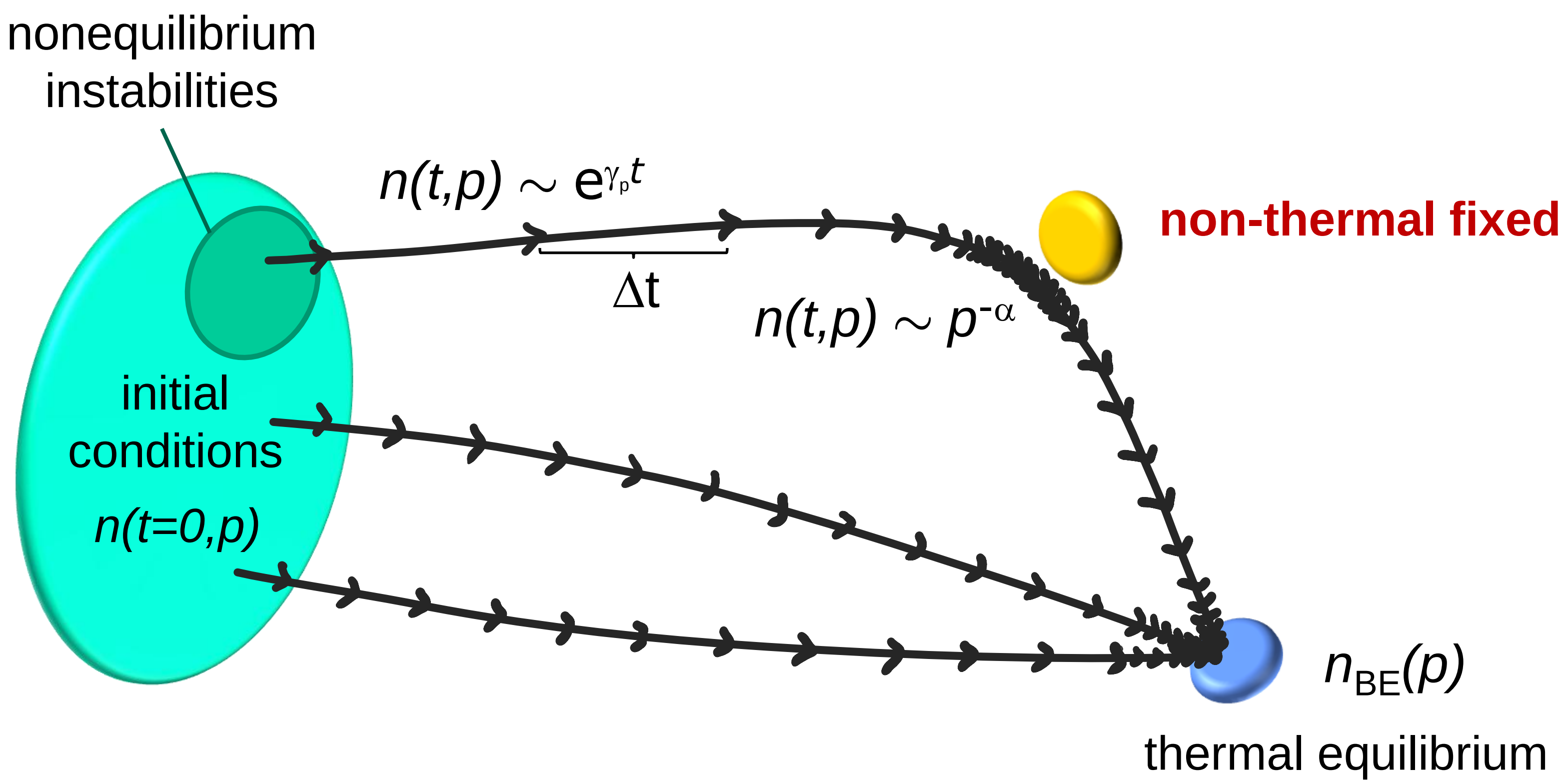




Conclusions: Universality far from equilibrium

Nonequilibrium theories can exhibit new infrared fixed points

- characterized by strongly enhanced infrared fluctuations ($\alpha = 4$!)
- approached from substantial class of initial conditions (no fine tuning!)
- subleading quantum corrections lead to asymptotic late-time thermalization



