Strong Coupling Regime of the Kardar-Parisi-Zhang Equation

Léonie Canet

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In Collaboration with

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M. A. Moore	*University of Manchester*

Outline

- The Kardar-Parisi-Zhang (KPZ) Equation
  - Physics of Interface Growth
  - Numerical Approaches
  - Analytical Approaches

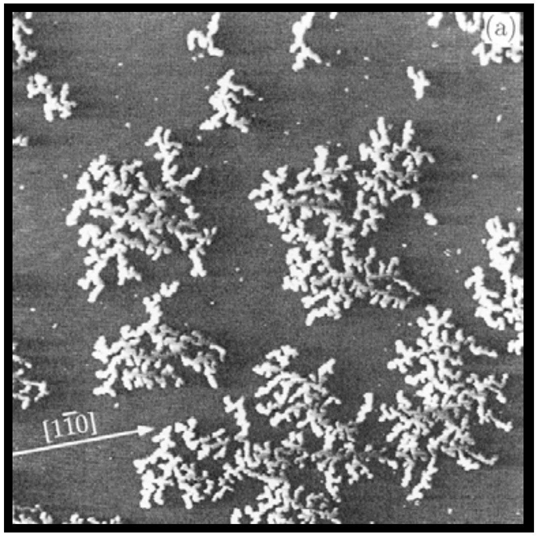
- ERG Approach to the KPZ Equation
  - Symmetries of the KPZ Equation
  - Derivative Expansion
  - 'Vertex' Approximation
  - Results

- Summary and outlook

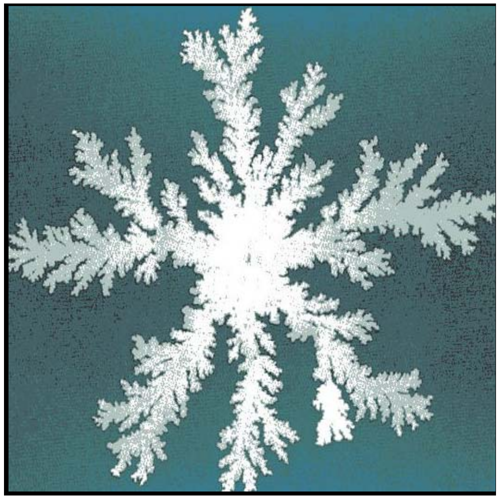
The KPZ Equation

Surface Growth

Molecular Beam Epitaxy

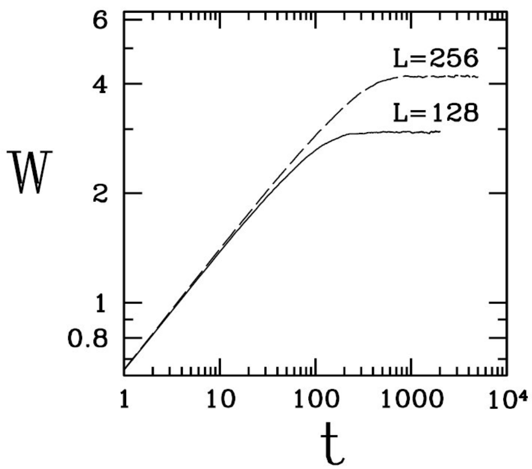


Bacterial Growth



Snow Deposition

Kinetic Roughening



- Kinetic roughening of the interface $h(\vec{x},t)$

width ($\equiv$ rms): $W(L,t) = \left(\frac{1}{L^{d-1}} \int_0^L d^{d-1}\vec{x} \left\langle [h(\vec{x},t) - \bar{h}]^2 \right\rangle \right)^{\frac{1}{2}}$

growth regime: $W(L,t) \sim t^{\beta}$

saturation regime: $W(L,t \to \infty) \sim L^{\chi}$

generic scaling: $W(L,t) = L^{\chi} f(t/L^{z})$

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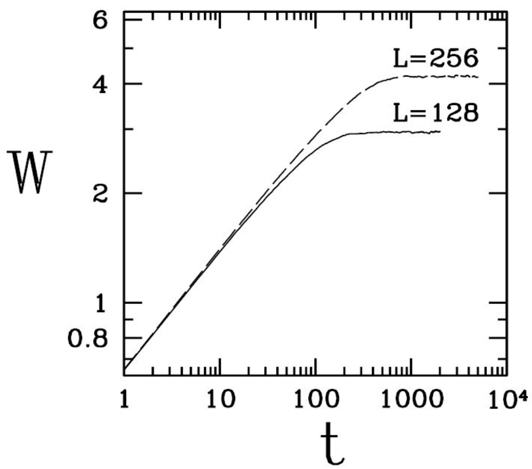

