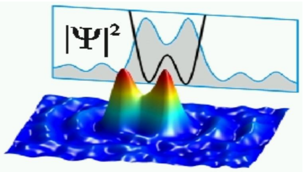
Functional RG approach to Far-From-Equilibrium Quantum Field Dynamics

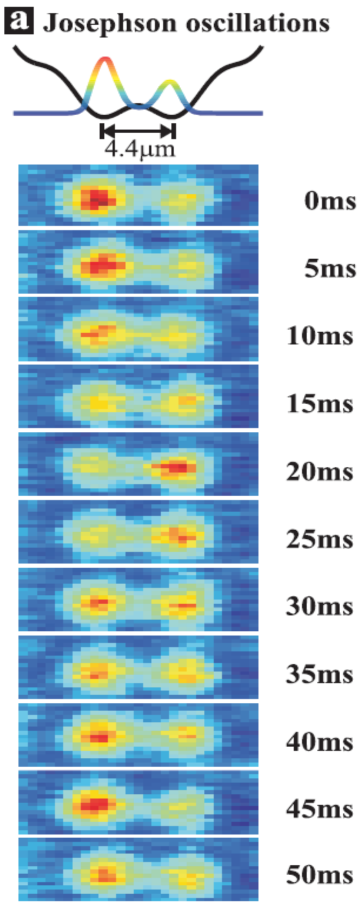
Thomas Gasenzer
Institut für Theoretische Physik
Philosophenweg 16
69120 Heidelberg

email: t.gasenzer@thphys.uni-heidelberg.de
www: www.thphys.uni-heidelberg.de/~gasenzer

Preface

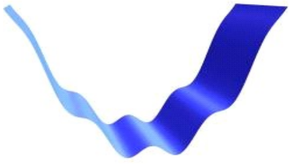
Cold-gases livestream on TV

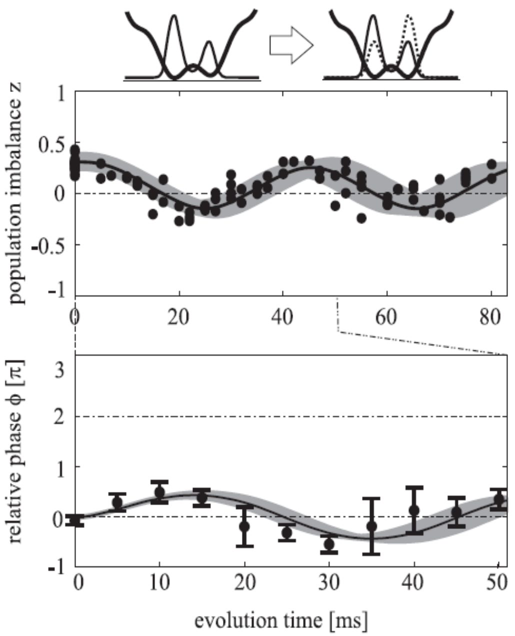




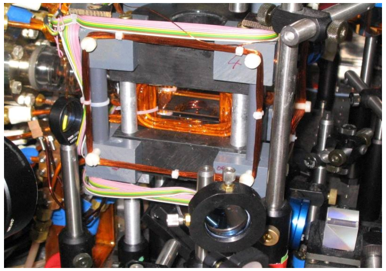
Experimenters can now...

- ✔ observe evolution in real time
- ✔ model freely initial state
- ✔ change boundary conditions
- ✔ measure mean densities, phases, fluctuations
- ✔ reduce atom numbers to a few hundreds & less





Oberthaler Labs (Heidelberg)



Quantum fields far away from equilibrium:

Theory – Status quo

- Mean-field theories [Gross-Pitaevskii, Hartree-Fock(-Bogoliubov)]
- Kinetic approaches [Quantum Boltzmann, ...] (**close** to equilibrium)

- (Semi-)classical simulations [Truncated Wigner Approximation, ...]
- Exactly solvable models

  [Lieb & Liniger, Girardeau, McGuire, Gaudin, Minguzzi, Buljan, ...]
- Quantum-Information inspired: tDMRG, MPS/PEPS

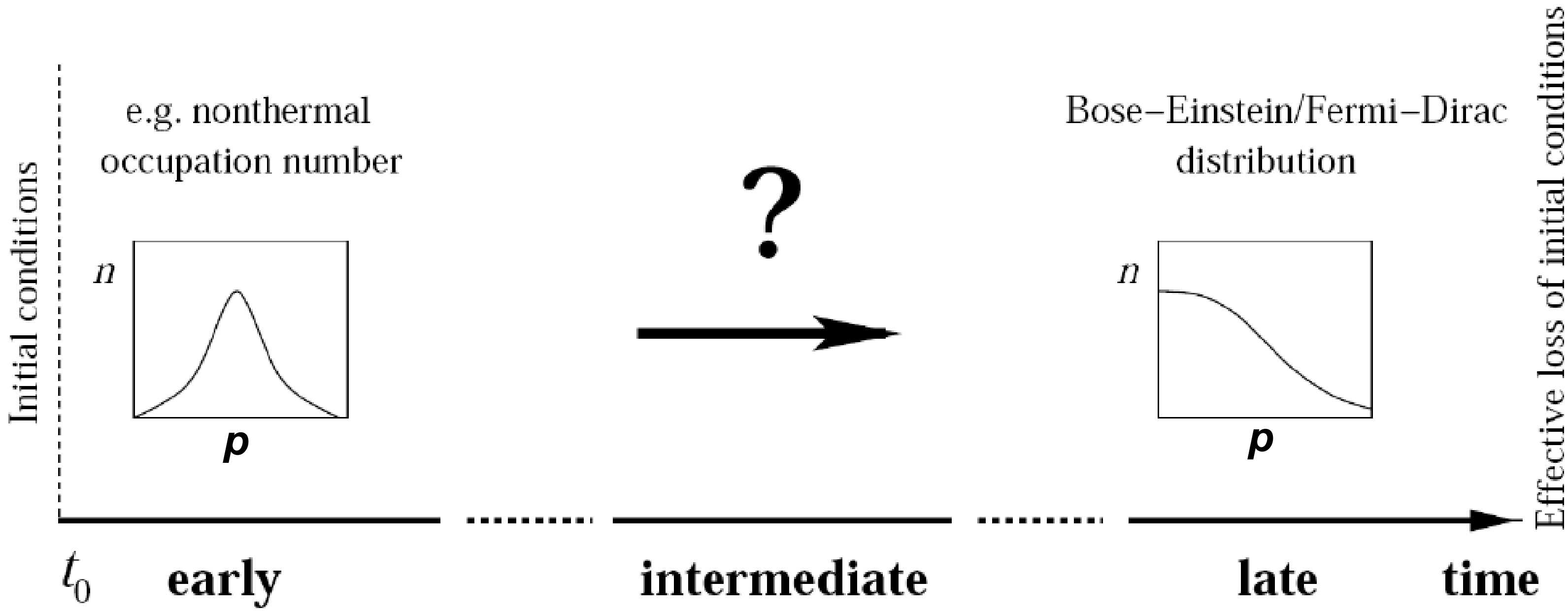
  [Vidal, Schollwöck, Kollath, White, Feiguin, Manmana, Muramatsu, Wolf, Cirac, ...]
- Quantum Monte Carlo, stochastic Quantisation

  [Mak, Egger, Berges & Stamatescu, ...]

- **Functional QFT:** 2PI effective action, $1/\mathcal{N}$ expansion

  [Berges, Aarts, Serreau, Baier, Cooper, Dawson, Mihaila, Borsanyi, Wetterich, Smit, Tranberg, ...]

Far-from-equilibrium dynamics



[J. Berges]

Thermal equilibrium: Loss of information about prior evolution.

Only a few conserved quantities persist.

Dynamical Field Theory

We will be interested, in particular, in the explicit time dependence of the lowest-order correlation functions:

$$\phi_a(x) = \langle \Phi_a(x) \rangle \qquad \text{(mean field)}$$

$$G_{ab}(x,y) = \langle \mathcal{T} \Phi_a(x) \Phi_b(y) \rangle_c \qquad \text{(density matrix, 2-point function, propagator)}$$

where $x = (\mathbf{x}, t)$

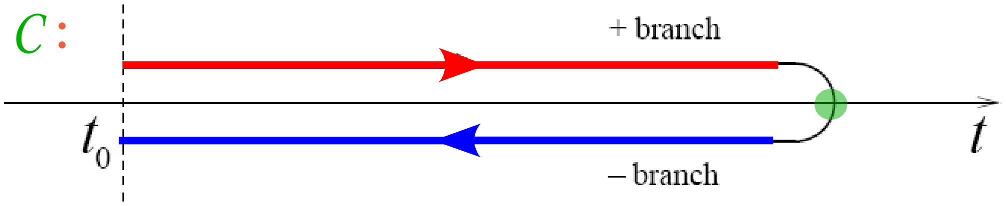
connected, i.e., $= \langle \mathcal{T} \Phi_a \Phi_b \rangle - \phi_a \phi_b$

Initial value problems...

...require the Schwinger-Keldysh closed time path (CTP):

$$\langle t\,|O|\,t\,\rangle = \langle t_0\,|U^\dagger(t)O\,U(t)|\,t_0\,\rangle$$
$$= \mathrm{Tr}\left[\rho(t_0)\,U^\dagger(t)O\,U(t)\,\right]$$

CTP C:



Initial value problems...

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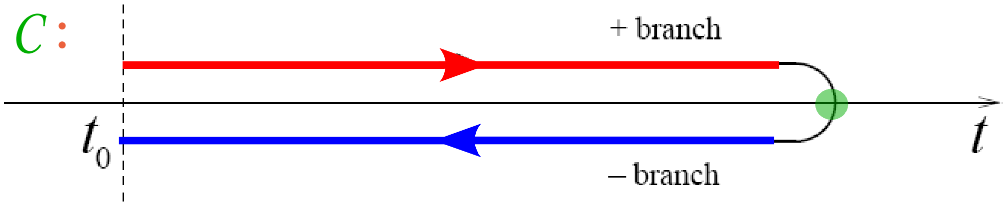
$$\begin{aligned}\langle t\,|O|\,t\rangle &= \langle t_0\,|U^\dagger(t)O\,U(t)|\,t_0\rangle \\ &= \mathrm{Tr}\left[\rho(t_0)\,U^\dagger(t)O\,U(t)\,\right] \\ &= \int \mathcal{D}\varphi_0\mathcal{D}\varphi_0\,\rho\,[\varphi_0,\varphi_0]\int \mathcal{D}\varphi'\mathcal{D}\varphi'\;O\;e^{i(S[\varphi]-S[\varphi])/\hbar}\end{aligned}$$

CTP C:



Functional RG approach

[with Jan M. Pawlowski, cond-mat/0710.4627]

RG applied to non-equilibrium

Presented methods:

- Classical statistical evolution, Wilsonian RG in momentum space
  [Canet, Delamotte, Deloubrière, & Wschebor (04); Mitra, Takei, Kim, & Willis (06); Zanella & Calzetta (06), Mattarese & Pietroni (07)]
- (Wegner) flow eqs. for Hamiltonian
  [Kehrein (04)]
- FRG with complex-valued cutoff in frequency space
  [Jakobs, Meden, & Schoeller (07); Korb, Reininghaus, Schoeller, & König (07)]
- Density-Matrix RG: tDMRG, MPS/PEPS
  [Vidal (03), Schollwöck, Kollath, White, Feiguin, Manmana, Muramatsu, Wolf, Cirac, ...]

Compare, also:

- Evolution equations for effective action
  [Wetterich (97), Bettencourt & Wetterich (98)]

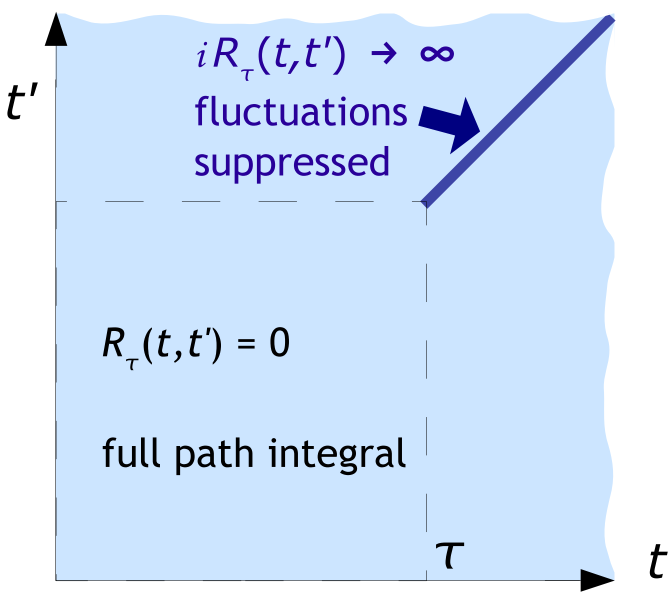
RG approach
to far-from-equilibrium dynamics

[TG & J.M. Pawlowski, cond-mat/0710.4627]

- Regularise generating functional

$$Z[J;\rho_0] = \int \mathcal{D}\varphi\,\rho_0 \exp\left\{iS[\varphi] + i\int_{x,\mathcal{C}} J_a\varphi_a\right\}$$

$$Z_\tau = \exp\left\{i\int_{x,y;\mathcal{C}} \frac{\delta}{\delta J_a(x)} R_{\tau,ab}(x,y)\frac{\delta}{\delta J_b(y)}\right\} Z$$



RG approach

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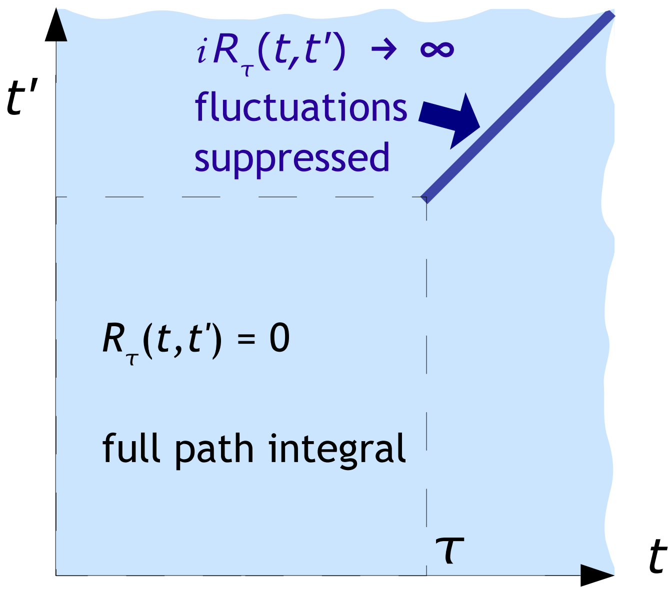


- Functional RG equation [C. Wetterich (92)]

$$\partial_\tau \Gamma_\tau = \frac{i}{2}\int_{\mathcal{C}} \left[\frac{1}{\Gamma_\tau^{(2)} + R_\tau}\right]_{ab} \partial_\tau R_{\tau,ab}$$

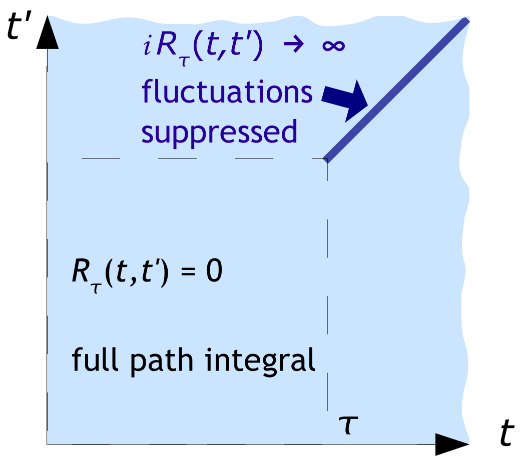
$$\Gamma_\tau[\phi, R_\tau] = W_\tau[J,\rho_0] - \int_{\mathcal{C}} J_a\phi_a - \frac{1}{2}\int_{\mathcal{C}} \phi_a R_{\tau,ab}\phi_b$$

$$(W_\tau = -i \ln Z_\tau)$$

RG approach

to far-from-equilibrium dynamics

[TG & J.M. Pawlowski, cond-mat/0710.4627]



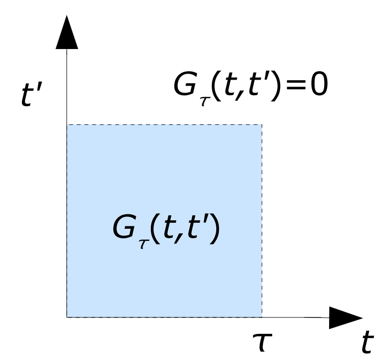
- Functional RG equation [C. Wetterich (92)]

$$\partial_\tau \Gamma_\tau = \frac{i}{2} \int_{\mathcal{C}} \left[\frac{1}{\Gamma_\tau^{(2)} + R_\tau} \right]_{ab} \partial_\tau R_{\tau,ab}$$

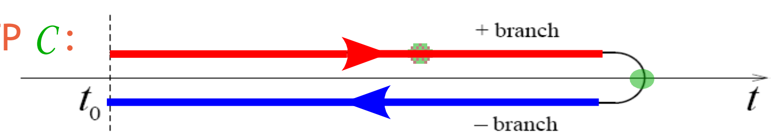
- Propagator:

$$\left[\frac{1}{\Gamma_\tau^{(2)} + R_\tau} \right]_{ab} = G_{\tau,ab}\,\theta(\tau - t_a)\theta(\tau - t_b)$$

$$= G_{\max(t_a,t_b),ab}\,\theta(\tau - t_a)\theta(\tau - t_b)$$



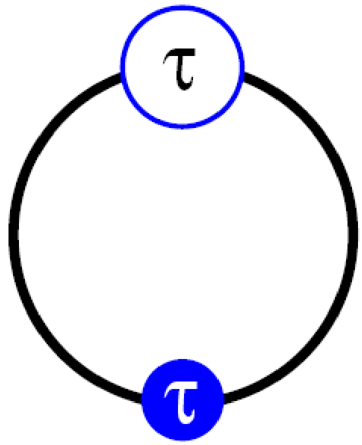
CTP $\mathcal{C}$:

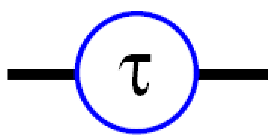


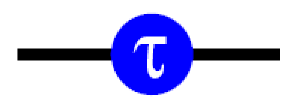
Real-time Functional RG equation

graphically

$$\partial_\tau \Gamma_\tau = \frac{i}{2} \int_{\mathcal{C}} \left[\frac{1}{\Gamma_\tau^{(2)} + R_\tau} \right]_{ab} \partial_\tau R_{\tau,ab}$$

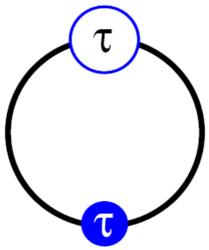
$$\partial_\tau \Gamma_\tau = \frac{i}{2}$$ 