Role of quantum fluctuations

Non-thermal fixed point unstable under inclusion of quantum corrections

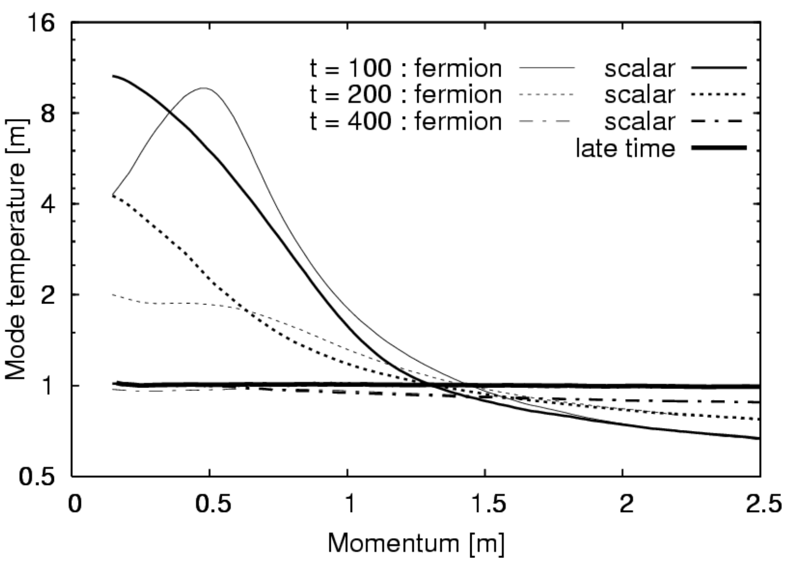
$\Rightarrow$ *late-time thermalization to Bose-Einstein/Fermi-Dirac distributions*

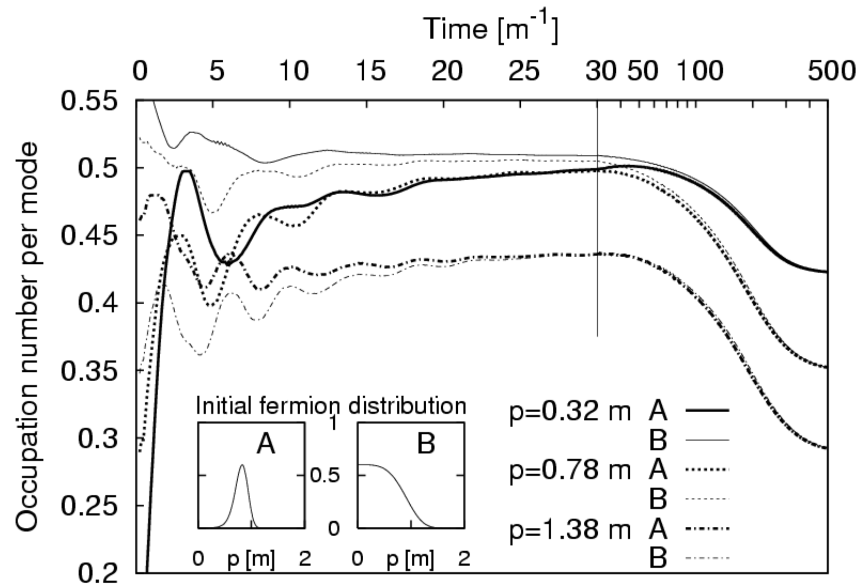
Thermalization example:

$SU(2)_L \times SU(2)_R$ fermionic model coupled to N=4 inflaton

'Mode temperature' T_p: $\left(n_p \sim \mathrm{tr}\,\frac{p^i\gamma^i}{p}\langle[\psi,\bar{\psi}]\rangle_p\right)$

$$n_p(t) \stackrel{!}{=} \frac{1}{\exp\left[\omega_p(t)/T_p(t)\right] \pm 1}$$





Emergence of BE/FD distribution with

$$\boxed{T_p^{(f)}(t) = T_p^{(s)}(t) = T_{\text{eq}}}$$

Berges, Borsányi, Serreau, NPB 660 (2003) 51

Nonequilibrium instabilities

Large class of possible instabilities:

Spinodal, Parametric, Plasma (Weibel) …

E.g. Weibel instability in electrodynamics:

Initial fluctuating current:

$\mathbf{j}(x) = j \cos(kx)\, \mathbf{e}_z$

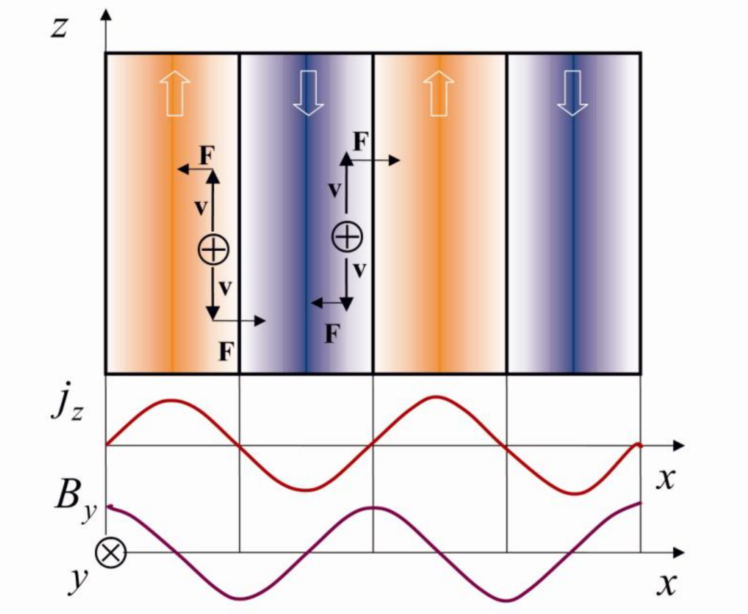
$\Rightarrow$ generated magnetic field:

$\mathbf{B}(x) = j \sin(kx)/k\, \mathbf{e}_y$

$\Rightarrow$ Lorentz force acts such that current grows:

$\mathbf{F}(x) = q\, \mathbf{v} \times \mathbf{B} = -\, q\, v_z\, j \sin(kx)/k\, \mathbf{e}_x$

$\Rightarrow$ B-field grows, etc.



Fast isotropization/thermalization due to instabilities?

Mrowczynski ’94; Romatschke, Strickland ‘03; Arnold, Lenaghan, Moore ‘03, Mrowczynski, Rebhan, Strickland ‘04; Rebhan, Romatschke, Strickland ’05; Dumitru, Nara ‘05; Romatschke, Venugopalan ‘06; Schenke, Strickland, Greiner, Thoma ‘06; Dumitru, Nara, Strickland ’07; Bödeker, Rummukainen ’07; Berges, Scheffler, Sexty ’08; Mrowczynski ’08 …

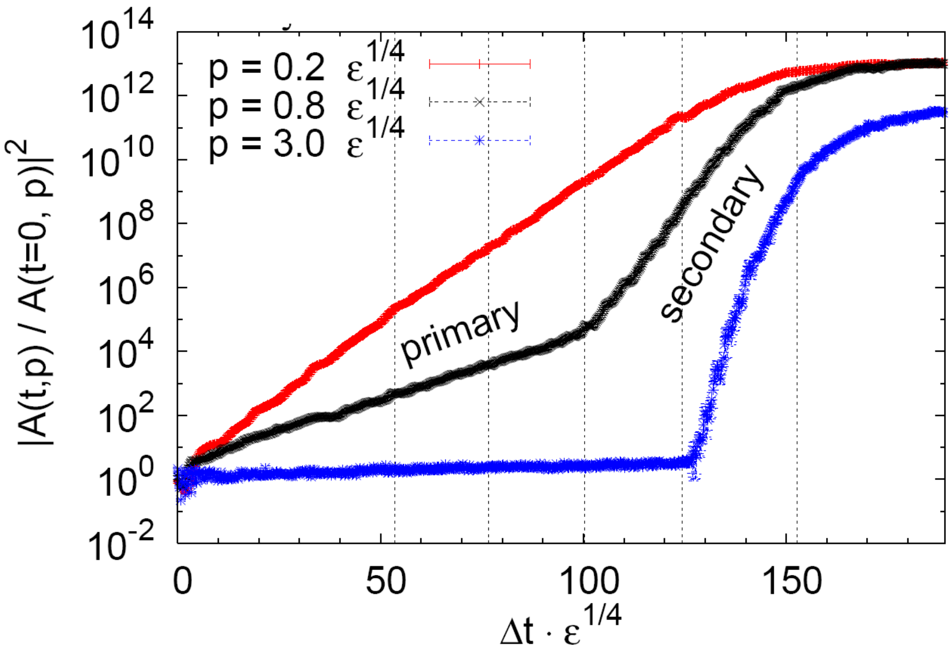
Characteristic time scales

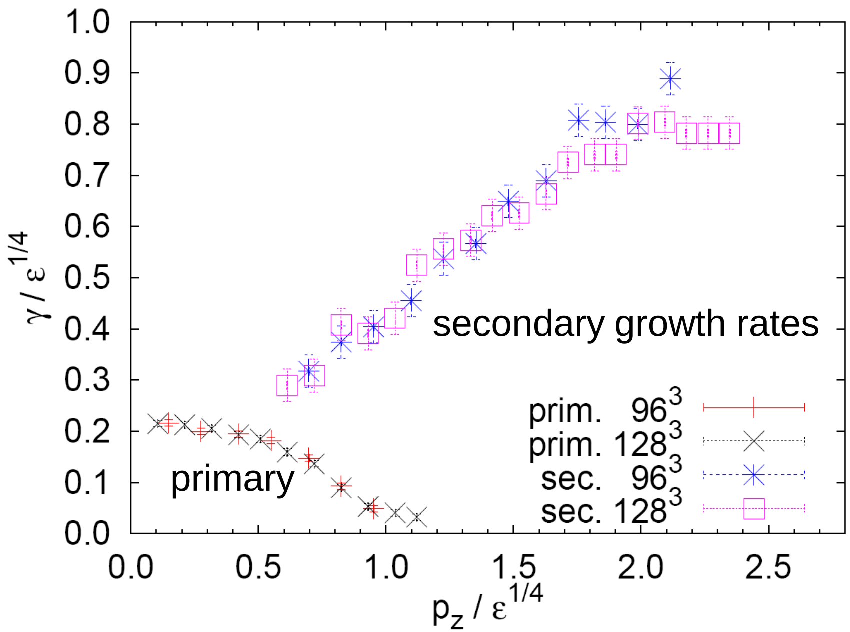
A) 'Soft' classical gauge fields + 'hard' classical particles