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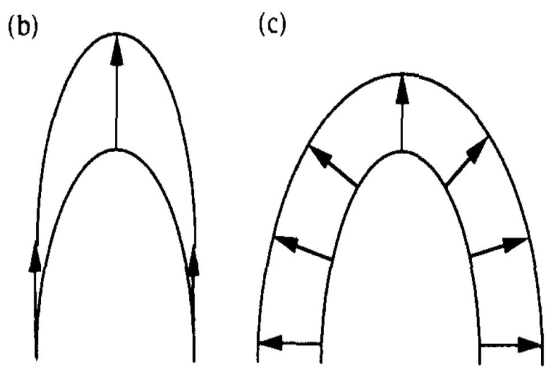

projection: $\sqrt{1+(\nabla h)^2}$

- Continous Langevin equation: the KPZ equation

$$\frac{\partial}{\partial t}h = \nu\,\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$$

$$\langle \eta(\vec{x},t)\eta(\vec{x}',t')\rangle = 2\,D\,\delta^d(\vec{x}-\vec{x}')\,\delta(t-t')$$

M. Kardar, G. Parisi and Y.-C. Zhang, *Phys. Rev. Lett.* **56** *(1986)*

non-linear, non-equilibrium, non-perturbative

Physics of the KPZ Equation

- Equivalence with many other problems

  $\longrightarrow$ directed polymer in random media DPRM

  $\longrightarrow$ randomly stirred fluids (noisy Burgers equation)

  $\longrightarrow$ . . .

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  $\longrightarrow$ ...

- Phenomenology

$d = 1$ interface always rough.

– exact exponents: $\chi = 1/2, \quad z = 3/2.$
– exact scaling function
H. Spohn, M. Prähofer, P. L. Ferrari, T. Sasamoto (2002-2005).

$d \geq 2$ phase transition

$\lambda \ll \lambda_c$: smooth phase: $\chi = (2-d)/2, \quad z = 2$
$\lambda \gg \lambda_c$: rough phase $\longrightarrow$ strong coupling regime

- Theoretical challenges:

  statistical properties of the rough phase
  existence of an upper critical dimension d_c

Numerical Approaches

- Discretization of the Langevin equation

  $\longrightarrow$ numerical instabilities $\Rightarrow$ need regularization

  T. J. Newman and A. J. Bray (1996) C. Dasgupta, J. M. Kim, M. Duta and S. Das Sarma (1997)

  $\longrightarrow$ infinite dimension limit ill-defined M. Marsili and A. J. Bray (1996)

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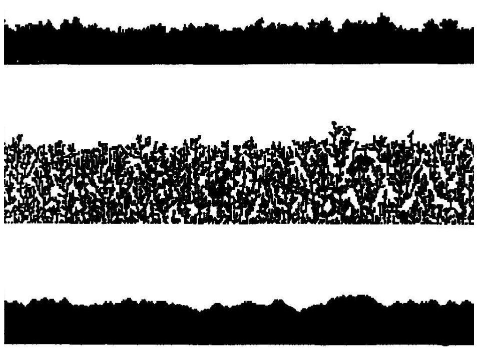
- Simulation of discrete growth models (RSOS, Eden growth, ballistic, NRG...)



d	1	2	3	4	5	...	9
z	1.5	1.62	1.70	1.75	1.80	...	1.86

L.-H. Tang *et al.* (1992), E. Marinari *et al.* (2000), Castellano *et al.* (1998-99)

many conjectures, *e.g.* $z = 2(d{+}2)/(d{+}3)$ J.M. Kim and J.M. Kosterlitz (1989)

numerical results suggest $d_c = \infty$

no formal proof of universality and of generic scaling

Analytical Approaches

- Perturbative RG: β-function for $g = \lambda^2 D/\nu^3$

  $\Longrightarrow$ no strong-coupling fixed point in $d \geq 2$

  – KPZ version: 2-loop order E. Frey and U. Taüber (1994), M. Lässig (1995), K. Wiese (1998)
  – DPRM version: all-order K. Wiese (1998)

- Mode-coupling theory: self-consistent equations for correlation functions

  $\Longrightarrow$ predicts $d_c = 4$ E. Frey, U. Taüber and T. Hwa (1996), M. Moore et al (2000)

- Functional RG: $\Longrightarrow$ predicts $d_c \simeq 2.4$ P. Le Doussal and K.J. Wiese (2005)

- Various arguments : $\Longrightarrow$ predicts $d_c = 4$ or $d_c = 2$ M. Lässig (2005), H. Fogedby (2005)

no controlled description of the strong-coupling fixed point!