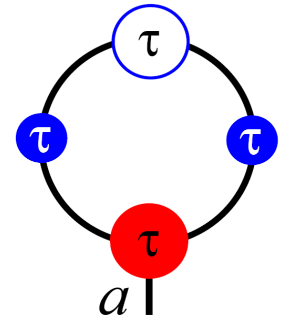
$$\dot{R}_{\tau,ab} =$$ 

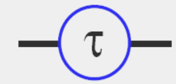
$$G_{\tau,ab} =$$ 

Exact flow equations

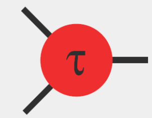
for moments $\Gamma^{(n)}$

$\partial_\tau \Gamma_\tau = \frac{i}{2}$ 

$\partial_\tau \Gamma^{(1)}_{\tau,a} = \frac{i}{2}$ 

$\dot{R}_{\tau,ab} =$ 

$G_{\tau,ab} =$ 

$\Gamma^{(3)}_{\tau,abc} =$ 

Exact flow equations

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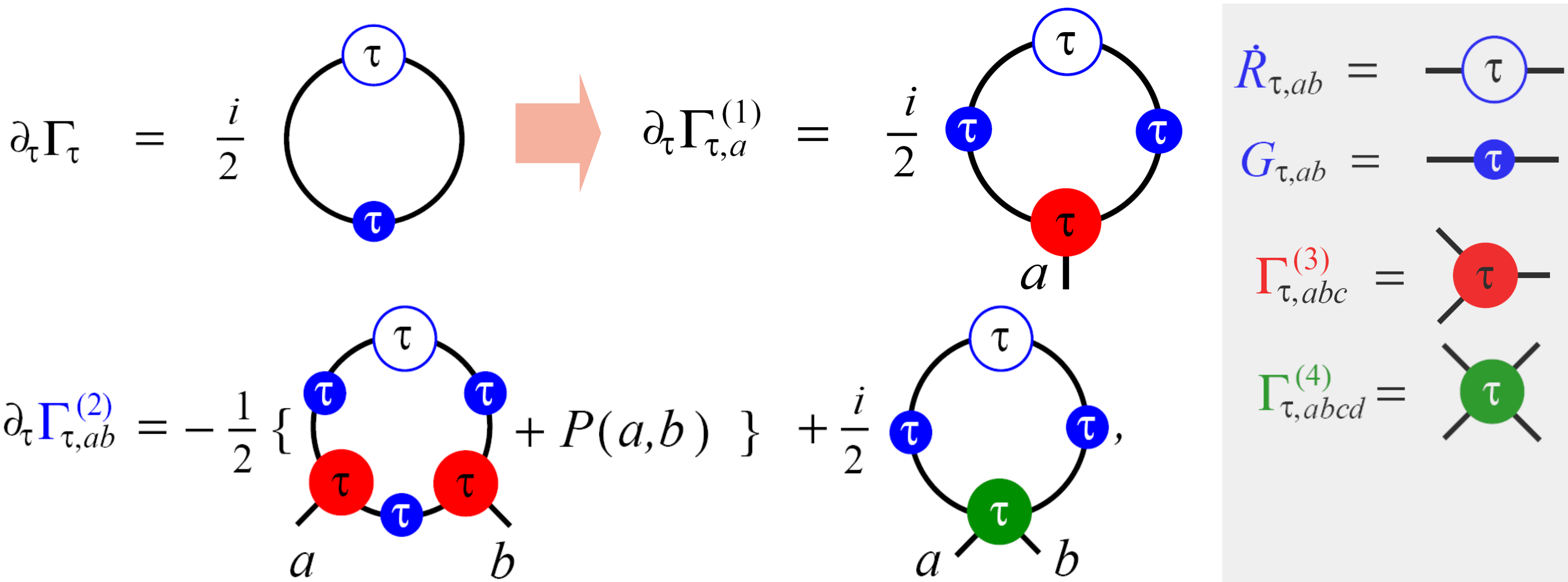
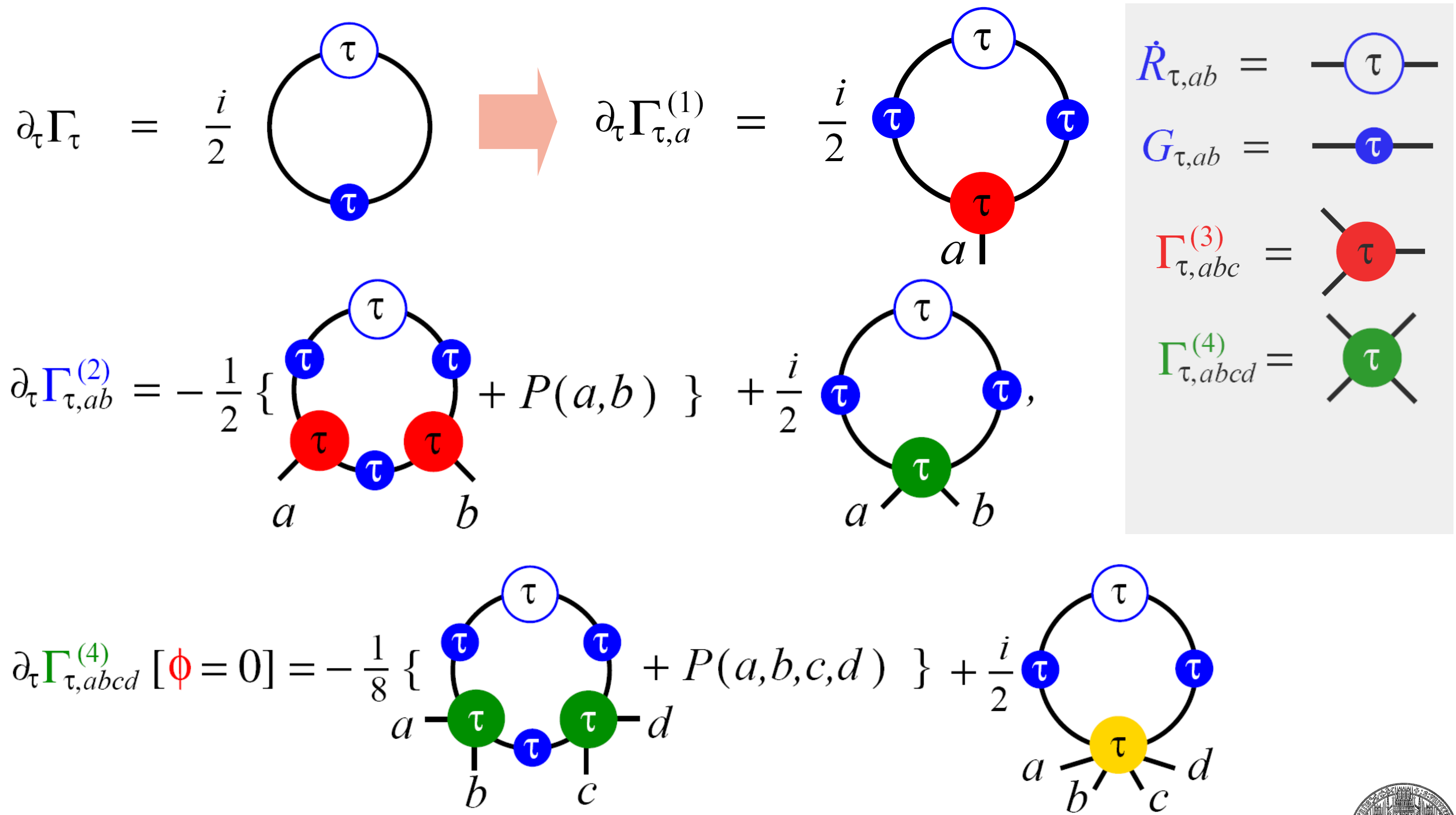


Exact flow equations

for moments $\Gamma^{(n)}$

…considering a scalar theory with φ^4 interaction



Exact flow equations

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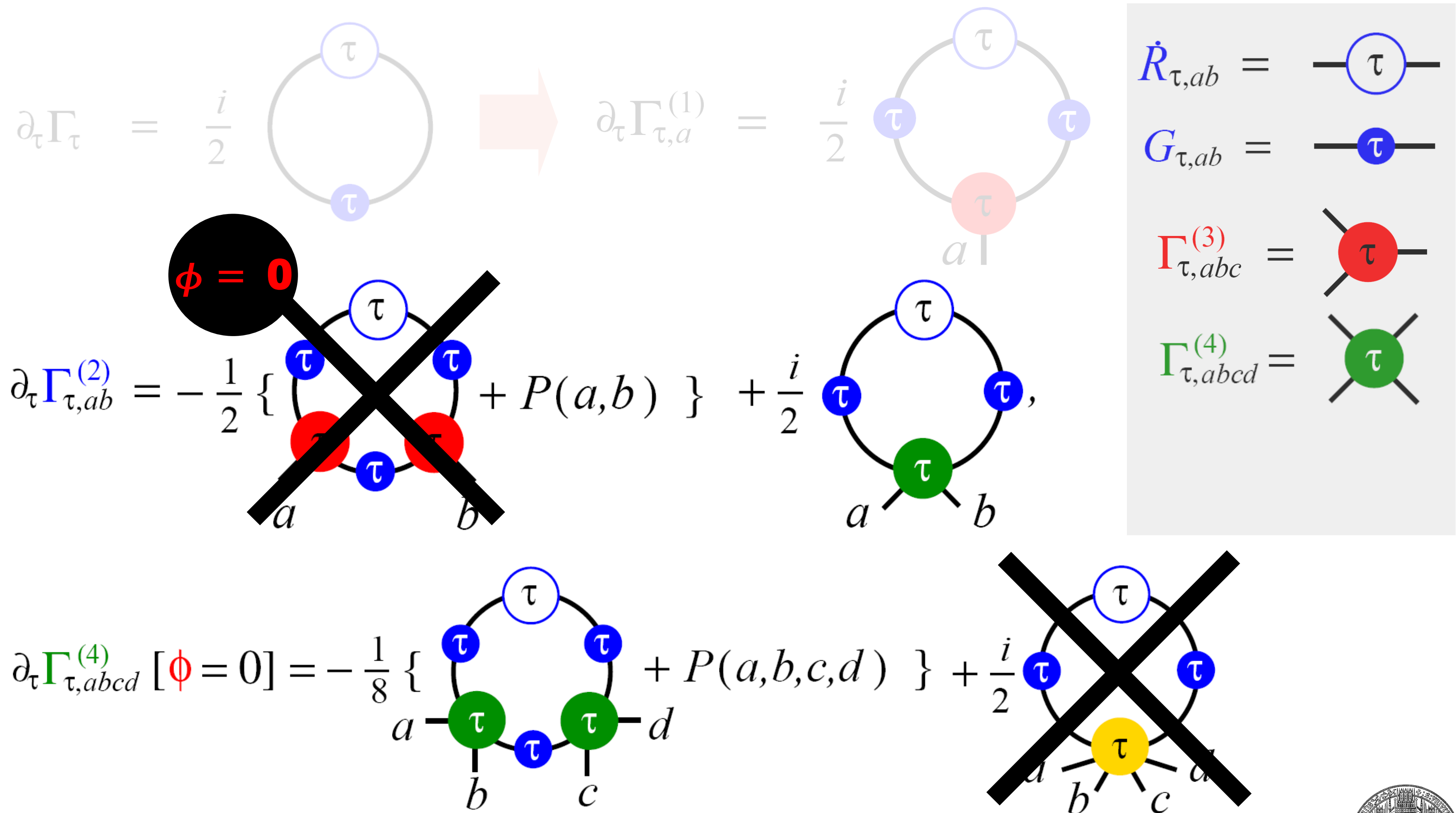
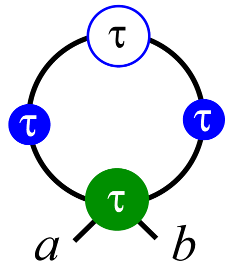


Integrated flows

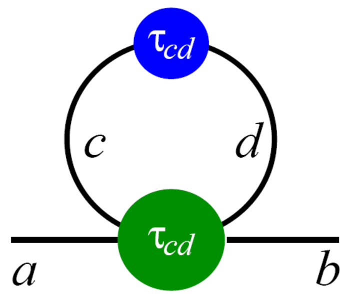
of moments $\Gamma^{(n)}[\phi \equiv 0]$

[TG & J.M. Pawlowski, cond-mat/0710.4627]

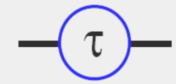
$$\partial_\tau \Gamma^{(2)}_{\tau,ab} = \frac{i}{2}$$



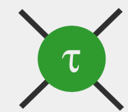
$$\Gamma^{(2)}_{\tau_{ab},ab} = \Gamma^{(2)}_{t_0,\,ab} + \frac{1}{2}$$



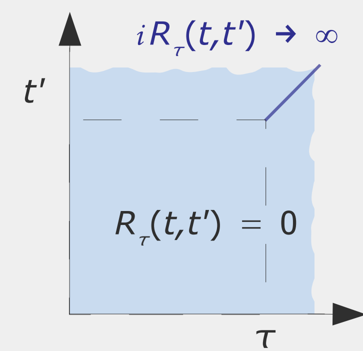
$t_c, t_d = t_0 \ldots \tau_{ab}$

$\dot{R}_{\tau,ab} =$ 

$G_{\tau,ab} =$ 

$\Gamma^{(4)}_{\tau,abc} =$ 

$\tau_{ab\ldots} = max(t_a, t_b, \ldots)$



Integrated flows

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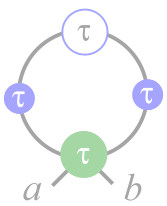
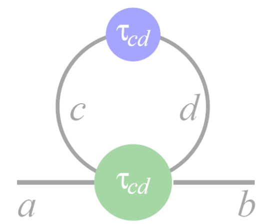
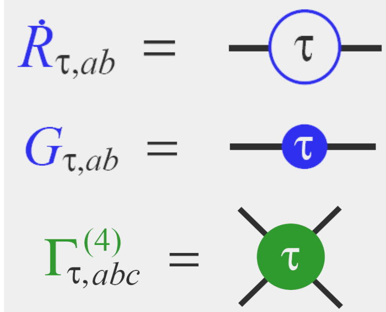
[TG & J.M. Pawlowski, cond-mat/0710.4627]

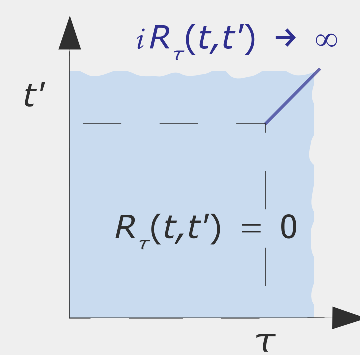
$\partial_\tau \Gamma^{(2)}_{\tau,ab} = \frac{i}{2}$ 

$\Rightarrow \quad \Gamma^{(2)}_{\tau_{ab},ab} = \Gamma^{(2)}_{t_0,ab} + \frac{1}{2}$ 

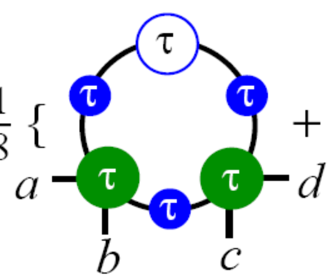
$t_c, t_d = t_0 \ldots \tau_{ab}$

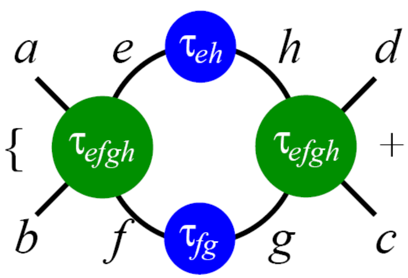


$\tau_{ab\ldots} = max(t_a, t_b, \ldots)$



similarly:

$\partial_\tau \Gamma^{(4)}_{\tau,abcd}[\phi = 0] = -\frac{1}{8}\{$  $+ P(a,b,c,d)\ \}$

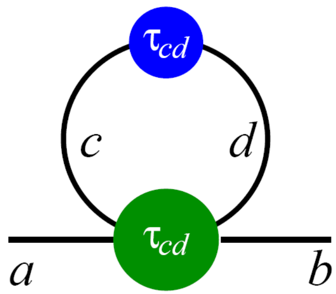
$\Rightarrow \quad \Gamma^{(4)}_{t,abcd} = \Gamma^{(4)}_{t_0,abcd} + \frac{i}{2}\{$  $+ P(a,b,c,d)\}$

$t_{e\ldots h} = t_0 \ldots t$

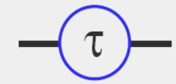
Dynamic equations

[TG & J.M. Pawlowski, cond-mat/0710.4627]

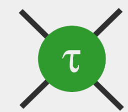
$$\Gamma^{(2)}_{\tau_{ab},ab} = \Gamma^{(2)}_{t_0,\,ab} + \frac{1}{2}$$



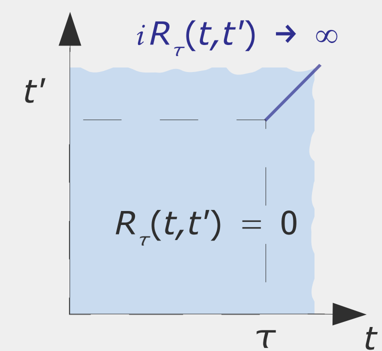
$t_c, t_d = t_0 \ldots \tau_{ab}$

$\dot{R}_{\tau,ab} =$ 

$G_{\tau,ab} =$ 

$\Gamma^{(4)}_{\tau,abc} =$ 

$\tau_{ab\ldots} = max(t_a, t_b, \ldots)$



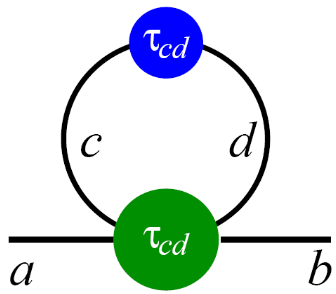
...gives the dynamic equation for *G*. We choose:

$$\Gamma^{(2)}_{t_0,ab}(x,y) = G^{-1}_{0,ab}(x,y) =$$

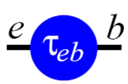
$$\delta(x-y)[-\sigma^2_{ab}\partial_{x_0} + iH_{1\mathrm{B}}(x)\delta_{ab}]$$

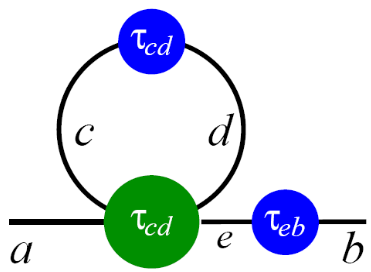
Dynamic equations

[TG & J.M. Pawlowski, cond-mat/0710.4627]

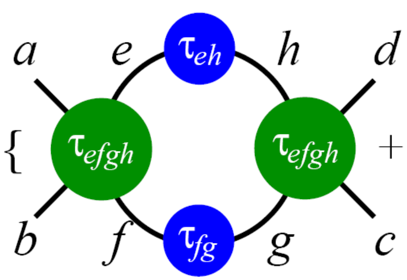
$\Gamma^{(2)}_{\tau_{ab},ab} = \Gamma^{(2)}_{t_0,\,ab} + \frac{1}{2}$ 

$t_c, t_d = t_0 \dots \tau_{ab}$

$G^{-1}_{0\,ae}$ 

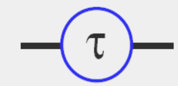
$= \delta_{ab} - \frac{1}{2}$ 

$t_c, t_d = t_0 \dots \tau_{ae}$

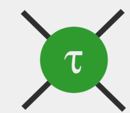
$\Gamma^{(4)}_{t,\,abcd} = \Gamma^{(4)}_{t_0,abcd} + \frac{i}{2}\{$ 

$+ P(a,b,c,d)\,\}$

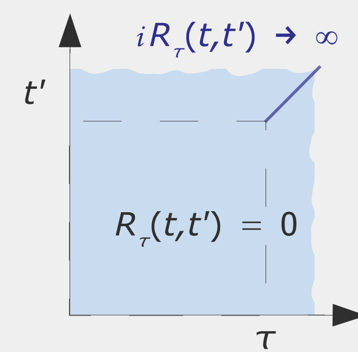
$t_{e\dots h} = t_0 \dots t$

$\dot{R}_{\tau,ab} =$ 

$G_{\tau,ab} =$ 

$\Gamma^{(4)}_{\tau,abc} =$ 

$\tau_{ab\dots} = max(t_a, t_b, \dots)$



2PI dynamics in NLO $1/\mathcal{N}$ as an RG truncation

2PI Effective Action (Φ-Functional)

[Luttinger, Ward (60); Baym (62); Cornwall, Jackiw, Tomboulis (74)]

- Double **Legendre transform**:

$$\Gamma[\phi, G] = -i \ln Z[J, K] - \phi_i J_i - \frac{1}{2}(\phi_i \phi_j + G_{ij}) K_{ij},$$

$$-i \left.\frac{\delta \ln Z[J, K]}{\delta J_i}\right|_{J=K\equiv 0} = \phi_i = \langle \hat{\Phi}_i \rangle,$$

$$-2i \left.\frac{\delta \ln Z[J, K]}{\delta K_{ij}}\right|_{J=K\equiv 0} = \phi_i \phi_j + G_{ij} = \langle \mathcal{T} \hat{\Phi}_i \hat{\Phi}_j \rangle.$$

2PI Effective Action (Φ-Functional)

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- Double **Legendre transform**:

$$\Gamma[\phi, G] = -i \ln Z[J, K] - \phi_i J_i - \frac{1}{2}(\phi_i \phi_j + G_{ij}) K_{ij},$$

$$-i \left. \frac{\delta \ln Z[J, K]}{\delta J_i} \right|_{J=K\equiv 0} = \phi_i = \langle \hat{\Phi}_i \rangle,$$

$$-2i \left. \frac{\delta \ln Z[J, K]}{\delta K_{ij}} \right|_{J=K\equiv 0} = \phi_i \phi_j + G_{ij} = \langle \mathcal{T} \hat{\Phi}_i \hat{\Phi}_j \rangle.$$

- Dynamic equations: [closed for Gaussian initial conditions (only ϕ, G ≠ 0 @ t = 0)]

(@ J = 0, K = 0)

$$\frac{\delta \Gamma[\phi, G]}{\delta \phi_x} = 0, \qquad \frac{\delta \Gamma[\phi, G]}{\delta G(x, y)} = 0$$

2PI Effective Action...