Arnold, Moore, Yaffe; Rebhan, Romatschke, Strickland; Dumitru, Nara, Strickland; Bödeker, Rummukainen

B) Classical-statistical gauge field evolution (here)

Romatschke, Venugopalan; Berges, Scheffler, Sexty





Inverse primary growth rate:

$\Rightarrow$ **$1/\gamma_{max} \simeq$ 1.1 fm/c** for ε = 30 GeV/fm^3

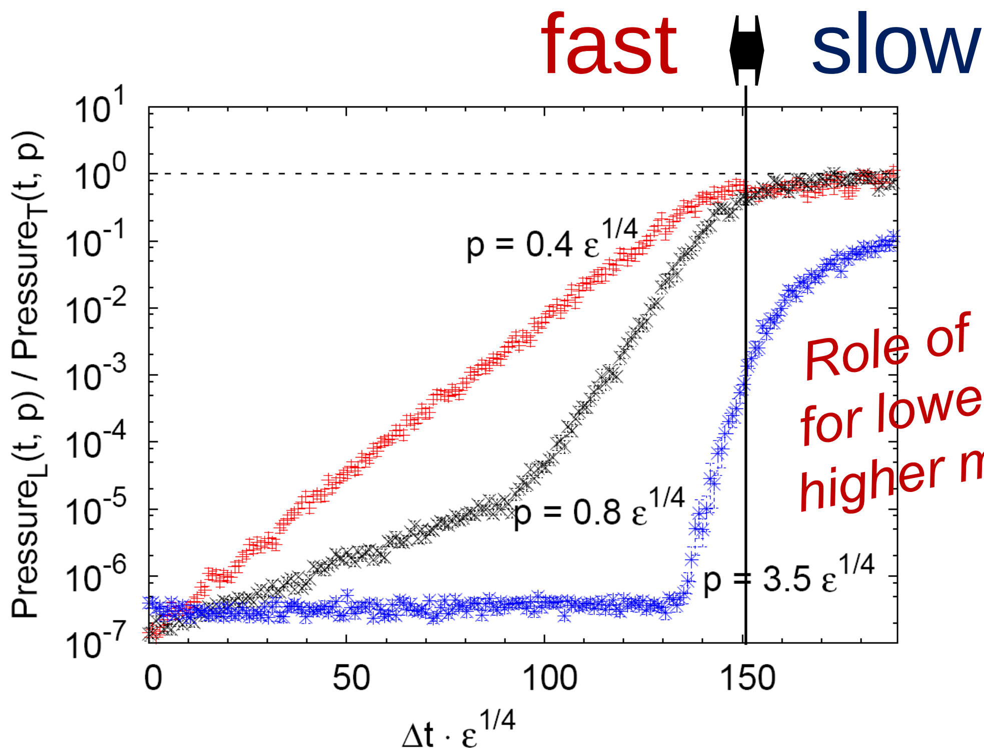
What energy density would be required to get $1/\gamma_{max} \simeq$ 0.1 fm/c ?

$\Rightarrow$ ε = 300 TeV/fm^3 (!)

Bottom-up isotropization of pressure

Spatial Fourier transform of the energy-momentum tensor $T^{\mu\nu}(x)$:

$P_L(t,p)$ for $\mu=\nu=3$, $P_T(t,p)$ from transversal components



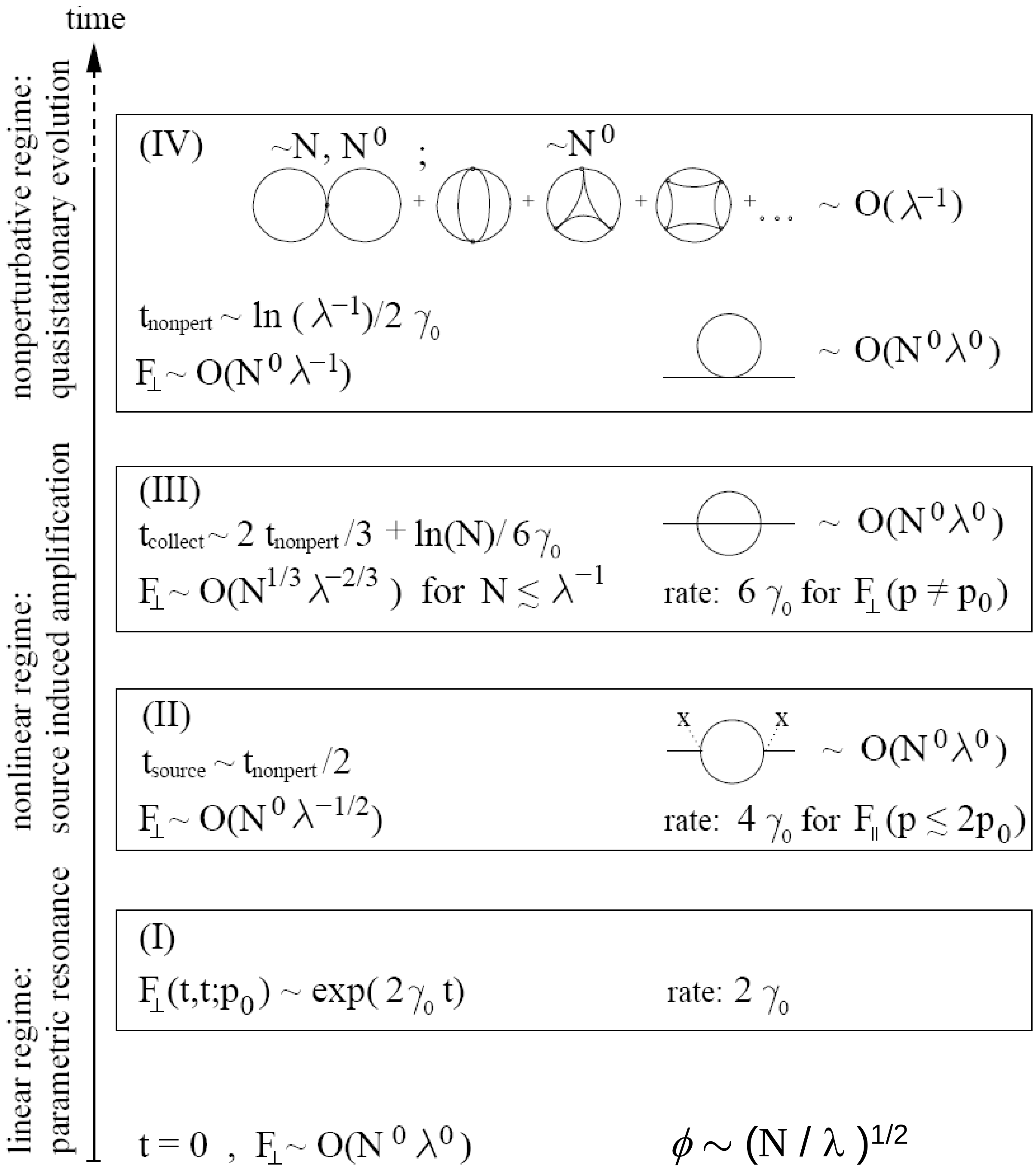
Role of quantum corrections for lower occupied modes at higher momenta?

For what p is $P_L(p)/P_T(p) \gtrsim 0.6$ at end of exponential growth? $\Rightarrow p_z \lesssim 1.4\, \varepsilon^{1/4}$

$p_z \lesssim 1$ GeV for $\varepsilon = 30$ GeV/fm^3

'enough' for hydro?

BUT: Isotropization time of dominant higher momenta consistent with 'infinity'



$\phi \sim (N/\lambda)^{1/2}$

slow

Nonperturbative: saturated occupation numbers $\sim 1/\lambda$

$\rightarrow$ *universal:* λ drops out

$\rightarrow$ *all processes* $\mathcal{O}(1)$

Effective weak coupling!