ERG Approach to the KPZ Equation

The KPZ Action and its Symmetries

- Field theory H. K. Janssen (1976), C. de Dominicis (1976)

Langevin dynamics: response field Response functional:

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta \quad \Longleftrightarrow \quad \mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi}\, e^{-\mathcal{S}[\phi, \tilde{\phi}]}$$

with $$\mathcal{S}[\phi, \tilde{\phi}] = \int d^d\vec{x}\, dt \left\{ \tilde{\phi} \left[\partial_t \phi - \nu \nabla^2 \phi - \frac{\lambda}{2} (\nabla \phi)^2 \right] - D\, \tilde{\phi}^2 \right\}$$

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- Galilean symmetry:

Invariance under field transformation $$\mathcal{T}_{\rm Gal} = \left\{ \begin{array}{lcl} \vec{x} & \rightarrow & \vec{x} + \lambda \vec{v}.\vec{x} \\ \phi & \rightarrow & \phi + \vec{v}.\vec{x} \\ \tilde{\phi} & \rightarrow & \tilde{\phi} \end{array} \right.$$

enforces identity $z + \chi = 2$

- 'Time Gauged' symmetry:

Invariance of $\left[\mathcal{S} - \int \tilde{\phi}\partial_t \phi\right]$ under field transformation $$\mathcal{T}_{\rm TG} = \left\{ \begin{array}{lcl} \phi & \rightarrow & \phi + f(t) \\ \tilde{\phi} & \rightarrow & \tilde{\phi} \end{array} \right.$$

First Strategy: Derivative Expansion

- Exact equation $$\partial_k \Gamma_k[\psi, \tilde{\psi}] = \frac{1}{2} \int_{x,t} \partial_k \mathcal{R}_k \left[\Gamma_k^{(2)} + \mathcal{R}_k \right]^{-1}$$

C. Wetterich (1993), T. Morris (1994)

- Cutoff term $$\Delta \mathcal{S}_k[\psi, \tilde{\psi}] = \frac{1}{2} \int_{x,t} \left(\psi \; \tilde{\psi} \right) \mathcal{R}_k(\nabla^2, \partial_t) \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}$$

with in Fourier space $$\mathcal{R}_k = \begin{pmatrix} 0 & q^2 \nu_k R_k(q^2) \\ q^2 \nu_k R_k(q^2) & -2 D_k R_k(q^2) \end{pmatrix}$$

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- Derivative expansion: expand Γ_k in powers of ∇^2 and ∂_t
- Very efficient for reaction-diffusion processes

⟶ non-perturbative phase diagram L. C., H. Chaté, B. Delamotte and N. Wschebor (2004)

⟶ non-perturbative fixed point L. C., H. Chaté, B. Delamotte, I. Dornic and M. Munoz (2005)

$$\Gamma_k[\psi,\tilde\psi] = \int_{x,t}\left\{D_k\tilde\psi\,\partial_t\,\psi - Z_k\,\tilde\psi\,\Delta\psi + U_k[\psi,\tilde\psi]\right\}$$