...now reads:

$$\Gamma[\phi,G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\, G_0^{-1}(\phi)\, G + \Gamma_2[\phi,G] + \mathrm{const}$$

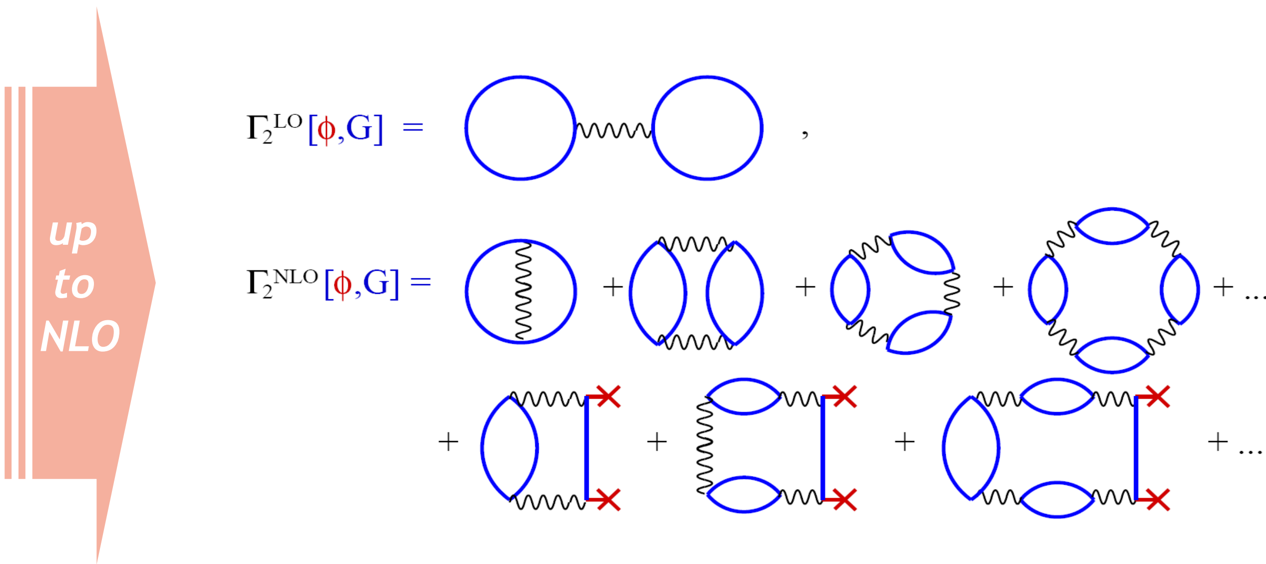
Variation w.r.t. ***G*** gives the Schwinger-Dyson equation:

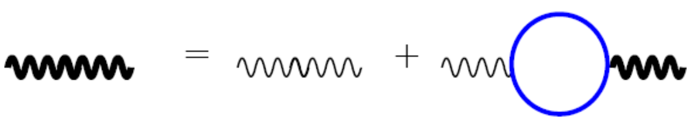
$$G^{-1}_{ab}(x,y) \;=\; G^{-1}_{0,ab}(x,y;\phi) \;-\; \Sigma_{ab}(x,y;\phi,G)$$

self energy

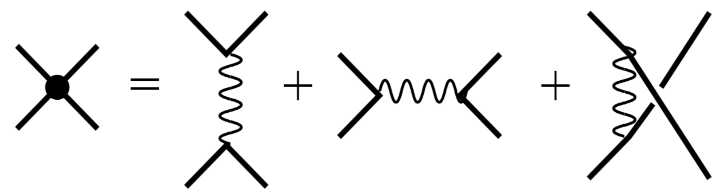
2PI 1/$\mathcal{N}$ Expansion

[Berges, NPA **699** (02) 847; Aarts, Ahrensmaier, Baier, Berges, & Serreau, PRD **66** (02) 45008]





Vertex resummation

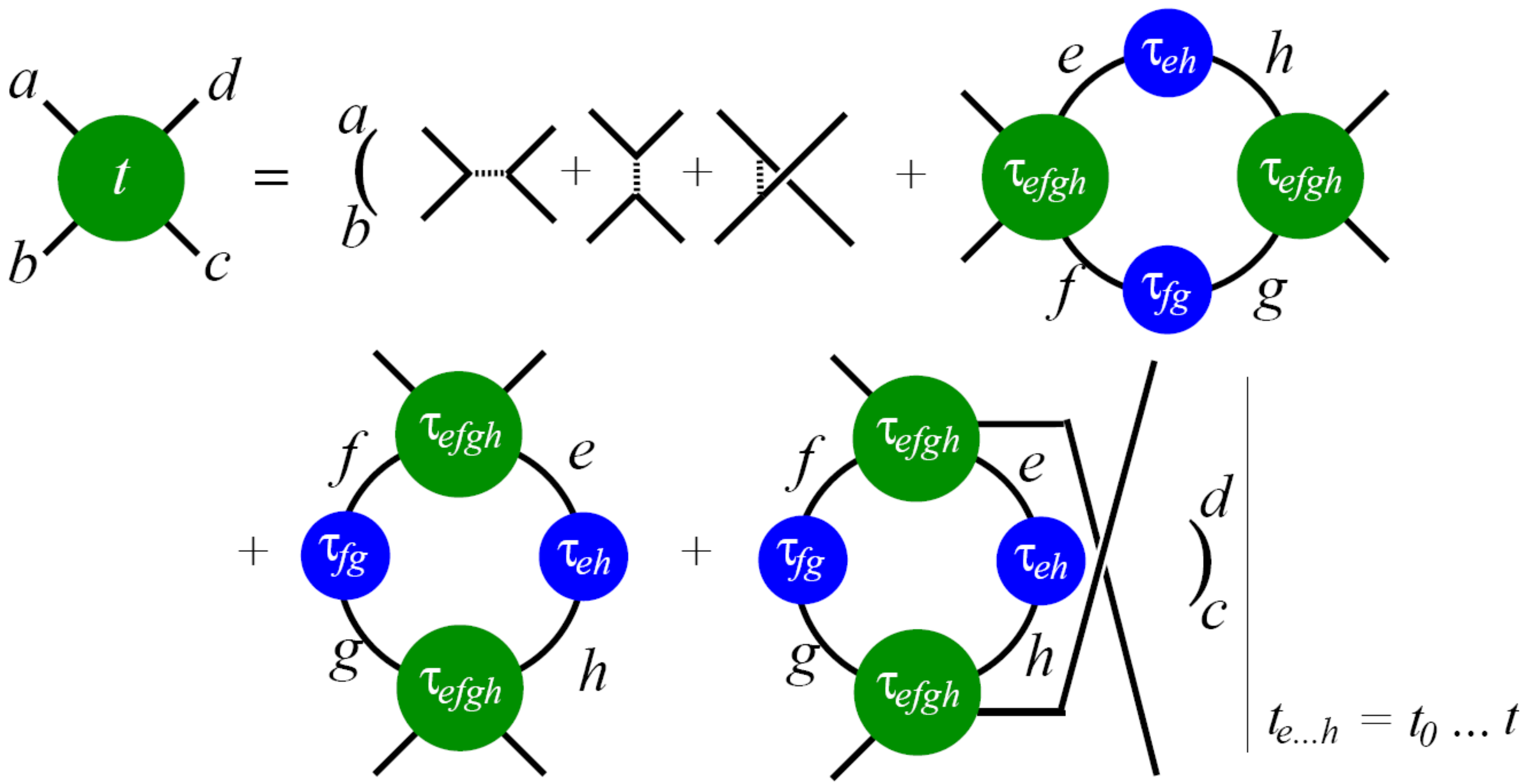


bare vertex

[Also: Mihaila, Dawson & Cooper (01); Cooper, Dawson & Mihaila (03); Berges & Serreau (03); Berges, Borsanyi & Serreau (03); Berges, Borsanyi & Wetterich (04); Alford, Berges & Cheyne (04); Arrizabalaga, Smit & Tranberg (04); ...; Rey, Hu, Calzetta (04); Baier & Stockamp (04); TG, Berges, Schmidt & Seco (05); ...]

Back to: Integrated flow
of 4-point function $\Gamma^{(4)}[\phi \equiv 0]$

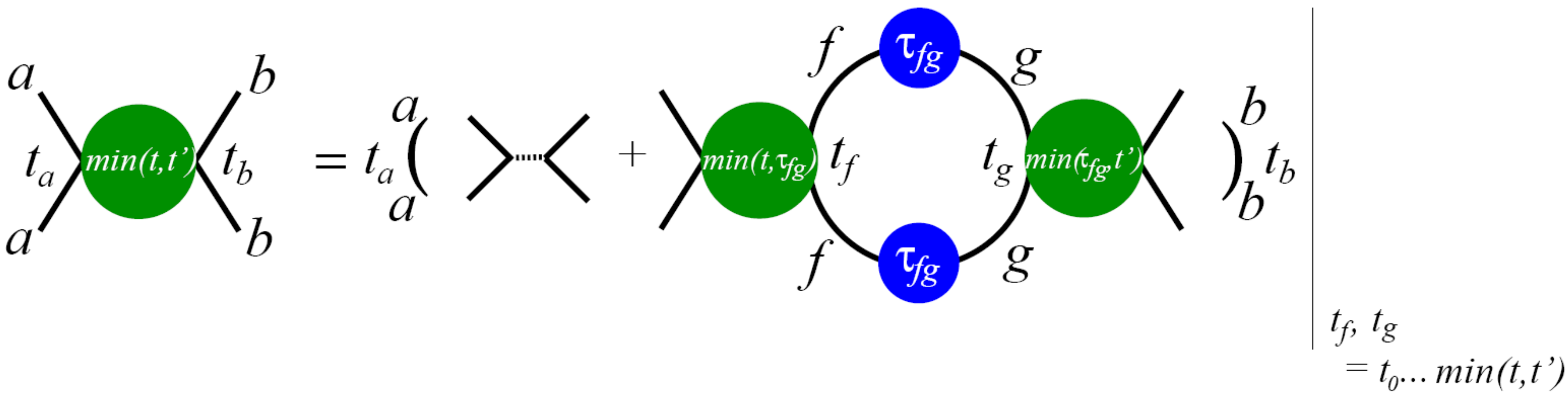
[TG & J.M. Pawlowski, cond-mat/0710.4627]



Integrated flow

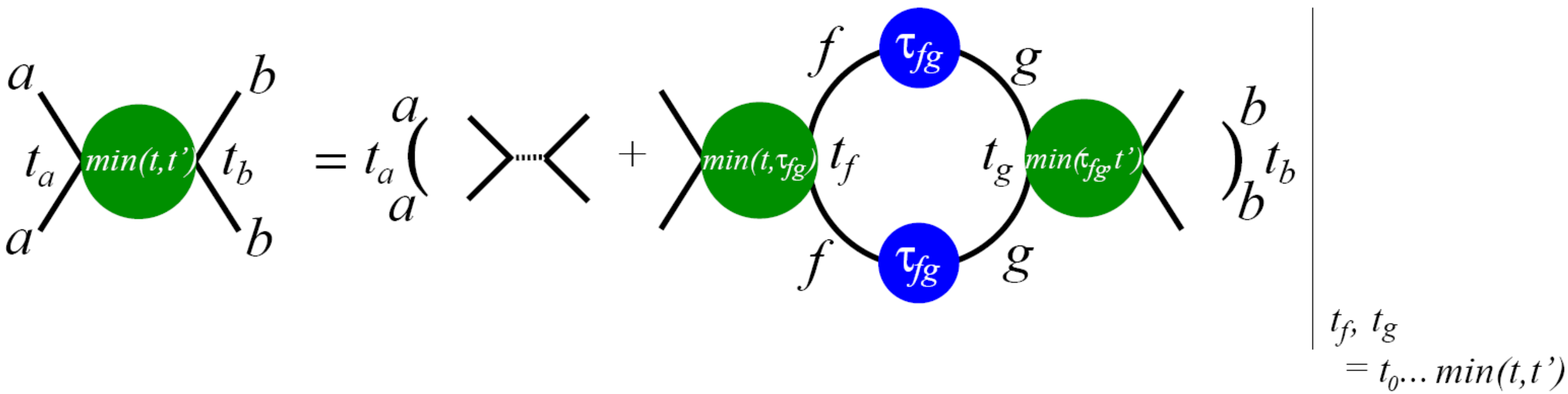
in s-channel approximation

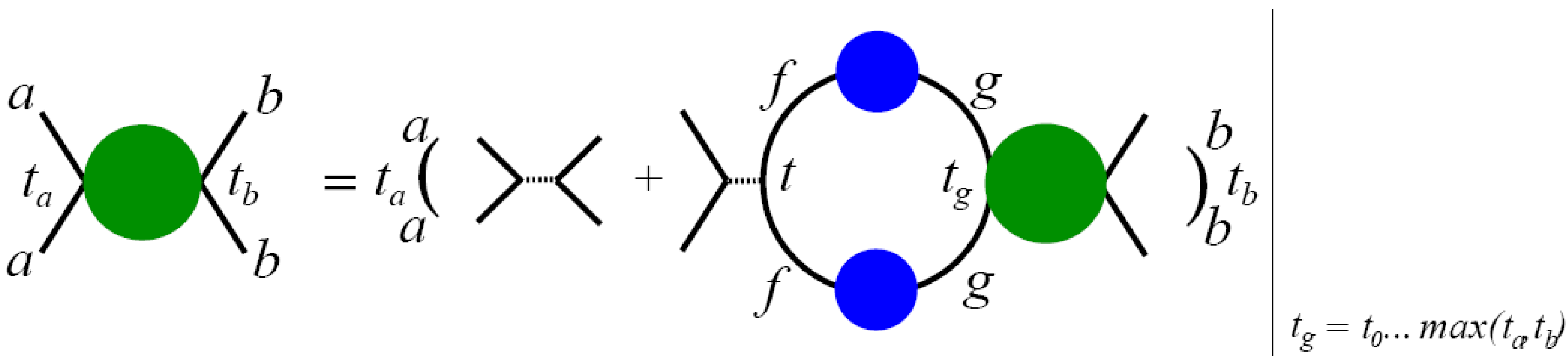
[TG & J.M. Pawlowski, cond-mat/0710.4627]



From s-channel RG to 2PI NLO $1/\mathcal{N}$

[TG & J.M. Pawlowski, cond-mat/0710.4627]





Renormalisation-group approach
to far-from-equilibrium dynamics

[TG & J.M. Pawlowski, cond-mat/0710.4627]

The dynamic equations derived from the Functional RG equation...

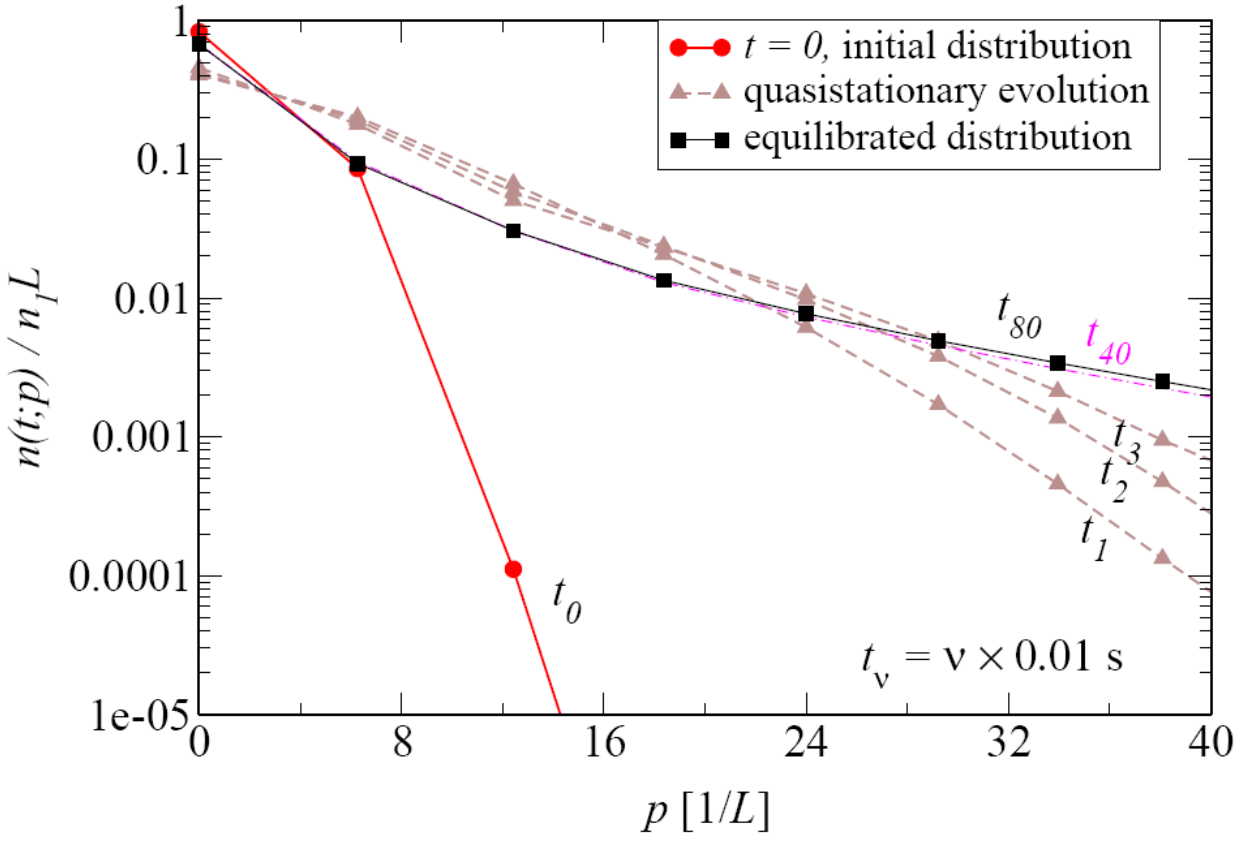
- ... can be solved iteratively in time,
- ... provide a resummed 4-vertex beyond 2PI NLO $1/\mathcal{N}$,
- ... allow non-perturbative truncations neglecting higher n-vertices,
- ... provide handle to study the quality of the truncation.

Equilibration of a 1D Bose gas

from 2PI NLO $1/\mathcal{N}$ / s-channel RG

Equilibration of a 1D Bose gas

Momentum distribution for different times:

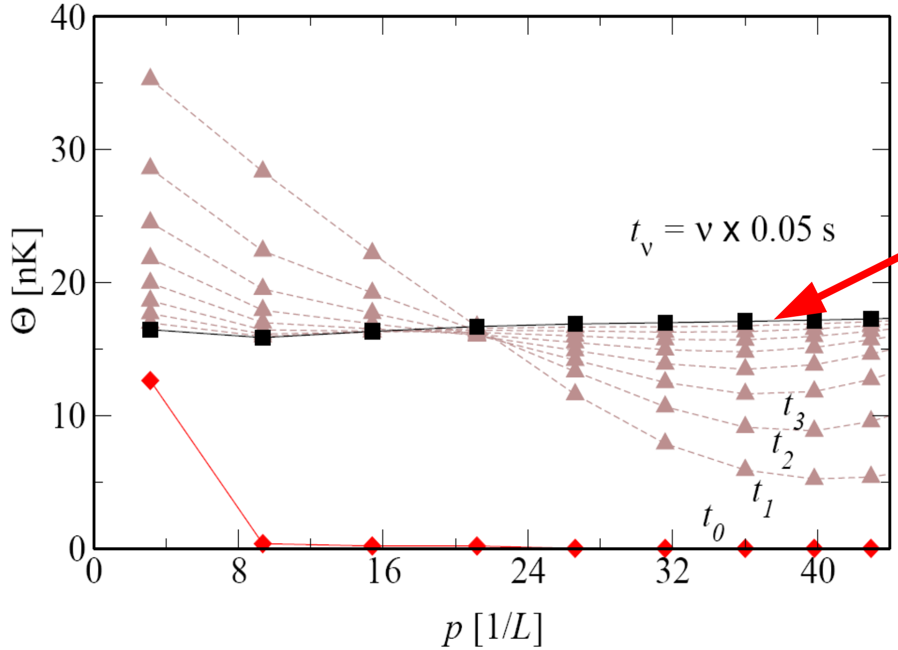


[TG, J. Berges, M. Seco & M.G.Schmidt, PRA **72** (05); J. Berges & TG, PRA **76** (07)]

Temperature appears

[Berges & Cox (01), Berges (01), Berges, Borsanyi, & Serreau (03), Berges, Borsanyi, & Wetterich (04)]

'Temperature' parameter $\Theta(p)$ at t_n



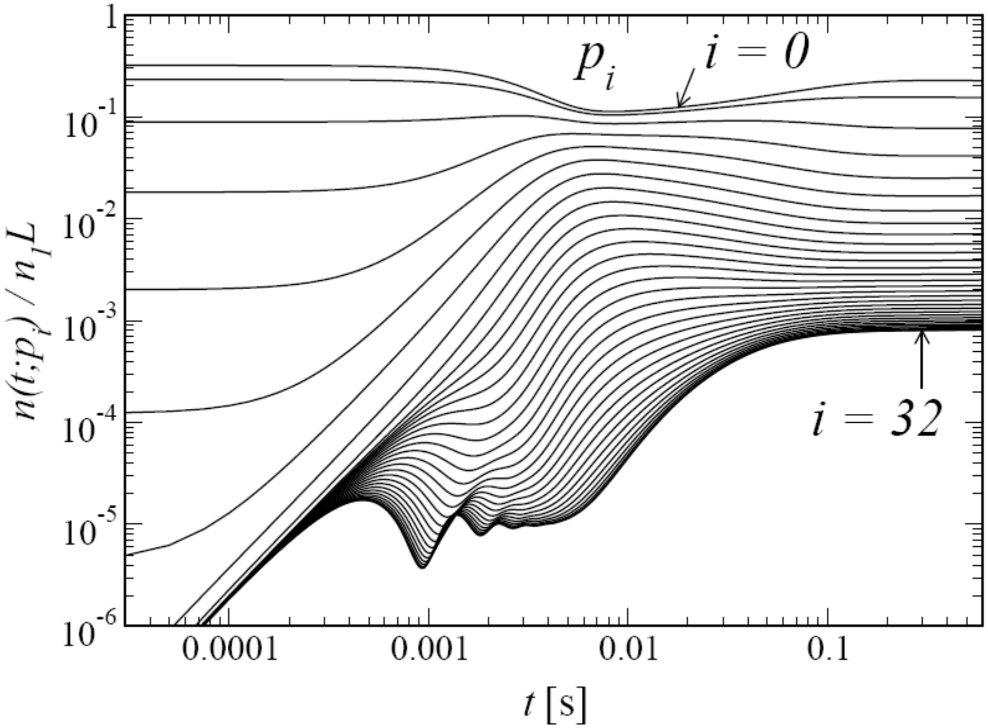
flat = temperature

[J. Berges & TG, PRA **76** (07)]

$$n(t;p) = \frac{1}{e^{\frac{1}{k_B \Theta(p)}\left(\frac{p^2}{2m} - \mu\right)} - 1}$$