fast

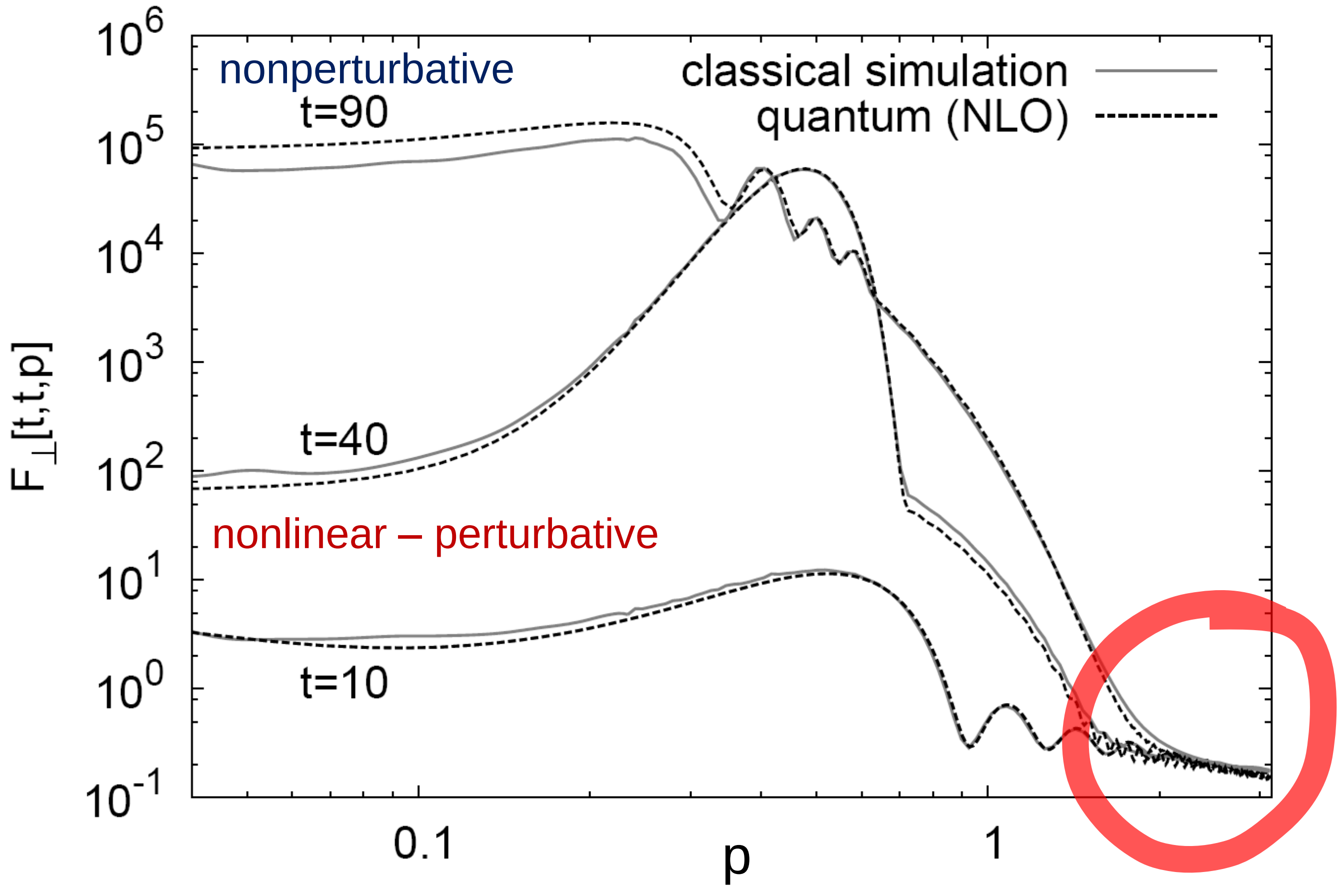
Nonlinear – perturbative: occupation numbers $< 1/\lambda$

secondary growth rates $c\,(2\gamma_0)$ with $c = 2,3,\ldots$

Classical/linear: primary growth rate

Comparison quantum/classical dynamics

Classical-statistical simulations: Khlebnikov, Tkachev ‘96; Prokopec,Roos ‘97; Tkachev, Khlebnikov, Kofman, Linde ’98; …



Berges, Rothkopf, Schmidt ‘08

Practically no quantum corrections at the end of preheating

Accurate nonperturbative description by 2PI 1/*N* to NLO

Nonequilibrium evolution equations

Propagator: spectral function $\sim \langle[\Phi,\Phi]\rangle$

$$\boxed{G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0 - y^0)}$$

statistical propagator $\sim \langle\{\Phi,\Phi\}\rangle$

Tremendous simplification if thermal equilibrium $G^{(eq)}(x,y)=G^{(eq)}(x-y)$ with

$$F^{(\mathrm{eq})}(\omega,\mathbf{p}) = -i\left(\frac{1}{2} + n_{\mathrm{BE}}(\omega)\right)\rho^{(\mathrm{eq})}(\omega,\mathbf{p})$$ *“ fluctuation-dissipation relation“*