First Strategy: Derivative Expansion

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- Ansatz at leading order

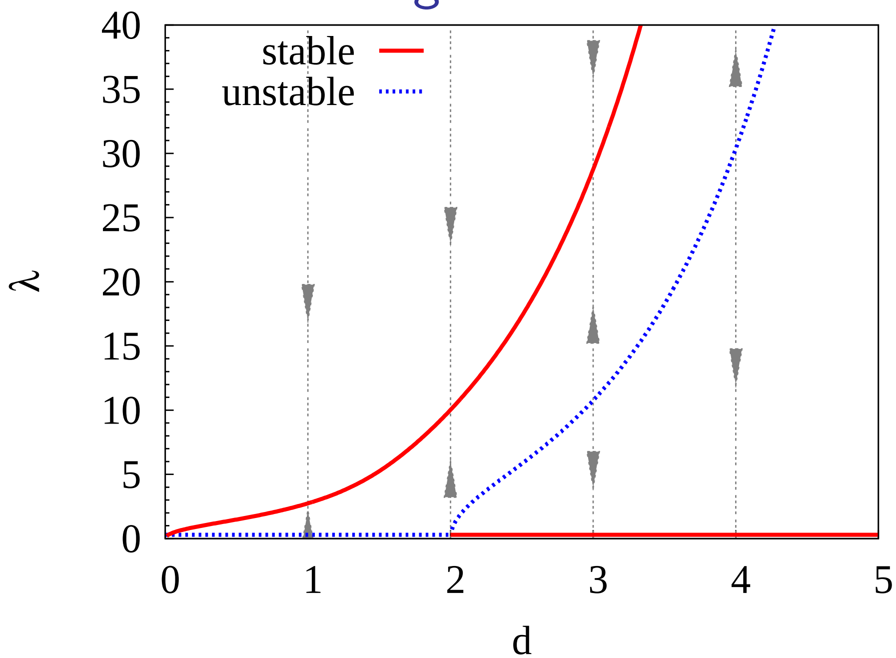
$$\Gamma_k[\psi,\tilde\psi] = \int_{x,t}\left\{\tilde\psi\,\partial_t\,\psi - \nu_k\,\tilde\psi\,\Delta\psi - \frac{1}{2}\lambda\,\tilde\psi\,(\nabla\psi)^2 - D_k\,\tilde\psi^2\right\}$$

- Critical exponents

$$\left\{\begin{array}{lcl} \eta_\nu & = & -\partial_s \ln \nu_k \\ \eta_D & = & -\partial_s \ln D_k \end{array}\right. \quad\Longrightarrow\quad \left\{\begin{array}{lcl} z & = & 2-\eta_\nu \\ \chi & = & (2-d+\eta_D-\eta_\nu)/2 \end{array}\right.$$

Phase Diagram at Leading Order

- Phase diagram



L. C. (cond-mat/0509541)

but KPZ interaction is derivative!

- Edwards-Wilkinson fixed point

smooth phase, $z = 2$ and $\chi = (2 - d)/2$

- Unstable fixed point

transition, $\chi = 0$

- Stable KPZ fixed point

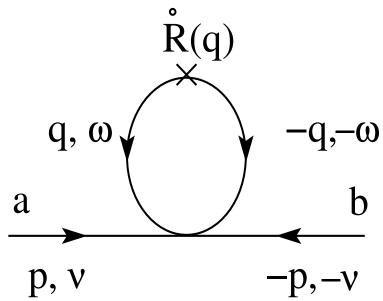
rough phase for $d \geq 2$
exact result in $d = 0$: $z = 4/3$
exact result in $d = 1$: $z = 3/2$

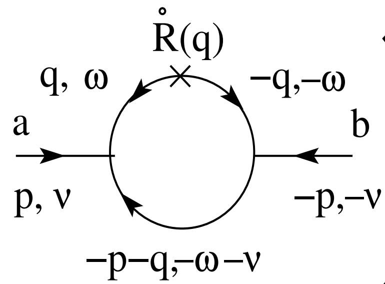
but very poor z

Alternative Strategy: Vertex Approximation

- Three exact coupled flow equations for the $\Gamma^{(2)}_{k_{ab}}$

$$\partial_k \Gamma^{(2)}_{k_{ab}}(p, -p, \nu, -\nu) = \Big(- \quad [\text{diagram 1}] \quad +2 \quad [\text{diagram 2}] \Big)$$





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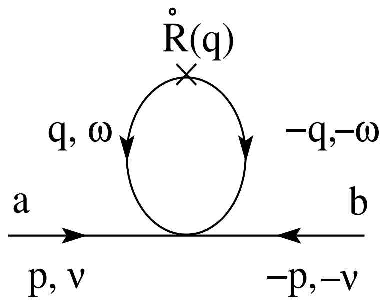
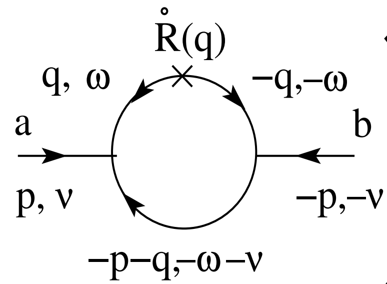


$$+2$$



$$\Bigg)$$

- 'Vertex' approximation J.-P. Blaizot, R. Mendez-Galain, N. Wschebor (2005)