Far-from-equilibrium evolution

Time evolution of mode occupation no^s:

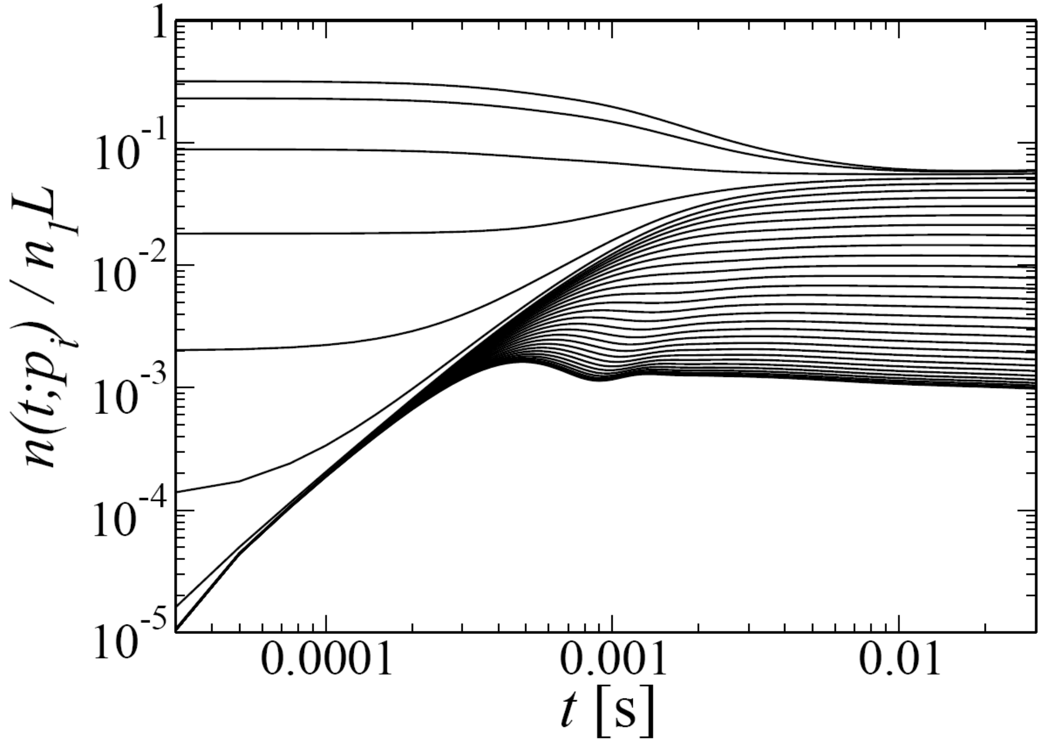


initial state:
- ^{23}Na atoms in 1D, $n_1 = 10^7$ m^{-1}
- interaction parameter $\gamma = \lambda m/(\hbar^2 n_1) = 7.5 \cdot 10^{-4}$
- Gaussian momentum distribution

[J. Berges & TG, PRA **76** (07)]

Far-from-equilibrium evolution

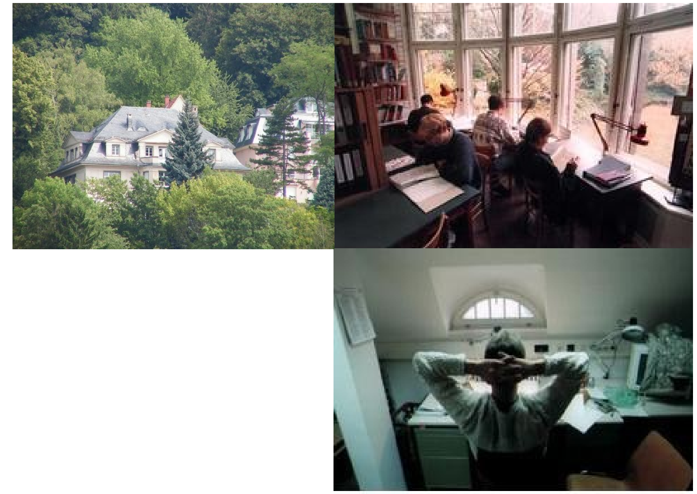
Time evolution of mode occupation no^s^:



initial state:

- ^{23}Na atoms in 1D, $n_1 = 10^5\,\mathrm{m}^{-1}$
- interaction parameter $\gamma = \lambda m/(\hbar^2 n_1) = 15$
- Gaussian momentum distribution

Thanks & credits to...



...my work group in Heidelberg:

Cédric Bodet
Alexander Branschädel (→ Karlsruhe)
Stefan Keßler
Matthias Kronenwett
Philipp Struck (→ Konstanz)
Kristan Temme (→ Vienna)
Martin Trappe

...my collaborators

Jürgen Berges • Darmstadt
Hrvoje Buljan • Zagreb
Markus Oberthaler • Heidelberg
Jan M. Pawlowski • Heidelberg
Robert Pezer • Sisak
Michael G. Schmidt • Heidelberg
Marcos Seco • Santiago de Compostela

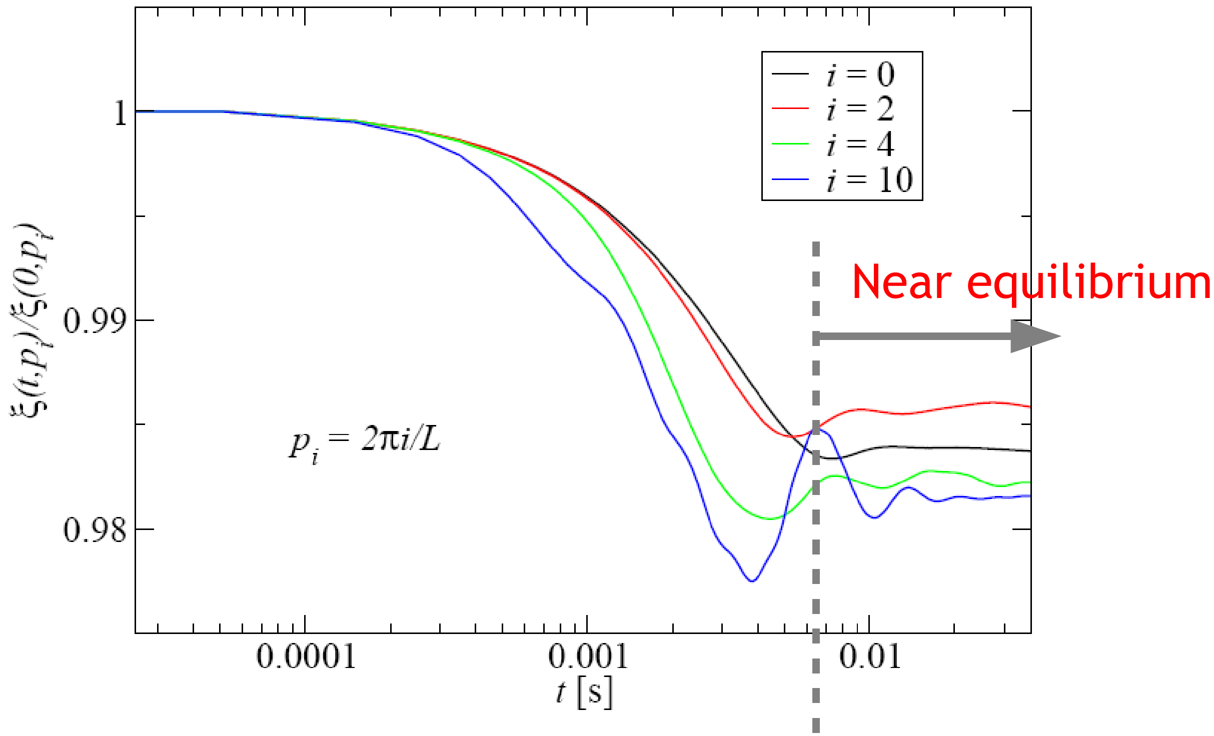
...also to

Keith Burnett • Oxford/Sheffield • David Hutchinson • Otago • Thorsten Köhler • UCL • Paul S. Julienne • NIST Gaithersburg • David Roberts • ENS Paris • Janne Ruostekoski • Southampton

Supplementary slides

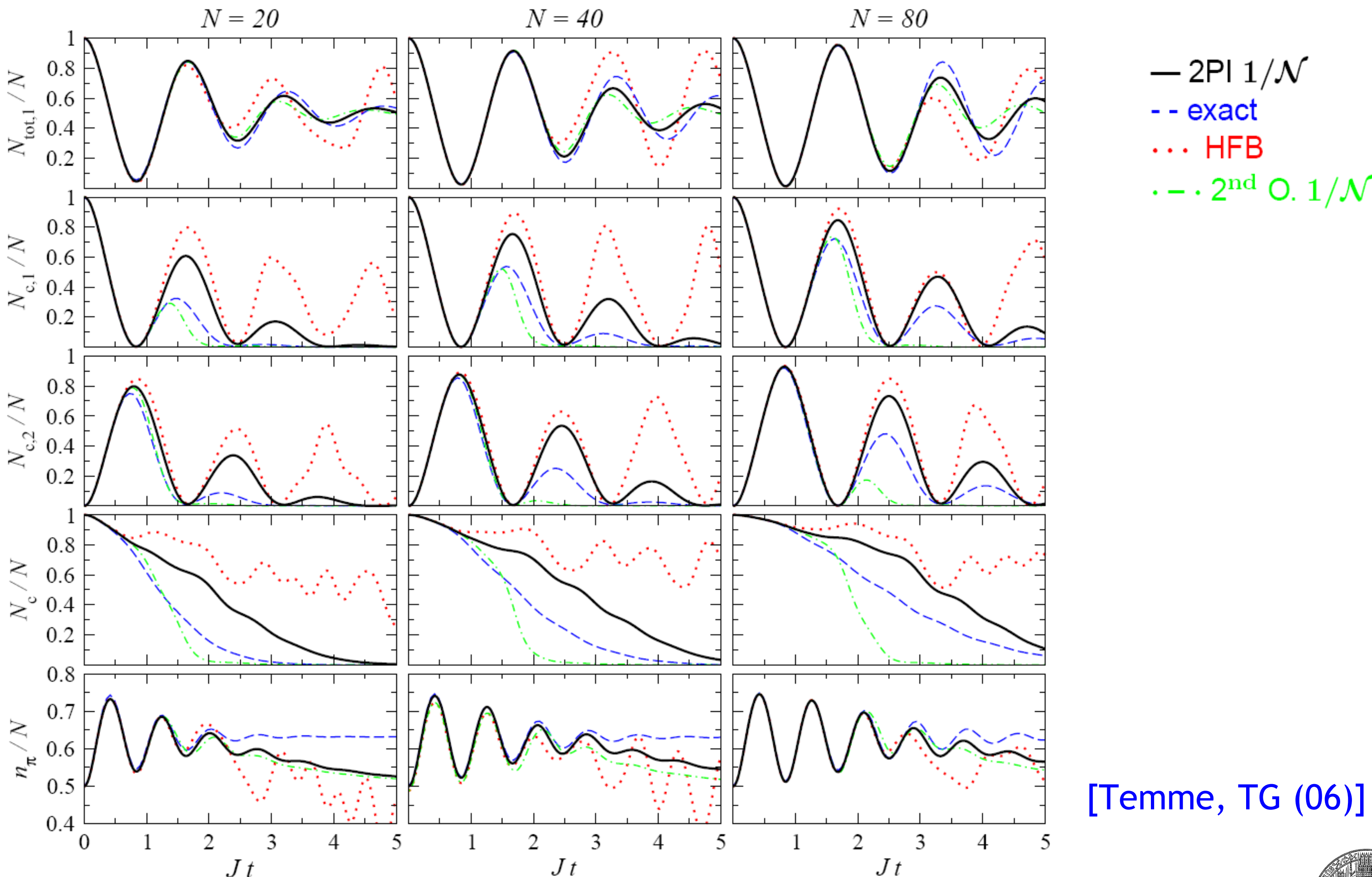
Onset of near-equilibrium evolution

Time evolution of temporal correlations $\xi(t,p) = F(t,0;p)/\rho(t,0;p)$:



(Fluctuation-Dissipation rel.: $F^{(\mathrm{eq})}_{\omega\mathbf{p}} = -i\left(n(\omega,T)+\frac{1}{2}\right)\rho^{(\mathrm{eq})}_{\omega\mathbf{p}}$)

Bose-Einstein condensate in lattice potential



[Temme, TG (06)]

Overview

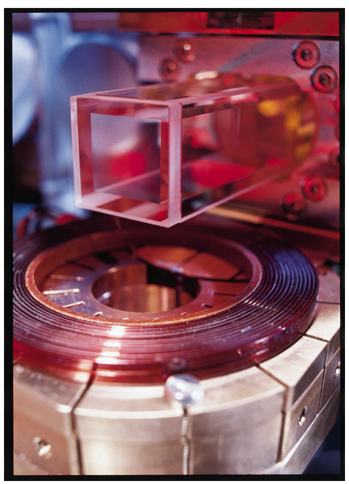
- Preface
  Ultracold gases out of equilibrium
- Non-equilibrium quantum field theory
  Functional RG approach
- Equilibration of a 1D Bose gas
  s-Channel approximation and 2PI NLO 1/N

ULTRAcool...

... atoms @ nanokelvins –

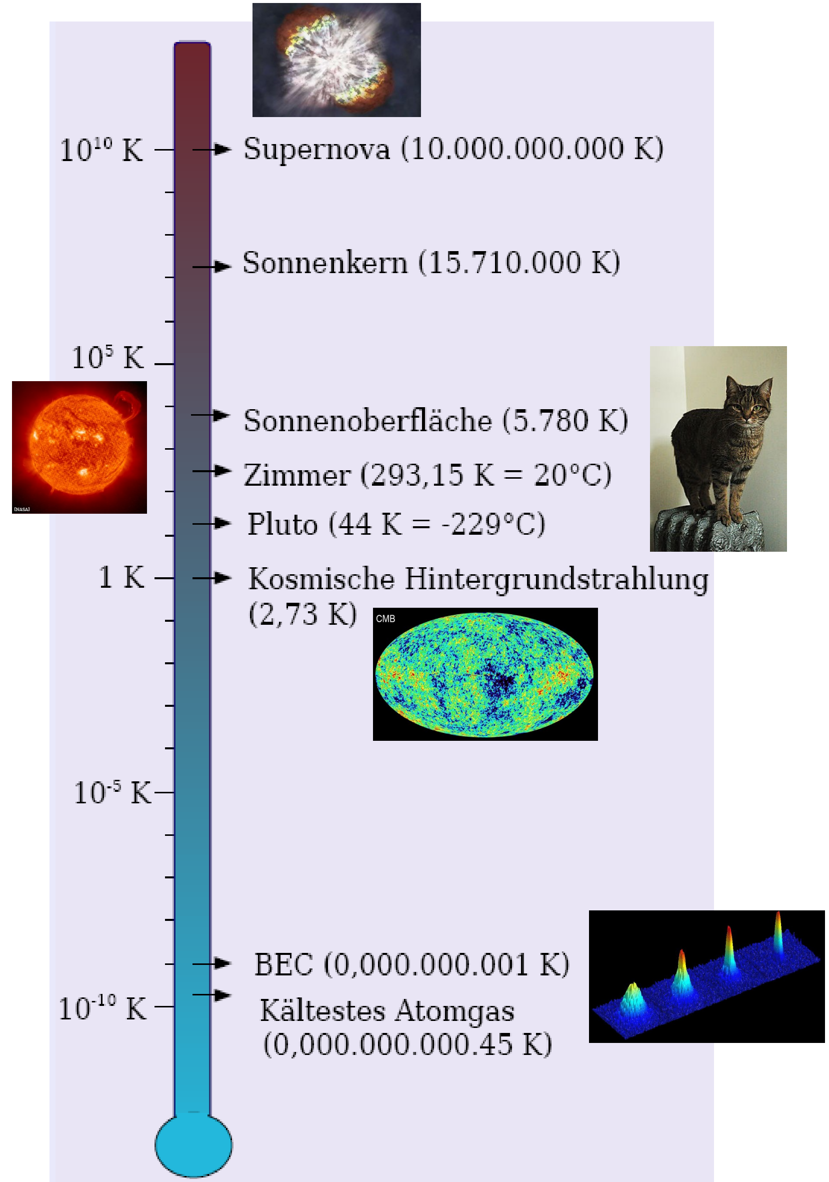
trapped only a few mm away from

glass cell @ room temperature

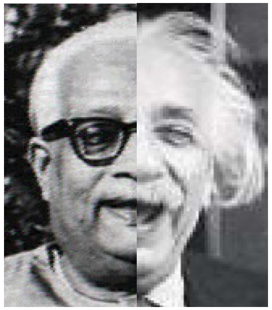


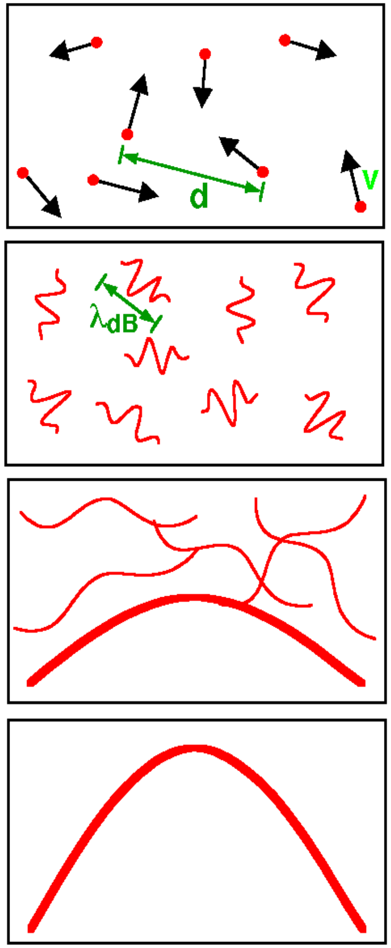
(vacuum of 10^{-12} Torr,
i.e. 10^{-15} bar,
or 10^{-10} Pa,

≈ atmospheric
pressure on the moon)



Bose-Einstein condensation





High Temperature T:
thermal velocity v
density d^{-3}
"Billiard balls"

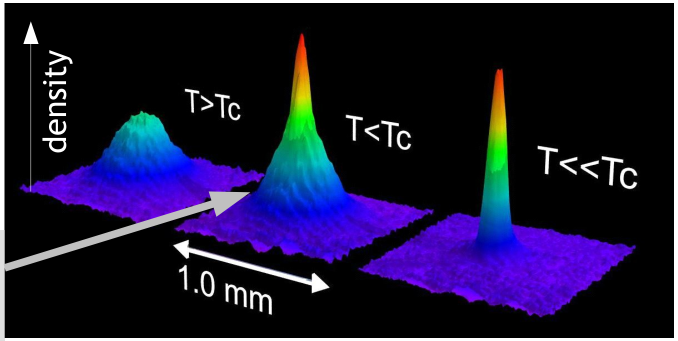
Low Temperature T:
De Broglie wavelength
$\lambda_{dB}=h/mv \propto T^{-1/2}$
"Wave packets"

$T=T_{crit}$:
Bose-Einstein Condensation
$\lambda_{dB} \approx d$
"Matter wave overlap"

T=0:
Pure Bose condensate
"Giant matter wave"

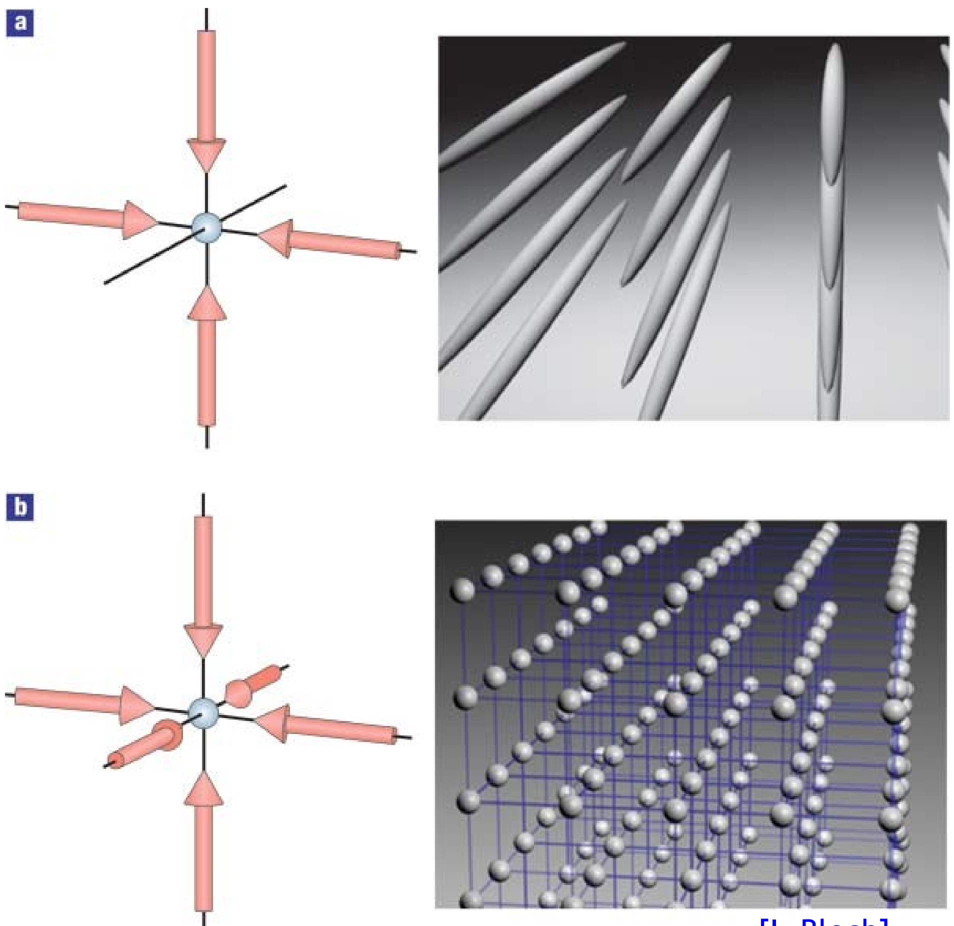
Experimental picture after free expansion of the trapped cloud:

Bose-Einstein condensation (BEC)



bimodal distribution

1D traps and lattices

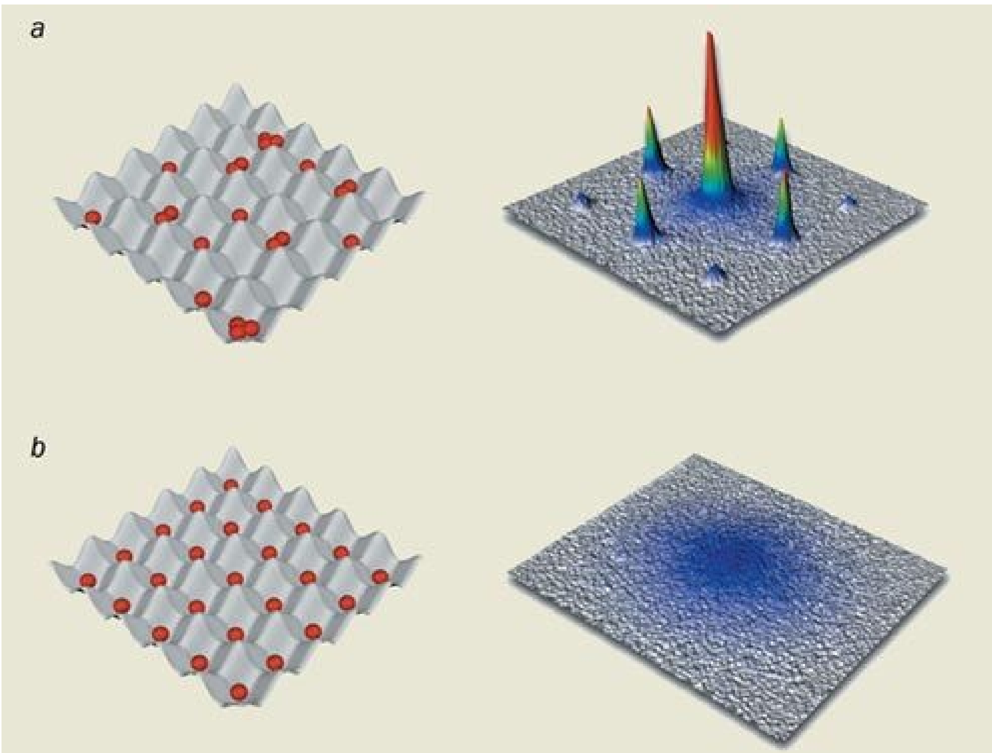


[I. Bloch]

Lasers allow to create lower dimensional traps and lattices

No restrictions to magnetic low-field seeking hyperfine states!

Optical lattices



Superfluid - Mott-insulator quantum phase transition

[M. Greiner et al., Nature 415 (02)]

Optical lattices allow

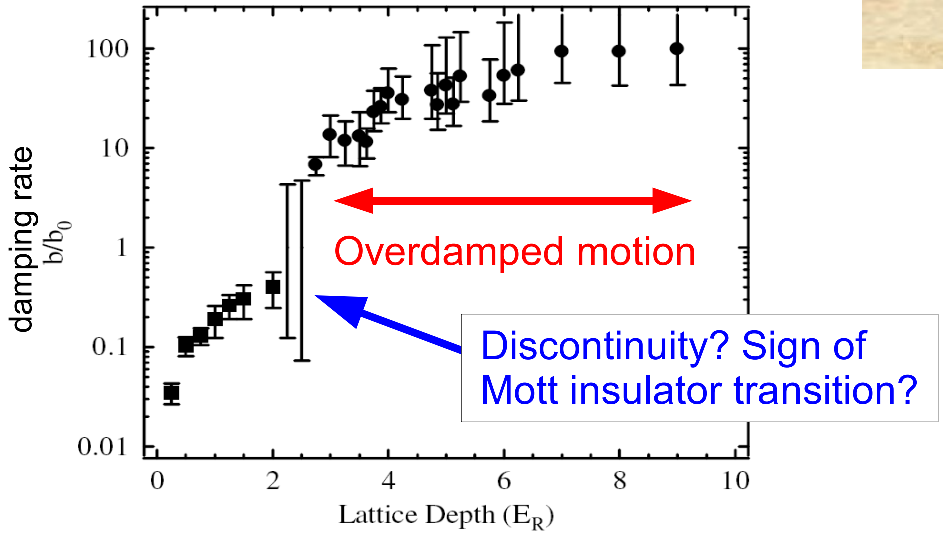
- simulation of solid state systems,
- study of quantum phase transitions,
- fast changes in long-range correlations

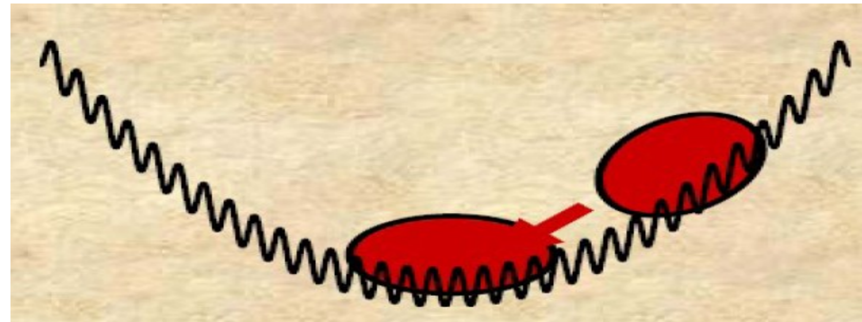
Nonequilibrium dynamics in lattices

[with P. Struck]

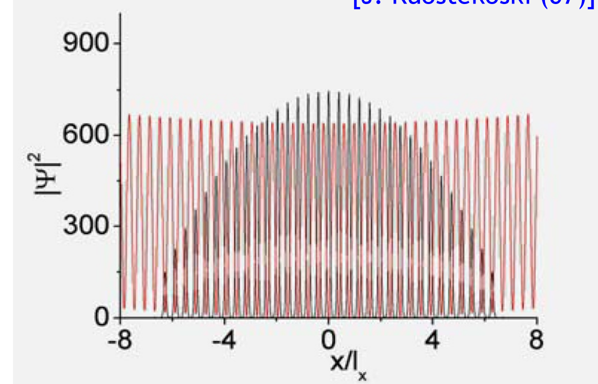
Dipole oscillations in lattices:
Damping rates?

[Expt. @ NIST: Fertig et al. PRL94 (05)]

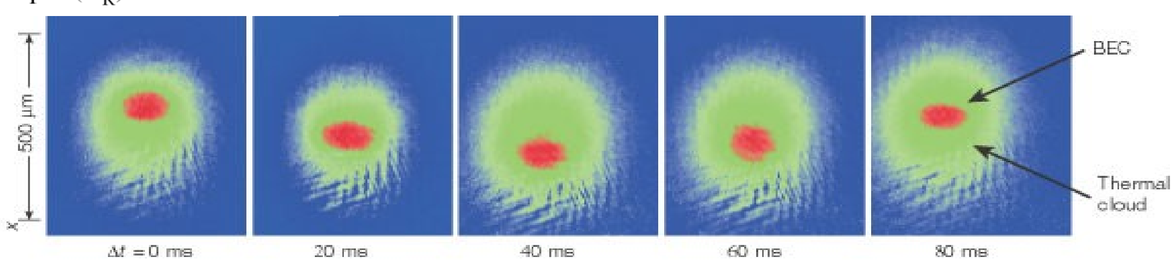




[J. Ruostekoski (07)]



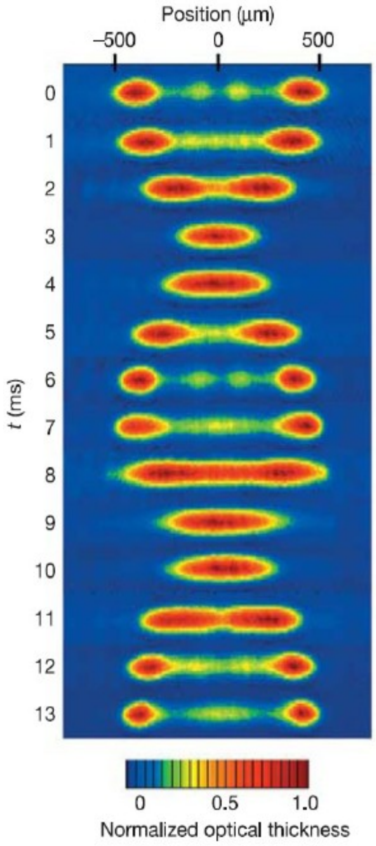
[Anglin & Ketterle (02)]



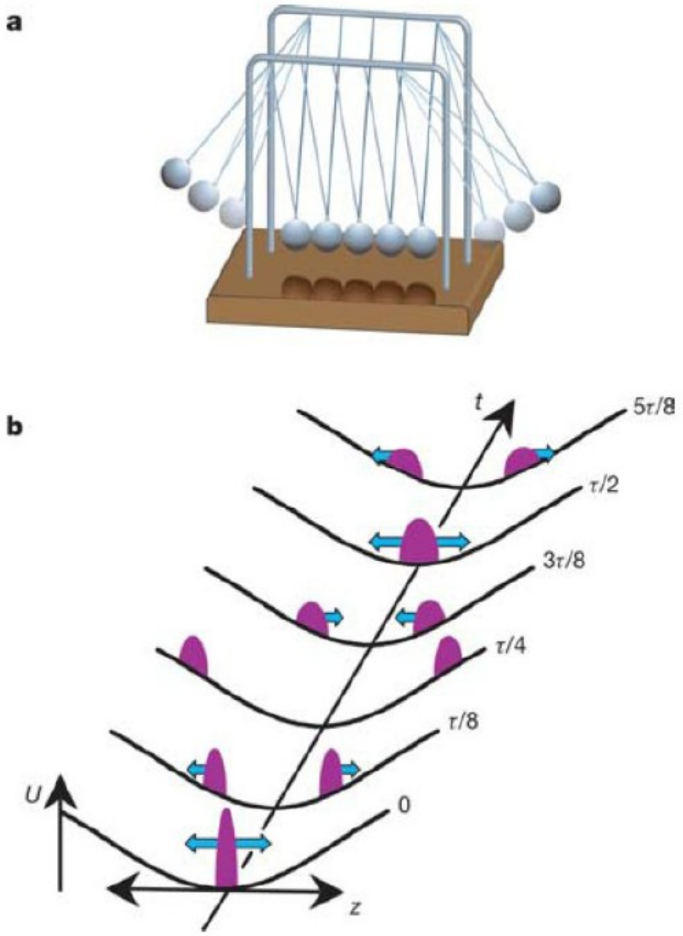
Long-time dynamics of ultracold gases

A quantum Newton's cradle.

[T. Kinoshita et al. Nature 440 (06)]



Indication for strong suppression of damping

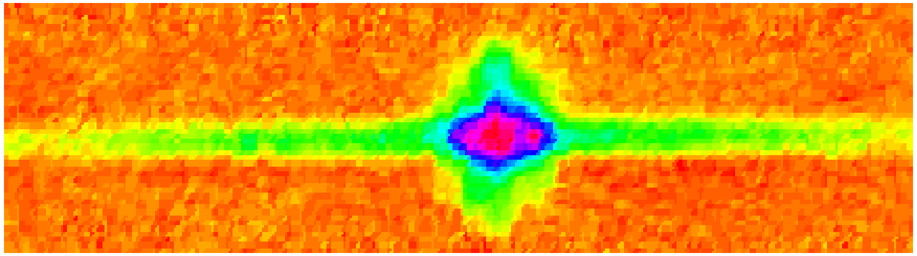


Strong Feshbach-induced dynamics

[with M. Kronenwett]

Feshbach resonances
allow fast changes of
collisional interaction strengths

"Bosenovae" [C. Wieman]



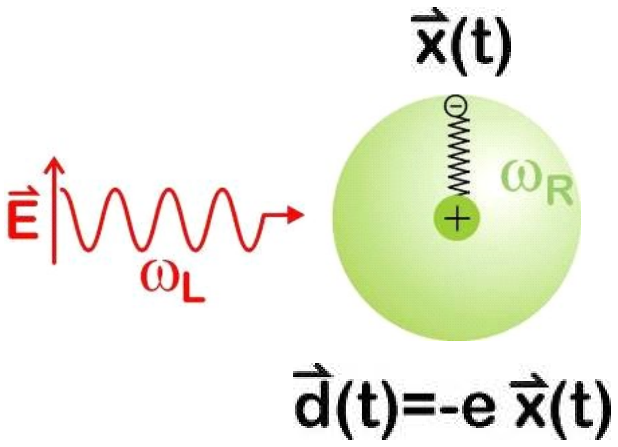
Molecule formation in a
Bose-Einstein condensate

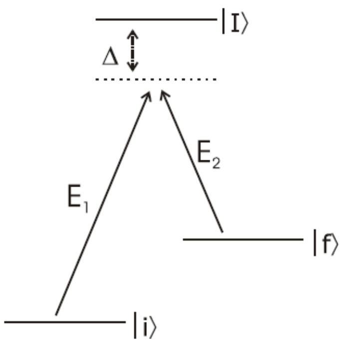
Crossover from a Superfluid of Bosons
to a "Superconductor" of Fermion pairs

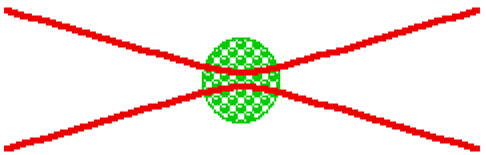
[JILA, Boulder]

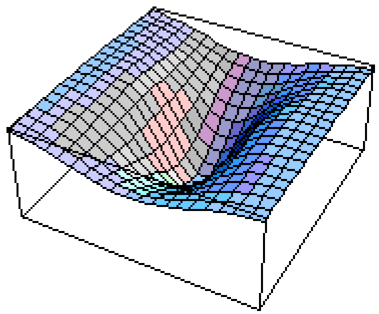
Atom-light-interactions -

the way to produce almost arbitrary trapping potentials









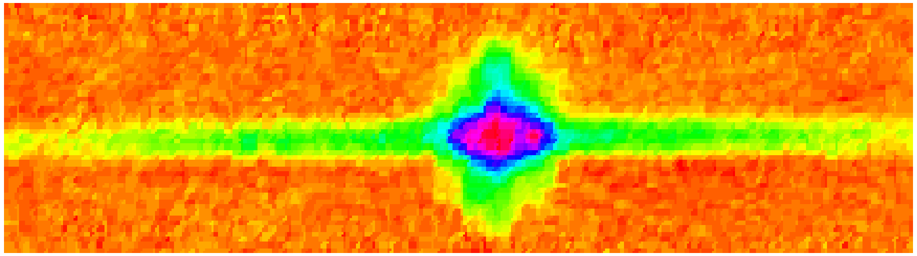
$$\alpha = -\frac{|\langle e|\vec{d}|g\rangle|^2}{\Delta}$$

$$U = -\alpha|\vec{E}|^2 \propto \frac{I}{\Delta}$$

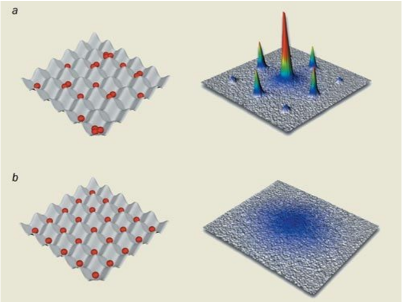
Simple example: Focused laser beam

"Strong" dynamics of ultracold gases

Feshbach resonances
allow fast changes of
collisional interaction strengths



"Bosenovae" [C. Wieman]



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[I. Bloch]

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allow fast changes in
long-range correlations

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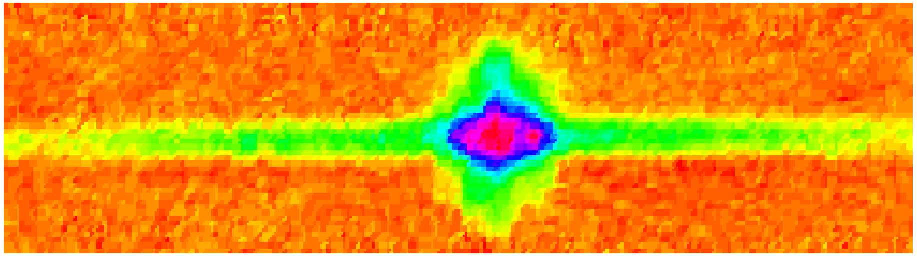


Molecule formation in a
Bose-Einstein condensate

Crossover from a Superfluid of Bosons
to a "Superconductor" of Fermion pairs

[JILA, Boulder]

SSB dynamics

Quench dynamics leading to the formation of ferromagnetic domains in a spinor BEC of ^{87}Rb

[Expt. @ UC Berkeley: Sadler et al. Nature 443 (06)]

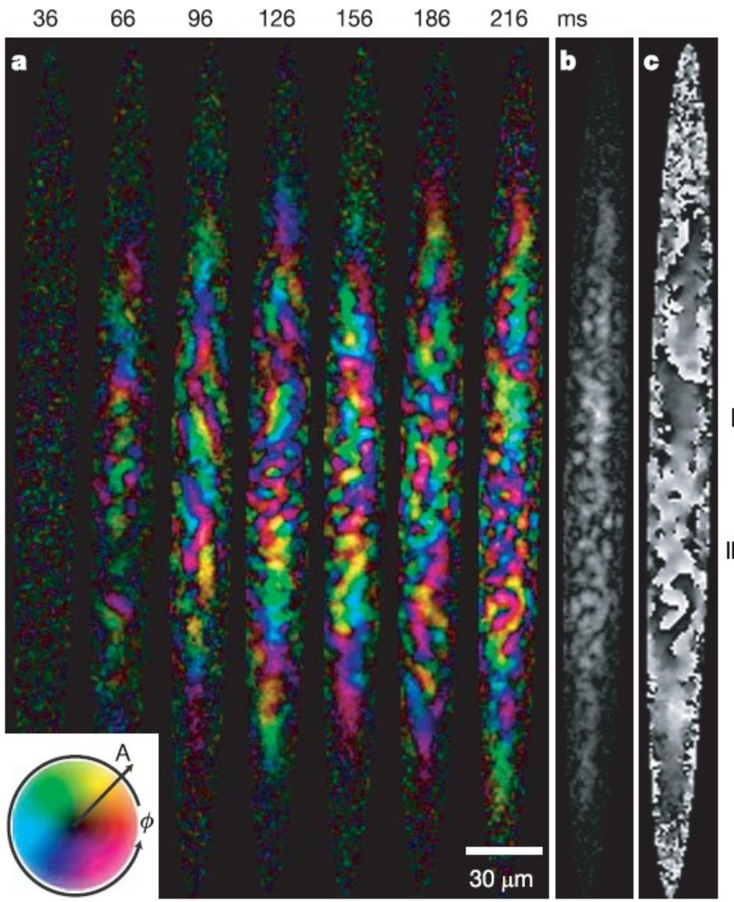
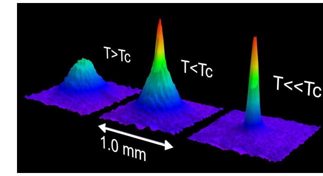


Figure 2 | *In situ* images of ferromagnetic domains and domain walls.

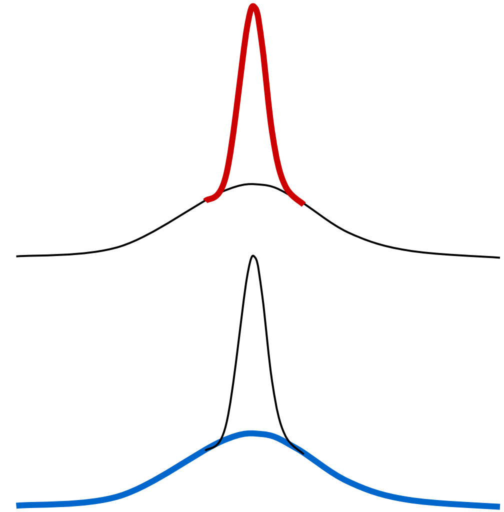
How to describe a condensate?