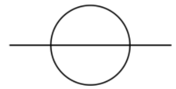
Nonequilibrium:

$$F \not\sim \rho$$

$$\begin{aligned}
\left[\Box_x\delta_{ac} + M^2_{ac}(x)\right]\rho_{cb}(x,y) &= -\int_{y^0}^{x^0} \mathrm{d}z\,\Sigma^{\rho}_{ac}(x,z)\rho_{cb}(z,y)\\
\left[\Box_x\delta_{ac} + M^2_{ac}(x)\right]F_{cb}(x,y) &= -\int_{0}^{x^0} \mathrm{d}z\,\Sigma^{\rho}_{ac}(x,z)F_{cb}(z,y)\\
&\quad + \int_{0}^{y^0} \mathrm{d}z\,\Sigma^{F}_{ac}(x,z)\rho_{cb}(z,y)
\end{aligned}$$

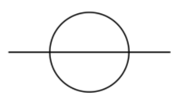
$$\left(\left[\Box_x + \frac{\lambda}{6N}\phi^2(x)\right]\delta_{ab} + M^2_{ab}(x;\phi=0,F)\right)\phi_b(x)$$
$$= -\int_0^{x^0}\mathrm{d}y\,\Sigma^{\rho}_{ab}(x,y;\phi=0,F,\rho)\,\phi_b(y)$$

Quantum- vs. classical-statistical contributions

Example: Quantum  (Similar for $1/N$ to NLO and $\phi \neq 0$)

$$\Sigma^F(t,t';\mathbf{p}) \;=\; -\frac{\lambda^2}{6}\int_{\mathbf{q},\mathbf{k}} F(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k}) \left[F(t,t';\mathbf{q})F(t,t';\mathbf{k}) - \frac{3}{4}\,\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})\right]$$

$$\Sigma^\rho(t,t';\mathbf{p}) \;=\; -\frac{\lambda^2}{2}\int_{\mathbf{q},\mathbf{k}} \rho(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k}) \left[F(t,t';\mathbf{q})F(t,t';\mathbf{k}) - \frac{1}{12}\,\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})\right]$$

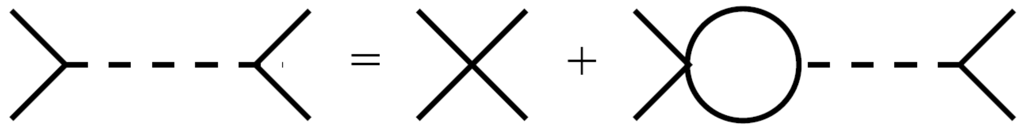
Classical 

$$\Sigma^F_{\mathrm{cl}}(t,t';\mathbf{p}) \;=\; -\frac{\lambda^2}{6}\int_{\mathbf{q},\mathbf{k}} F(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k})F(t,t';\mathbf{q})F(t,t';\mathbf{k})$$

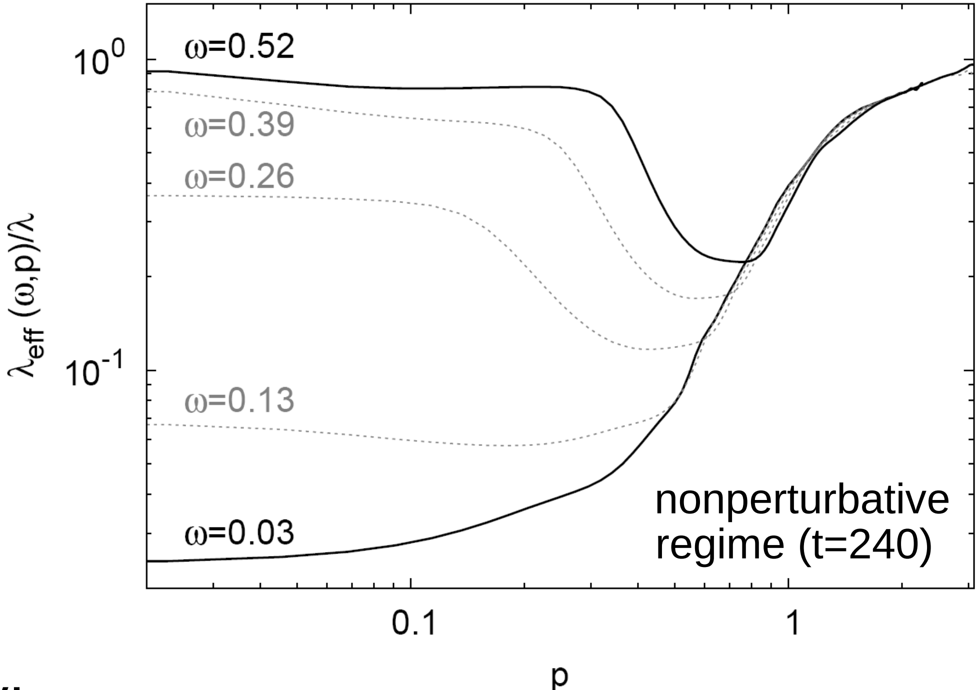
$$\Sigma^\rho_{\mathrm{cl}}(t,t';\mathbf{p}) \;=\; -\frac{\lambda^2}{2}\int_{\mathbf{q},\mathbf{k}} \rho(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k})F(t,t';\mathbf{q})F(t,t';\mathbf{k})$$