$\longrightarrow$ neglect loop momentum q in vertex functions

$$\Gamma^{(3)}_{k_{abc}}(p, q, -p-q) \simeq \Gamma^{(3)}_{k_{abc}}(p, 0, -p) = \frac{\partial \Gamma^{(2)}_{k_{bc}}}{\partial \psi_a}(p, -p)$$

Vertex Approximation: Final Ansatz

- Ward identities for n-point functions $\quad \Gamma_k^{(n,m)} \equiv \dfrac{\delta^{(n+m)}\Gamma}{\delta\psi^n\,\delta\tilde{\psi}^m}$

$\mathcal{T}_{\rm TG}$: $\quad \Gamma_k^{(n,m)}(p_1=0,\varpi_1;\ldots;p_{n+m},\varpi_{n+m})=0 \quad \Longrightarrow \quad \Gamma_k^{(n,m)}$ independent of ψ

$\mathcal{T}_{\rm TG}$: $\quad \Gamma_k^{(1,1)}(0,\varpi)=i\varpi$ and $\Gamma_k^{(2,0)}(0,\varpi)=0 \quad \Longrightarrow \quad$ no ϖ^n, at least ϖp^2

$\mathcal{T}_{\rm Gal}$: $\quad \partial_{\vec{p}_{|p=0}}\Gamma_k^{(1,1)}(p,0)=\vec{0}$ and $\partial_{\vec{p}_{|p=0}}\Gamma_k^{(2,0)}(p,0)=\vec{0} \quad \Longrightarrow \quad$ explicit p^2 dependence

$$\left\{\begin{array}{lcl}\Gamma_k^{(1,1)}(p,\varpi,\tilde{\psi}) & = & i\varpi+p^2\nu_k\gamma_k^{(1,1)}(p)\\ \Gamma_k^{(0,2)}(p,\varpi,\tilde{\psi}) & = & -2D_k\gamma_k^{(0,2)}(p)\\ \Gamma_k^{(2,0)}(p,\varpi,\tilde{\psi}) & = & -\lambda p^2\tilde{\psi}\end{array}\right.$$

$\longrightarrow$ Closed flow equations for $\gamma_k^{(1,1)}(p)$ and $\gamma_k^{(0,2)}(p)$

Strong-Coupling Phase of the KPZ Equation

- Strong-coupling fixed-point:
  - exists up to $d = 8$

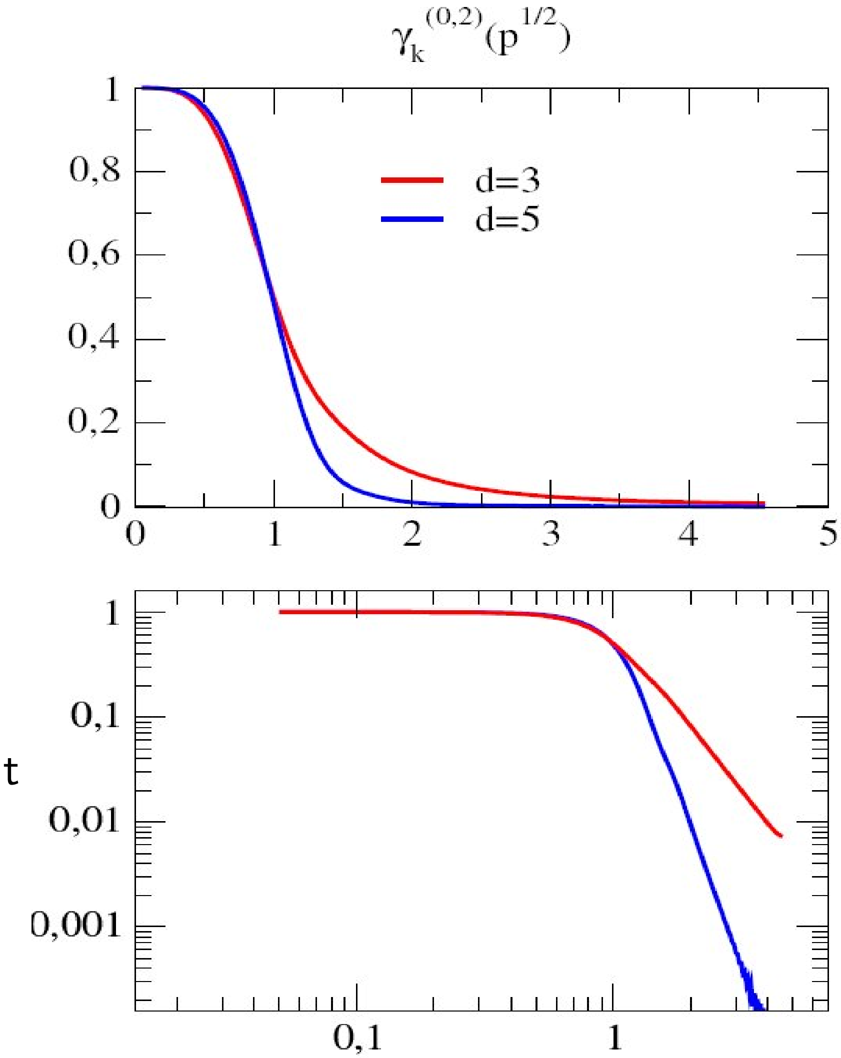
but z less accurate in higher d
(dependence on the cutoff R_k increases with d)