For **bosons**: $[\hat{\Phi}_{t,\mathrm{x}}, \hat{\Phi}^{\dagger}_{t,\mathrm{x}'}] = \delta(\mathrm{x}-\mathrm{x}')$

- Matter wave **mean field** $[x = (x_0, \mathrm{x}) = (t, \mathrm{x})]$

$$\phi_{\boldsymbol{x}} = \langle \hat{\Phi}_{\boldsymbol{x}} \rangle, \qquad |\phi_{\boldsymbol{x}}|^2 = n_c(x) = \text{condensate density},$$



- Density of **non-condensed atoms** $(\hat{\Phi} = \phi + \widetilde{\Phi},\ \phi = \langle \hat{\Phi} \rangle)$

$$\langle \widetilde{\Phi}^{\dagger}_{\boldsymbol{x}} \widetilde{\Phi}_{\boldsymbol{x}} \rangle = n_{\mathrm{nc}}(x) \equiv n(x) - n_c(x),$$

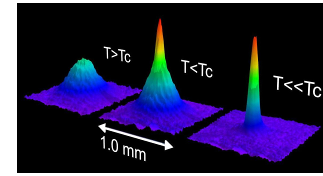
- Total one-body **density matrix**

$$G_{11}(x,y) = \langle \widetilde{\Phi}^{\dagger}_{x} \widetilde{\Phi}_{y} \rangle \Rightarrow \text{spatial Fourier transform: momentum distribution } n(\mathrm{p},t)$$
$$\Rightarrow 1^{\mathrm{st}}\text{-order phase coherence}$$

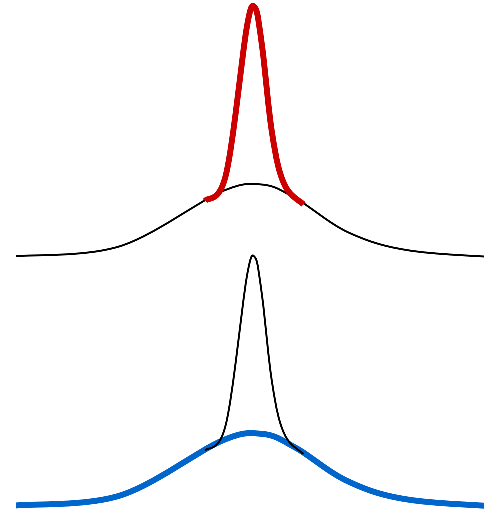
- **Anomalous one-body density** matrix

$$G_{12}(x,y) = \langle \widetilde{\Phi}_{x} \widetilde{\Phi}_{y} \rangle \Rightarrow \text{e.g., number of Bose-condensed bound pairs (molecules)}$$

How to describe a condensate?



For **bosons**: $[\hat{\Phi}_{t,\mathrm{x}}, \hat{\Phi}^{\dagger}_{t,\mathrm{x}'}] = \delta(\mathrm{x}-\mathrm{x}')$



- Matter wave **mean field** $[x = (x_0, \mathrm{x}) = (t, \mathrm{x})]$

$$\phi_{\boldsymbol{x}} = \langle \hat{\Phi}_{\boldsymbol{x}} \rangle, \qquad |\phi_{\boldsymbol{x}}|^2 = n_c(x) = \text{condensate density},$$

- Density of **non-condensed atoms** $(\hat{\Phi} = \phi + \widetilde{\Phi},\ \phi = \langle \hat{\Phi} \rangle)$

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- Total one-body **density matrix**

$$G_{11}(x,y) = \langle \widetilde{\Phi}^{\dagger}_{\boldsymbol{x}} \widetilde{\Phi}_{\boldsymbol{y}} \rangle \Rightarrow \text{spatial Fourier transform: momentum distribution } n(\mathrm{p}, t)$$

$\Rightarrow$ 1^{st}-order phase coherence

- **Anomalous one-body density** matrix

$$G_{12}(x,y) = \langle \widetilde{\Phi}_{\boldsymbol{x}} \widetilde{\Phi}_{\boldsymbol{y}} \rangle \Rightarrow \text{e.g., number of Bose-condensed bound pairs (molecules)}$$

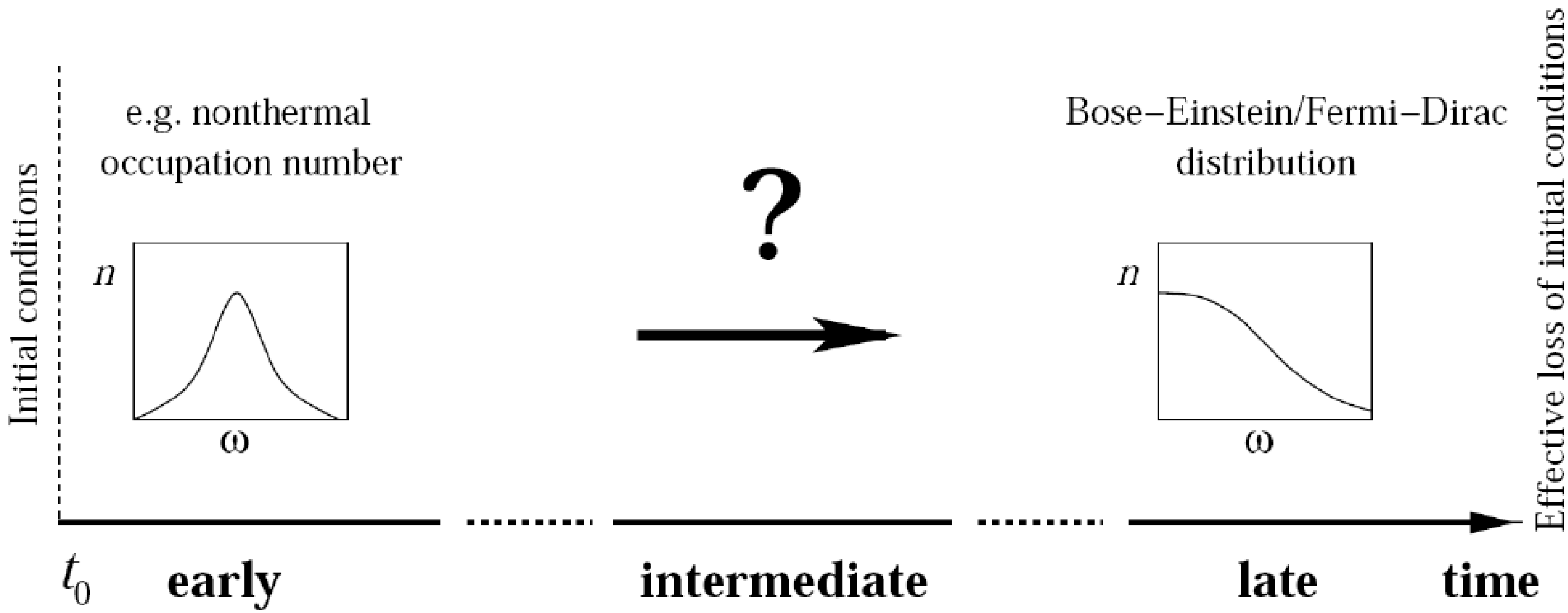
Bose-Einstein condensate in lattice potential

Remember the Bose-Hubbard Hamiltonian: (nearest neighbour hopping):

$$\hat{H}_{BH} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \hat{b}_{i+1}^\dagger \hat{b}_i \right) + \sum_i \epsilon_i \hat{b}_i^\dagger \hat{b}_i + \frac{U}{2} \sum_i \hat{b}_i^\dagger \hat{b}_i^\dagger \hat{b}_i \hat{b}_i$$

Non-equilibrium evolution in Quantum Field Theory

Far-from-equilibrium dynamics



Thermal equilibrium: Loss of information about prior evolution.

- Only a few conserved quantities persist.

Quantum Field Theory

Fields generally allow an effective description,

e.g. through

$$\rho(\mathbf{x},t),\ \mathbf{v}(\mathbf{x},t)$$

instead of coordinates/velocities of many particles;

similarly:

$$\phi_{\mathrm{i}}(x) = \langle \Phi_{\mathrm{i}}(x) \rangle \qquad \text{(mean field)}$$

$$G_{\mathrm{ij}}(x,y) = \langle \Phi_{\mathrm{i}}(x)\Phi_{\mathrm{j}}(y) \rangle \qquad \text{(density matrix)}$$

$$x = (\mathbf{x},t)$$

Observables

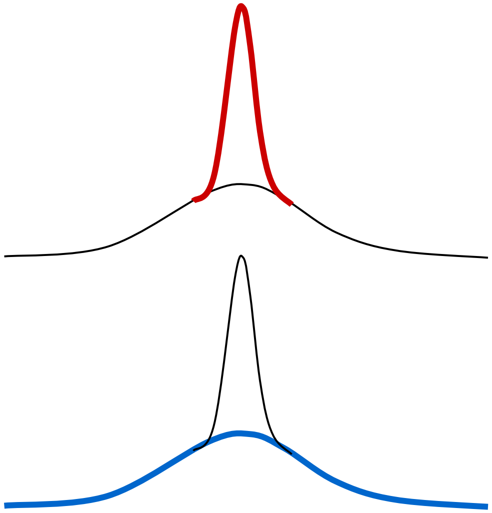
For **bosons**: $[\hat{\Phi}_{t,\mathrm{x}}, \hat{\Phi}^{\dagger}_{t,\mathrm{x}'}] = \delta(\mathrm{x}-\mathrm{x}')$

- Matter wave **mean field** $[x = (x_0, \mathrm{x}) = (t, \mathrm{x})]$

$$\phi_x = \langle \hat{\Phi}_x \rangle, \qquad |\phi_x|^2 = n_c(x) = \text{condensate density},$$

- Density of **non-condensed atoms** ($\hat{\Phi} = \phi + \widetilde{\Phi}$, $\phi = \langle \hat{\Phi} \rangle$)

$$\langle \widetilde{\Phi}^{\dagger}_x \widetilde{\Phi}_x \rangle = n_{\mathrm{nc}}(x) \equiv n(x) - n_c(x),$$



- Total one-body **density matrix**

$$G_{11}(x,y) = \langle \widetilde{\Phi}^{\dagger}_x \widetilde{\Phi}_y \rangle \Rightarrow \text{spatial Fourier transform: momentum distribution } n(\mathrm{p}, t)$$

$$\Rightarrow 1^{\mathrm{st}}\text{-order phase coherence}$$

- **Anomalous one-body density** matrix

$$G_{12}(x,y) = \langle \widetilde{\Phi}_x \widetilde{\Phi}_y \rangle \Rightarrow \text{e.g., number of Bose-condensed bound pairs (molecules)}$$

Path Integral Approach

Classical dynamics of φ from $\delta S[\varphi] = 0$.

Path Integral Approach

QM transition amplitude:

$$\langle t_{\text{fin}} | t_{\text{ini}} \rangle = \int \mathcal{D}\varphi \; e^{iS[\varphi]/\hbar}$$

$$\mathcal{D}\varphi = \prod_{x=x_{\text{ini}}}^{x_{\text{fin}}} \mathrm{d}\varphi(x)$$

Classical dynamics of φ from $\delta S[\varphi] = 0$.

Path Integral Approach

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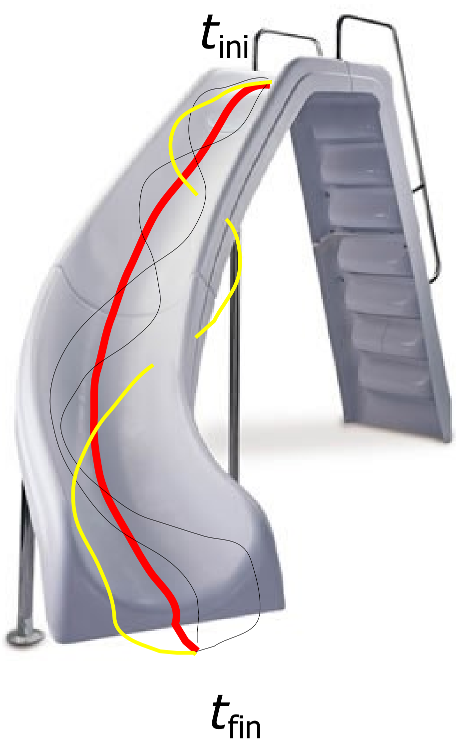
$$\mathcal{D}\varphi = \prod_{x=x_{ini}}^{x_{fin}} \mathrm{d}\varphi(x)$$

Classical dynamics of φ from $\delta S[\varphi] = 0.$

Path Integral Approach...
...in QFT:



Generating functional:

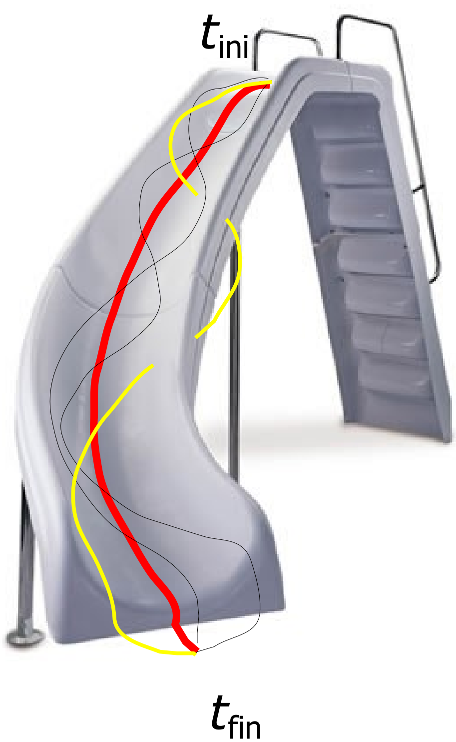
$$Z[J] = \int \mathcal{D}\varphi \; e^{iS[\varphi]/\hbar - i\int J\varphi}$$

$$\phi = i\left.\frac{\delta \ln Z}{\delta J}\right|_{J=0} = Z^{-1}\int \mathcal{D}\varphi \; \varphi \, e^{iS[\varphi]/\hbar}$$

Classical dynamics of φ from $\delta S[\varphi] = 0$.

Effective Action

Generating functional:



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$$\phi = i \left.\frac{\delta \ln Z}{\delta J}\right|_{J=0} = Z^{-1} \int \mathcal{D}\varphi \; \varphi \, e^{iS[\varphi]/\hbar}$$

Classical dynamics of φ from $\delta S[\varphi] = 0.$

Quantum dynamics of ϕ from variation of an **effective action,** $\delta\Gamma[\phi]/\delta\phi = -J$:

$$Z[J] = \int \mathcal{D}\varphi \; \delta[\varphi - \phi] e^{i\Gamma[\varphi]/\hbar - i\int J\varphi}$$

$$\Gamma[\phi] = -i\hbar \ln Z[J] - \int J\phi$$

Legendre transform

Effective Action

Generating functional:

$$Z[J] = \int \mathcal{D}\varphi \, e^{iS[\varphi]/\hbar - i\int J\varphi}$$

$$\phi = i \frac{\delta \ln Z}{\delta J}\Big|_{J=0} = Z^{-1} \int \mathcal{D}\varphi \, \varphi \, e^{iS[\varphi]/\hbar}$$

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$$Z[J] = \int \mathcal{D}\varphi \, \delta[\varphi - \phi] e^{i\Gamma[\varphi]/\hbar - i\int J\varphi}$$

$$\Gamma[\phi] = -i\hbar \ln Z[J] - \int J\phi$$

$$= S[\phi] - i/2 \, \mathrm{Tr} \ln G + \ldots$$

Gaussian integration

Initial value problems...

...require the Schwinger-Keldysh closed time path (CTP):

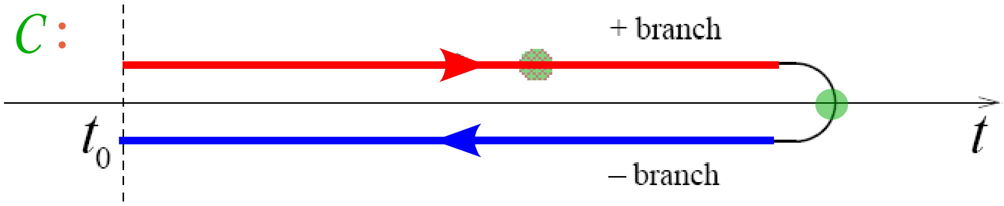
$$\langle t\,|O|\,t\rangle = \langle t_0\,|U^\dagger(t)O\,U(t)|\,t_0\rangle$$
$$= \mathrm{Tr}[\rho(t_0)\,U^\dagger(t)O\,U(t)\,]$$

time ordering along CTP $\mathcal{C}$

e.g.

$$G_{ab}(x,y) = \mathrm{Tr}[\rho(t_0)\,\mathcal{T}_{\mathcal{C}}U^\dagger(t)\Phi_a(x)\Phi_b(y)U(t)\,] - \text{disc.}$$

CTP $\mathcal{C}$:



Conserved quantities

- **Energy** conservation à la Emmy Noether according to

$$\delta\Gamma[\phi, G] = \int \left\{ \frac{\delta\Gamma[\phi, G]}{\delta\phi}\delta\phi + \frac{\delta\Gamma[\phi, G]}{\delta G}\delta G \right\} = 0,$$

  with time translations encoded into $\delta\phi$, δG.

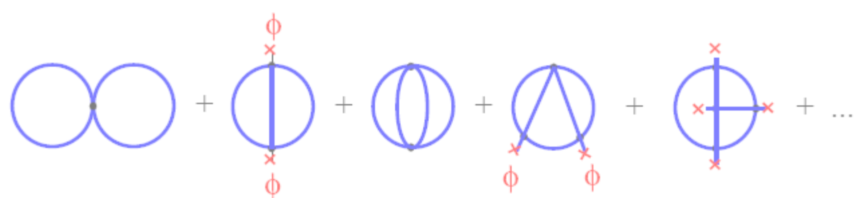
- **Number** conservation at any diagrammatic truncation of $\Gamma[\phi, G]$ as a consequence of $O(\mathcal{N})$ invariance.

2PI (2-particle irreducible) **Effective Action**:

$$\Gamma[\phi, G] = S[\phi] + \tfrac{i}{2}\mathrm{Tr}\ln(G^{-1} + G_0^{-1}(\phi)G) + \Gamma_2[\phi, G],$$

with $G_{0,ij}^{-1} = \delta^2 S[\phi]/(\delta\phi_i^\dagger \delta\phi_j)$

$\Gamma_2[\phi,G] =$



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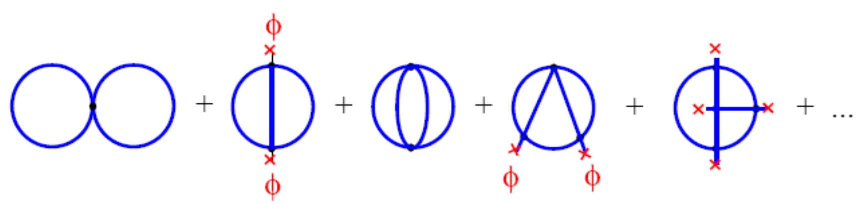


Dynamic Equations

From the stationarity condition $\frac{\delta S[\phi]}{\delta \phi_i} = 0$, one obtains, in leading order, the **Gross-Pitaevskii equation**:

$$\Big[-i\sigma_2 \partial_{x_0} - \frac{g}{2}\mathrm{tr}(\phi_x \phi_x) \Big] \phi_x - H_{1\mathrm{B}}(x)\, \phi_x = 0,$$

with

$$H_{1\mathrm{B}}(x) = -\frac{\Delta}{2m} + V_{\mathrm{ext}}(x),$$

$$\Gamma[\phi, G] = S[\phi]$$

Dynamic Equations

From the stationarity conditions $\frac{\delta\Gamma^{\text{HFB}}[\phi,G]}{\delta\phi_i}=0\,,\ \frac{\delta\Gamma^{\text{HFB}}[\phi,G]}{\delta G_{ij}}=0$ one obtains the **HFB dynamic equations**:

$$\Big[-i\sigma_2\partial_{x_0}-\frac{g}{2}\mathrm{tr}(\phi_x\phi_x)\Big]\phi_x-\int_y M_{xy}[0,F]\,\phi_y=0,$$

$$\Big[-i\sigma_2\delta_{xz}\partial_{z_0}-\int_z M_{xz}[\phi,F]\Big]\begin{pmatrix}F_{zy}\\ \rho_{zy}\end{pmatrix}=0$$

Note:

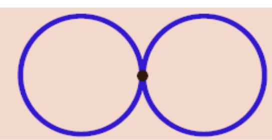
$$\langle\mathcal{T}\tilde{\Phi}_x\tilde{\Phi}_y\rangle=G_{xy}=F_{xy}-\tfrac{i}{2}\mathrm{sign}_{\mathcal{C}}(x_0-y_0)\,\rho_{xy}$$

$$\Longrightarrow\quad F_{xy}=\langle\{\tilde{\Phi}_x,\tilde{\Phi}_y\}\rangle,\quad \rho_{xy}=\langle[\tilde{\Phi}_x,\tilde{\Phi}_y]\rangle.$$

with

$$(M_{xy}[\phi,F])_{ab}=\delta_{xy}\delta_{ab}\Big[H_{1\mathrm{B}}(x)+\frac{g}{2}\mathrm{tr}\Big(\phi_x\phi_x+F_{xx}\Big)\Big]+2\Big(\phi_x\phi_x+F_{xx}\Big)_{ab},$$

$$H_{1\mathrm{B}}(x)=-\frac{\Delta}{2m}+V_{\mathrm{ext}}(x),$$

$$\Gamma[\phi,G]=S[\phi]+\frac{i}{2}\mathrm{Tr}(\ln G^{-1}+G_0^{-1}(\phi)G)+$$ 

Dynamic Equations (⊇ Kadanoff-Baym)