Effective coupling

Graphically:



$$\boxed{\lambda_{\text{eff}}(p) = \frac{\lambda}{|1+\Pi_R(p)|^2}}$$



'One-loop' retarded self-energy:

$$\Pi_R(p) = \frac{\lambda}{3(2\pi)^4}\int \mathrm{d}^4q\, F(q) G_R(p-q) \quad ; \quad \Delta(\phi,p) = \frac{2\lambda\phi^2}{3N}\mathrm{Re}\left[\frac{G_R(p)}{1+\Pi_R(p)}\right]$$

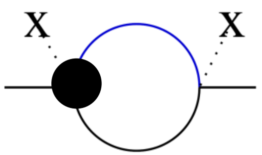
Fixed point condition

Time and space translation invariant solutions require:

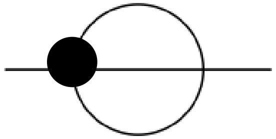
$$\Sigma^{\rho}_{ab}(\phi,p)F_{bc}(p) - \Sigma^{F}_{ab}(\phi,p)\rho_{bc}(p) = J^{(3)}_{ac}(\phi,p) + J^{(4)}_{ac}(\phi,p) \equiv 0$$

Neglecting quantum corrections and $F_{ab} \sim \delta_{ab} F$, $\rho_{ab} \sim \delta_{ab}\rho$, $1/N$ to NLO:

$$\begin{aligned} J^{(3)}_{aa}(\phi,p) &= \frac{\lambda\phi^2}{18N(2\pi)^4}\int \mathrm{d}^4k\,\mathrm{d}^4q\,\delta^4(p-q-k) \\ &\quad [\lambda_{\text{eff}}(k)+\lambda_{\text{eff}}(q)+\lambda_{\text{eff}}(p)]\,[\rho(k)F(q)F(p) \\ &\quad +F(k)\rho(q)F(p) - F(k)F(q)\rho(p)] \end{aligned}$$



$$\begin{aligned} J^{(4)}_{aa}(\phi,p) &= \frac{\lambda}{18(2\pi)^8}\int \mathrm{d}^4k\,\mathrm{d}^4q\,\mathrm{d}^4r\,\delta^4(p+k-q-r) \\ &\quad \lambda_{\text{eff}}(p+k)\Big\{[F(p)\rho(k)+\rho(p)F(k)]F(q)F(r) \\ &\quad -F(p)F(k)[F(q)\rho(r)+\rho(q)F(r)]\Big\} \\ &\quad (1-\Delta(\phi,p)) \end{aligned}$$



$\times$ (1-Δ(ϕ))

Berges, Rothkopf, Schmidt '08

Scaling solutions

$$\begin{aligned} F(p) &= s^{2+\alpha} F(sp) \\ \rho(p) &= s^2 \rho(sp) \\ \lambda_{\text{eff}}(p) &= s^\gamma \lambda_{\text{eff}}(sp) \end{aligned}$$

$\Rightarrow \quad \Pi_R(p) = s^\alpha \Pi_R(sp)$, i.e. λ_{eff} scales differently in UV and IR:

$\mathrm{I}: \quad \gamma = 0 \;$ for $\; \Pi_R(p) \ll 1$ **(UV)**

$$\begin{aligned} J^{(3)}_{aa}(\phi, p) &= s^{2\alpha} J^{(3)}_{aa}(s\phi, sp) \\ J^{(4)}_{aa}(0, p) &= s^{3\alpha} J^{(4)}_{aa}(0, sp) \end{aligned}$$

$\leftarrow$ dominates UV for $\alpha > 0$

$$J^{(3)}_{aa}(\phi, p) = 0 \quad \Rightarrow \quad \boxed{\alpha = 1\,,\ \alpha = \frac{3}{2}}$$

$\mathrm{II}: \quad \gamma = -2\alpha \;$ for $\; \Pi_R(p) \gg 1$ **(IR)**

$$\begin{aligned} J^{(3)}_{aa}(\phi, p) &= s^{0} J^{(3)}_{aa}(s\phi, sp) \\ J^{(4)}_{aa}(0, p) &= s^{\alpha} J^{(4)}_{aa}(0, sp) \end{aligned}$$

$\leftarrow$ dominates IR for $\alpha > 0$

$$J^{(4)}_{aa}(0, p) = 0 \quad \Rightarrow \quad \boxed{\alpha = 0\,,\ \alpha = 1\,,\ \alpha = 4\,,\ \alpha = 5}$$