  - is fully attractive

provides evidence of generic scaling

  - is non-perturbative

$i.e.$ not connected by a flow line to the EW fixed-point
thus not accessible perturbatively



L. C., H. Chaté, B. Delamotte and N. Wschebor (in preparation)

Critical Exponent and Phase Diagram

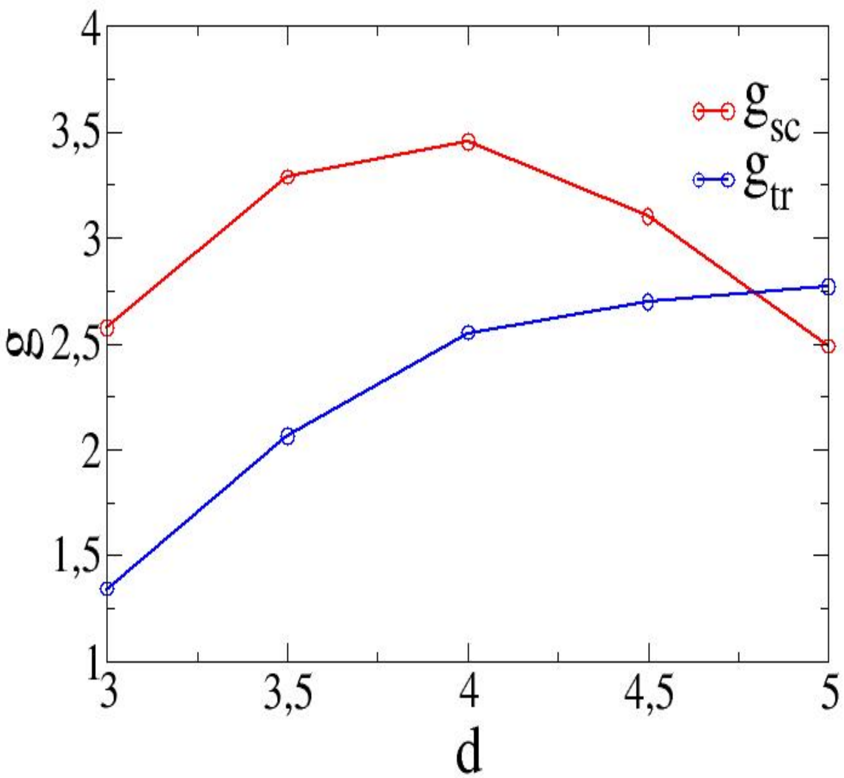
- Critical exponents

d	1	2	3	4
z (ERG)	1.5	1.67	1.85	1.92
z (num.)	1.5	1.62	1.70	1.75

- Possible Scenario

Hints of a FP collapse in $4 \leq d_c \leq 5$

⟶ compatible with numerics (and $d_c = \infty$)
⟶ compatible with analytics ($d_c = 4$)



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Summary

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- critical exponents in good agreements with numerics
- evidence for generic scaling and non-perturbative nature
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Outlook

- elucidate the properties of the rough phase(s) in higher d
- calculate full scaling functions $\gamma_k^{(1,1)}(p,\varpi)$ and $\gamma_k^{(0,2)}(p,\varpi)$
- other growth models (Kuramoto-Sivashinsky, depinning transition . . .)

Thank you !!!