From the stationarity conditions $\dfrac{\delta\Gamma[\phi,G]}{\delta\phi_x} = 0\,,\quad \dfrac{\delta\Gamma[\phi,G]}{\delta G(x,y)} = 0$ one obtains the **full dynamic equations**:

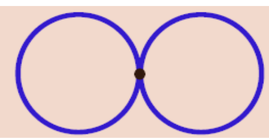
$$\Big[-i\sigma_2\partial_{x_0} - \frac{g}{2}\mathrm{tr}(\phi_x\phi_x)\Big]\phi_x - \int_y M_{xy}[0,F]\,\phi_y = \int_0^{x_0} dy\,\overline{\Sigma}^{\rho}_{xy}[0,G]\,\phi_y,$$

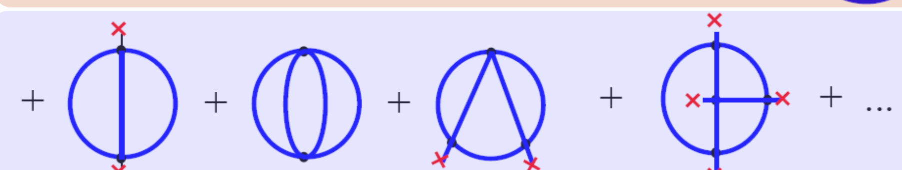
$$\Big[-i\sigma_2\delta_{xz}\partial_{z_0} - \int_z M_{xz}[\phi,F]\Big]\begin{pmatrix}F_{zy}\\ \rho_{zy}\end{pmatrix} = \begin{pmatrix}\int_0^{x_0} dz\,\overline{\Sigma}^{\rho}_{xz}[\phi,G] & -\int_0^{y_0} dz\,\overline{\Sigma}^{F}_{xz}[\phi,G]\\ 0 & \int_{y_0}^{x_0} dz\,\overline{\Sigma}^{\rho}_{xz}[\phi,G]\end{pmatrix}\begin{pmatrix}F_{zy}\\ \rho_{zy}\end{pmatrix}$$

with

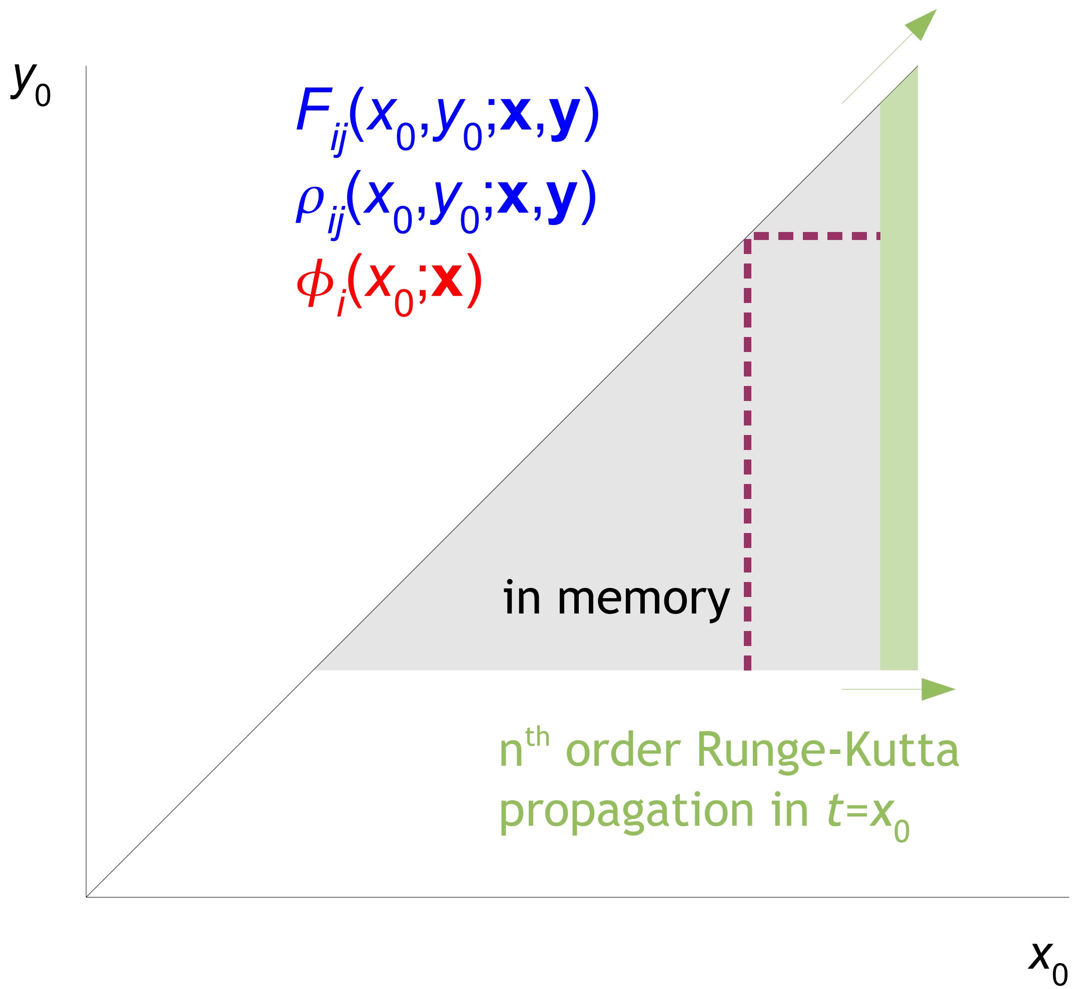
$$(M_{xy}[\phi,F])_{ab} = \delta_{xy}\delta_{ab}\Big[H_{1\mathrm{B}}(x) + \frac{g}{2}\mathrm{tr}\Big(\phi_x\phi_x + F_{xx}\Big)\Big] + 2\Big(\phi_x\phi_x + F_{xx}\Big)_{ab},$$

$$H_{1\mathrm{B}}(x) = -\frac{\Delta}{2m} + V_{\mathrm{ext}}(x),$$

$$\Gamma[\phi,G] = S[\phi] + \frac{i}{2}\mathrm{Tr}(\ln G^{-1} + G_0^{-1}(\phi)G) +$$ 



Numerical Demand



e.g.

- 16 x 16 spatial grid
- 1000 x 1000 temporal grid
- $\mathcal{N}$ x $\mathcal{N}$ index grid, $\mathcal{N}=2$

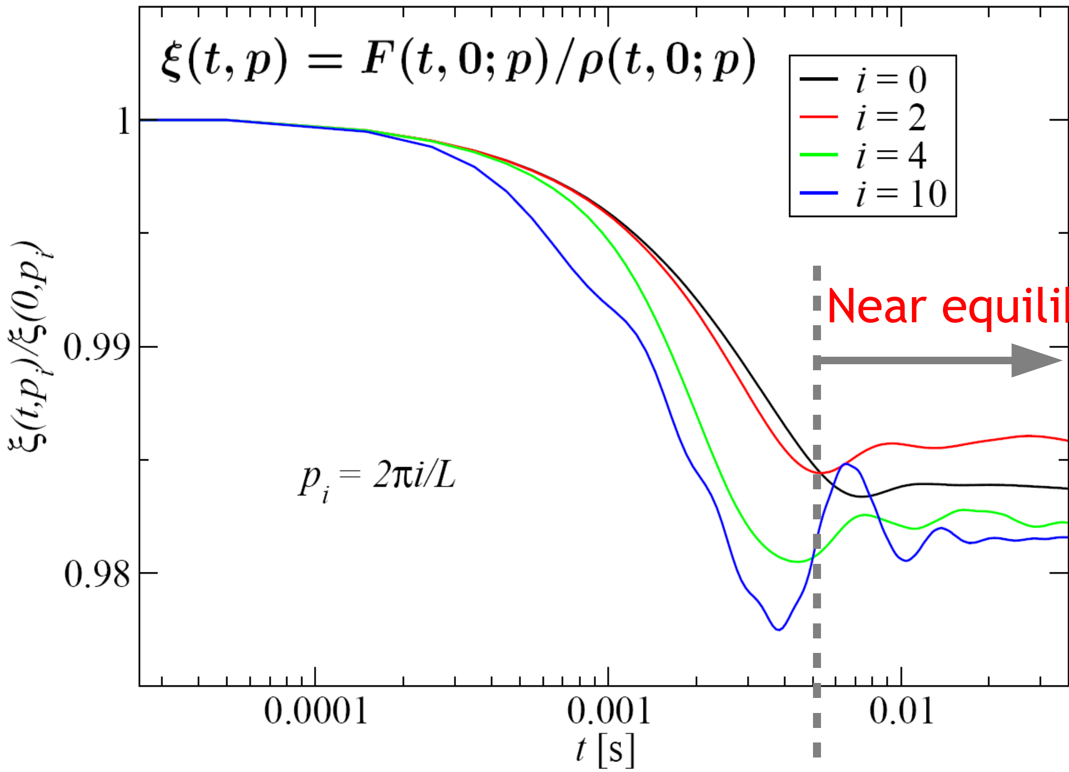
⇒ ~16 GB RAM

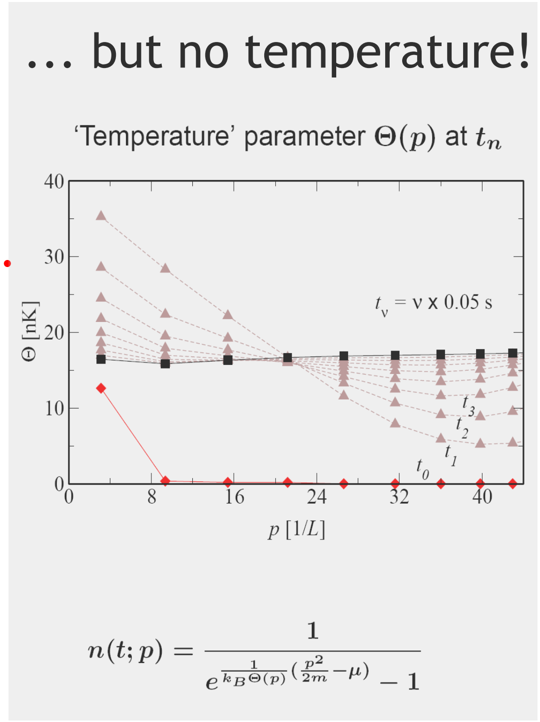
@ *ITP*:
10+ AMD DualCore
2 GB RAM/node

@ *IWR Heidelberg*:
8 x SPARC Ultra III 900 MHz
64 GB shared RAM

Onset of near-equilibrium evolution

Time evolution of temporal correlations





(Fluctuation-Dissipation rel.: $F^{(\mathrm{eq})}_{\omega\mathbf{p}} = -i\left(n(\omega,T) + \frac{1}{2}\right)\rho^{(\mathrm{eq})}_{\omega\mathbf{p}}$)

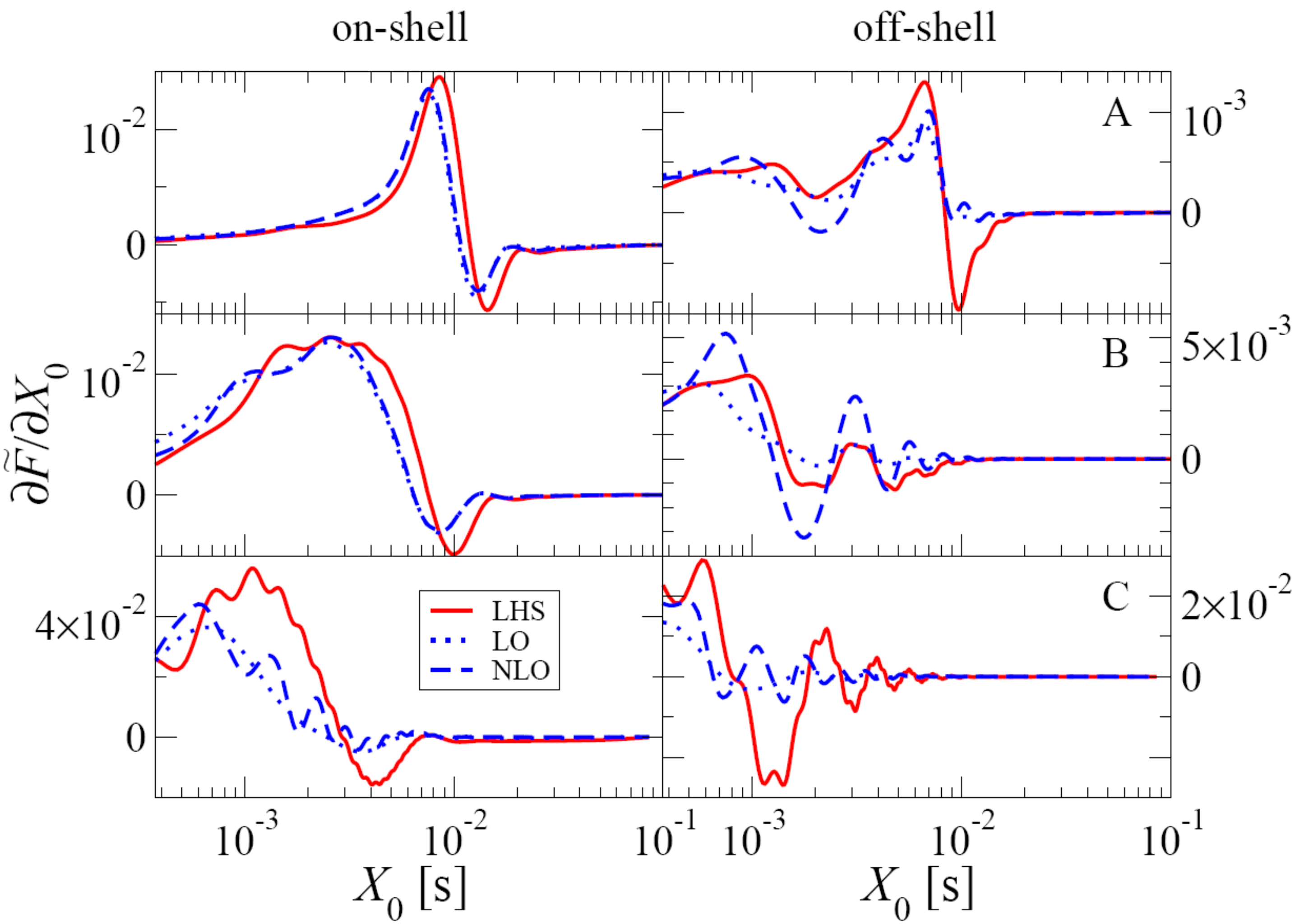
[J. Berges & TG, PRA **76** (07), cf. also Berges & Müller (03)]

Comparison with kinetic theory

- Go to Wigner representation (central & relative times)
- Send initial time to minus infinity
- Fourier transform w.r.t. relative time
- Gradient expansion (Markov approx. & corrections)

[in our context cf.: A.M. Rey *et al.*, PRA '05 - generally: Baym & Kadanoff (62), specifically: J. Berges & M.M. Müller (03), J. Berges, S. Borsányi & C. Wetterich (04), M. Lindner & M.M. Müller (06,07)]

Comparison with kinetic theory

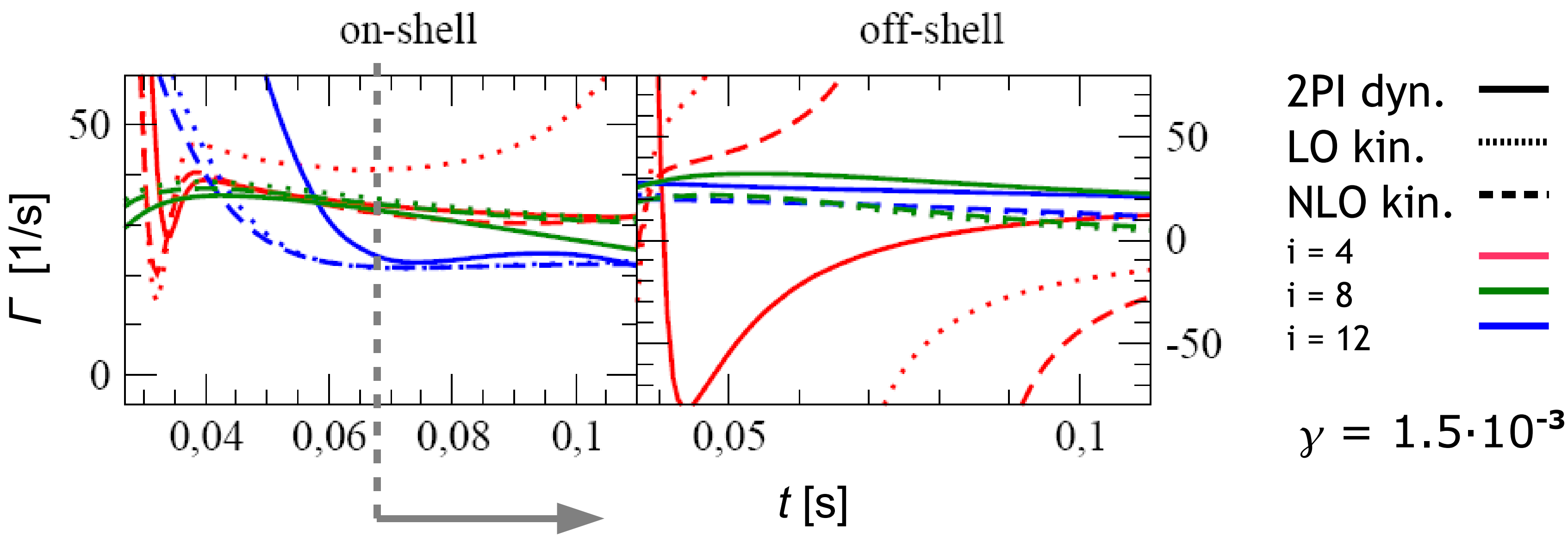


$\gamma = 1.5\cdot10^{-3}$

$n_p = 4$

[A. Branschädel & TG, cond-mat/0801.4466]

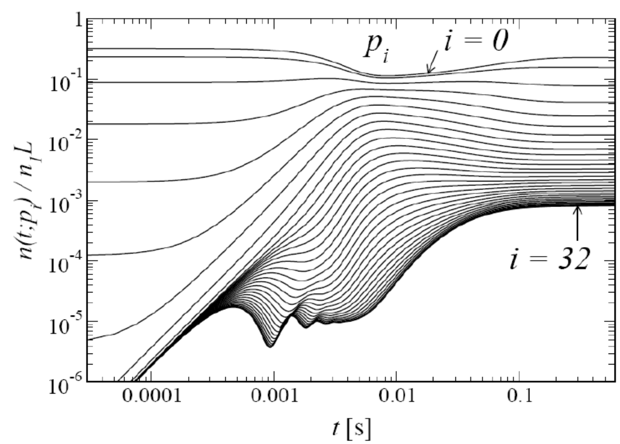
Comparison with kinetic theory



Damping rate

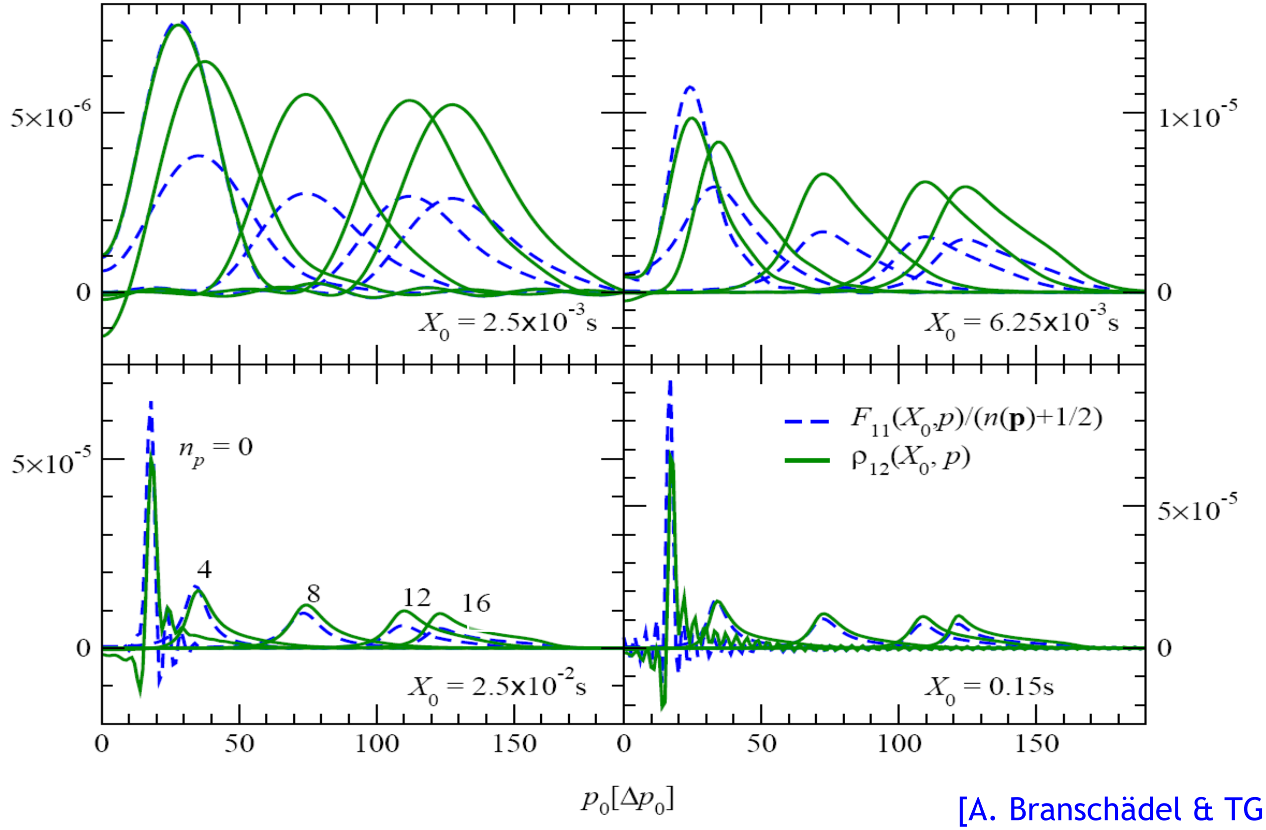
Time evol. of densities *n*:

$\Gamma = n''/n'$



[A. Branschädel & TG, cond-mat/0801.4466]

Fluctuation-dissipation relation



[A. Branschädel & TG, cond-mat/0801.4466]

F = statistical corr. fct. – ***ρ*** = spectral fct.

→ fulfill fluctuation-dissipation relation $F^{(\mathrm{eq})}_{\omega\mathbf{p}} = -i\left(n(\omega,T)+\frac{1}{2}\right)\rho^{(\mathrm{eq})}_{\omega\mathbf{p}}$

Extensive work on

Kinetic Theories for Ultracold Quantum Gases

e.g.

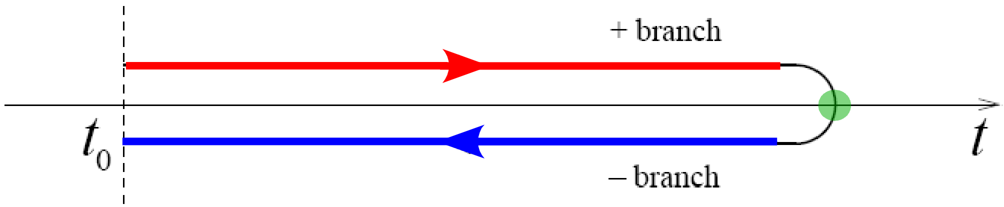
- Semiclassical hydrodynamics
  Griffin, Nikuni, Zaremba, ...
- Quantum-Boltzmann equations, linear response theory:
  Burnett, Giorgini, Proukakis, Rusch, Stoof, ...
- Generalized master equations, quantum Boltzmann equations, non-Markovian extensions:
  Bhongale, Cooper, Holland, Kokkelmans, Wachter, Walser, Williams, ...
- Quantum stochastic master equations, quantum Boltzmann equations, classical simulations:
  Ballagh, Burnett, Davis, Gardiner, Jaksch, Zoller, ...
- Fokker-Planck equation, Langevin field equation, quantum Boltzmann equations:
  Al Khawaja, Bijlsma, Proukakis, Stoof, ...
- Greens-function approaches:
  Boyanovsky, Griffin, Imamovic-Tomasovic, Clark, Rey, Hu, ...

+ many more on (finite-T) stationary properties
(Burnett, Castin, Clark, Fedichev, Griffin, Hutchinson, Morgan, Shlyapnikov, Stoof, Stringari, Williams, Zaremba, ...)

Initial value problem

$$\langle t | O | t \rangle = \langle t_0 | U^{\dagger}(t) O \, U(t) | t_0 \rangle = Z^{-1} \int \mathcal{D}\varphi \mathcal{D}\varphi \; O \; e^{i(S[\varphi]-S[\varphi])/\hbar}$$

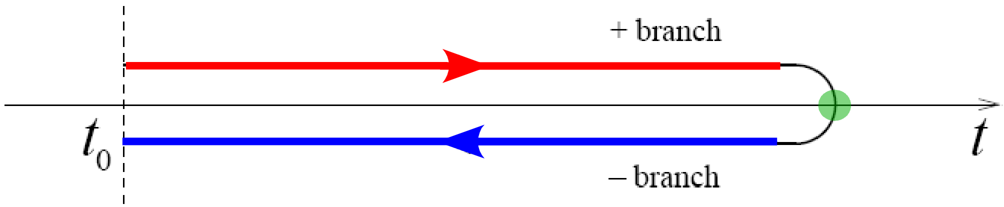
Schwinger-Keldysh closed time path:



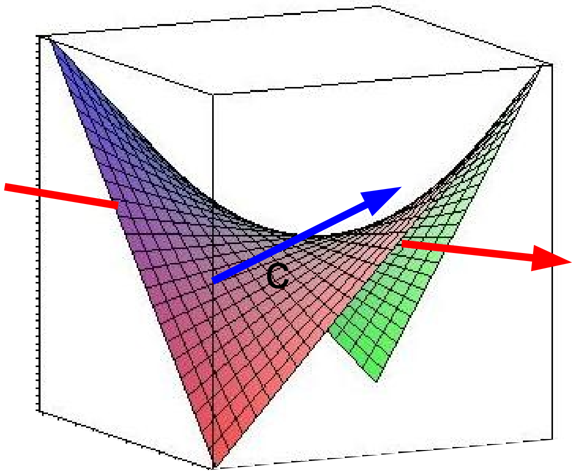
Initial value problem

$$\langle t\,|O|\,t\rangle = \langle t_0\,|U^\dagger(t) O\, U(t)|\,t_0\rangle \;=\; Z^{-1}\int \mathcal{D}\varphi\,\mathcal{D}\varphi\; O\; e^{i(S[\varphi]-S[\varphi])/\hbar}$$

Schwinger-Keldysh closed time path:



Quadratic action (QM Harm. Osc.): $S[\varphi]\sim\int dt\{(\partial_t\varphi)^2 - \varphi^2\}$:



$$\varphi^2 - \varphi^2$$
$$= (\varphi - \varphi)(\varphi + \varphi)$$
$$=: \tilde{\varphi}\varphi$$

Classical Path Integral

Consider QM Harm. Osc.:

$$S[\varphi] - S[\varphi] \sim -\int dt \{ \varphi(\partial_t^2 + \omega^2)\varphi^2 - \varphi(\partial_t^2 - \omega^2)\varphi^2 \}$$

$$\sim -\int dt\ \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi$$

[J. Berges, TG, PRA (07). ClPI goes back to Hopf (50), cf. also Phythian (75), DeDominicis et al. (76), Janssen et al. (76), Chou et al. (85), Blagoev et al. (01)]

Classical Path Integral

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$$S[\varphi] - S[\varphi] \sim -\int \mathrm{d}t \{ \varphi(\partial_t^2 + \omega^2)\varphi^2 - \varphi(\partial_t^2 - \omega^2)\varphi^2 \}$$

$$\sim -\int \mathrm{d}t \; \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi$$

Path integral evaluates to classical solution:

$$\int \mathcal{D}\varphi \mathcal{D}\varphi \; O \; \rho[\varphi_0, \varphi_0] \; e^{i(S[\varphi]-S[\varphi])/\hbar}$$

$$\sim \int \mathcal{D}\tilde{\varphi} \mathcal{D}\varphi \; O \; \rho[\tilde{\varphi}_0, \varphi_0] \; \exp[-\int \mathrm{d}t \; \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi/\hbar]$$

[J. Berges, TG, PRA (07). ClPI goes back to Hopf (50), cf. also Phythian (75), DeDominicis et al. (76), Janssen et al. (76), Chou et al. (85), Blagoev et al. (01)]

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$$\sim \int \mathcal{D}\varphi\ O\ \mathcal{W}[\pi_0,\varphi_0]\ \delta[(\partial_t^2 + \omega^2)\varphi]$$

[J. Berges, TG, PRA (07). ClPI goes back to Hopf (50), cf. also Phythian (75), DeDominicis et al. (76), Janssen et al. (76), Chou et al. (85), Blagoev et al. (01)]

Classical Path Integral

Consider QM Harm. Osc.:

$$S[\varphi] - S[\varphi] \sim -\int \mathrm{d}t\{\varphi(\partial_t^2 + \omega^2)\varphi^2 - \varphi(\partial_t^2 - \omega^2)\varphi^2\}$$

$$\sim -\int \mathrm{d}t\ \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi$$

Not with interactions!
$g(\varphi^4 - \varphi^4) = g(\tilde{\varphi}\varphi^3 + \tilde{\varphi}^3\varphi)$

Path integral evaluates to classical solution:

$$\int \mathcal{D}\varphi\mathcal{D}\varphi\ O\ \rho[\varphi_0,\varphi_0]\ e^{i(S[\varphi]-S[\varphi])/\hbar}$$

$$\sim \int \mathcal{D}\tilde{\varphi}\mathcal{D}\varphi\ O\ \rho[\tilde{\varphi}_0,\varphi_0]\ \exp[-\int \mathrm{d}t\ \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi/\hbar]$$

$$\sim \int \mathcal{D}\varphi\ O\ \mathcal{W}[\pi_0,\varphi_0]\ \delta[(\partial_t^2 + \omega^2)\varphi]$$

[J. Berges, TG, PRA (07). ClPI goes back to Hopf (50), cf. also Phythian (75), DeDominicis et al. (76), Janssen et al. (76), Chou et al. (85), Blagoev et al. (01)]

RG approach

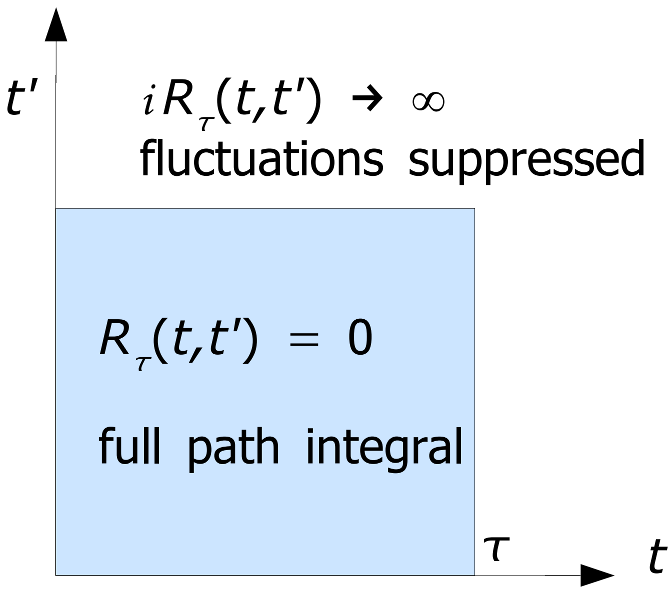
to far-from-equilibrium dynamics

[TG & J.M. Pawlowski, cond-mat/0710.4627]

- Regularise generating functional

$$Z_\tau = \exp\left\{ i \int_{x,y;\mathcal{C}} \frac{\delta}{\delta J_a(x)} R_{\tau,ab}(x,y) \frac{\delta}{\delta J_b(y)} \right\} Z$$

$$Z[J;\rho_0] = \int \mathcal{D}\varphi \, \rho_0 \exp\left\{ iS[\varphi] + i \int_{x,\mathcal{C}} J_a \varphi_a \right\},$$



- Functional RG equation [C. Wetterich (92)]

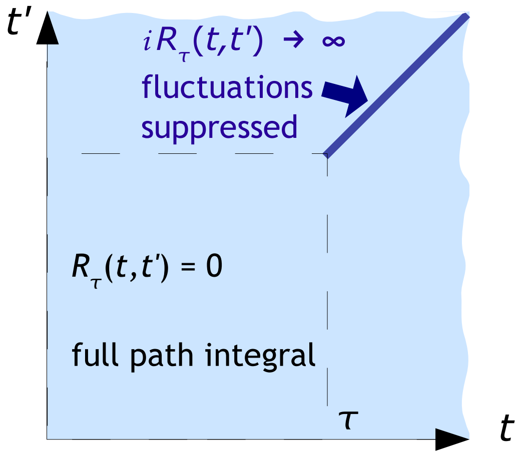
$$\partial_\tau \Gamma_\tau = \frac{i}{2} \int_{\mathcal{C}} \left[\frac{1}{\Gamma_\tau^{(2)} + R_\tau} \right]_{ab} \partial_\tau R_{\tau,ab}$$

$$\Gamma_\tau[\phi, R_\tau] = W_\tau[J, \rho_0] - \int_{\mathcal{C}} J_a \phi_a - \frac{1}{2} \int_{\mathcal{C}} \phi_a R_{\tau,ab} \phi_b$$

RG approach

to far-from-equilibrium dynamics

[TG & J.M. Pawlowski, cond-mat/0710.4627]



- Functional RG equation [C. Wetterich (92)]

$$\partial_\tau \Gamma_\tau = \frac{i}{2}\int_{\mathcal{C}} \left[\frac{1}{\Gamma^{(2)}_\tau + R_\tau}\right]_{ab} \partial_\tau R_{\tau,ab}$$

$$\Gamma_\tau[\phi, R_\tau] = W_\tau[J, \rho_0] - \int_{\mathcal{C}} J_a \phi_a - \frac{1}{2}\int_{\mathcal{C}} \phi_a R_{\tau,ab} \phi_b$$

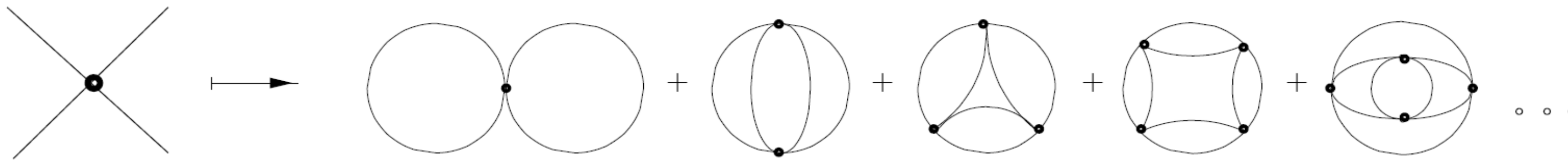
- Compare to 1-loop effective action (with additional source R):

$$\Gamma[\phi, R_\tau] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln\left(S^{(2)}[\phi] + R_\tau\right) + \ldots$$

t

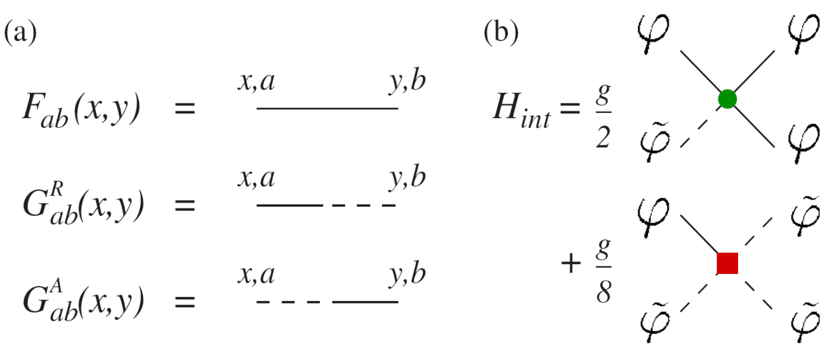
Classical vs. Quantum diagrams

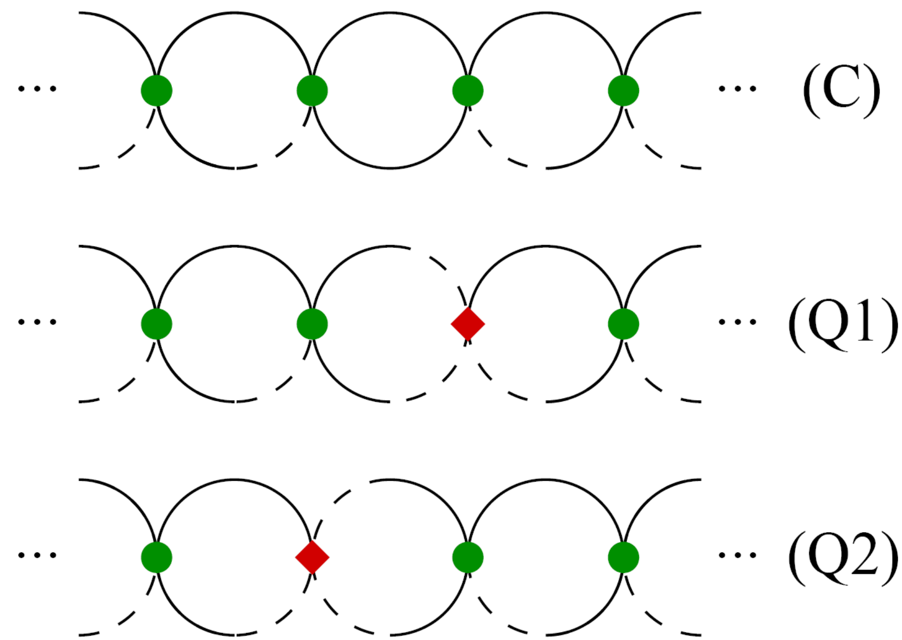
(2PI $1/\mathcal{N}$)



Interactions:

$g\,(\varphi^4 - \varphi^4) = g\,(\tilde{\varphi}\varphi^3 + \tilde{\varphi}^3\varphi)$





Classicality condition

[J. Berges, TG, PRA **76**, 033604 (07)]

Under the rescaling
$$\varphi_a(x) \longrightarrow \varphi'_a(x) = \sqrt{g}\,\varphi_a(x),$$
$$\tilde{\varphi}_a(x) \longrightarrow \tilde{\varphi}'_a(x) = (1/\sqrt{g})\,\tilde{\varphi}_a(x)$$

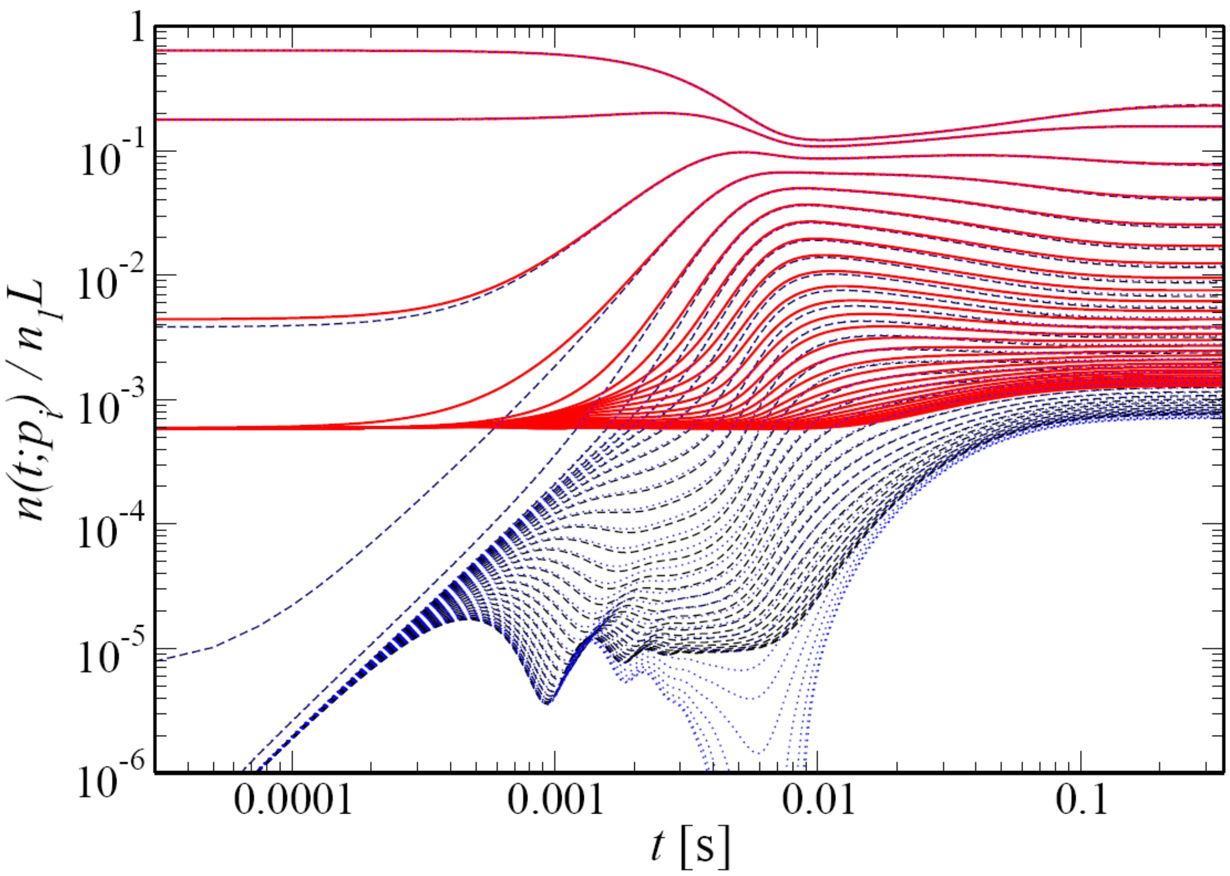
the interaction part becomes $\tilde{\varphi}'\varphi'^3 + g\,\tilde{\varphi}'^3\varphi'$

Classicality condition:

$$|F'_{ab}(x,y)F'_{cd}(z,w)| \gg \frac{3}{4}g^2\,|\rho_{ab}(x,y)\rho_{cd}(z,w)|$$

Classical vs. quantum evolution

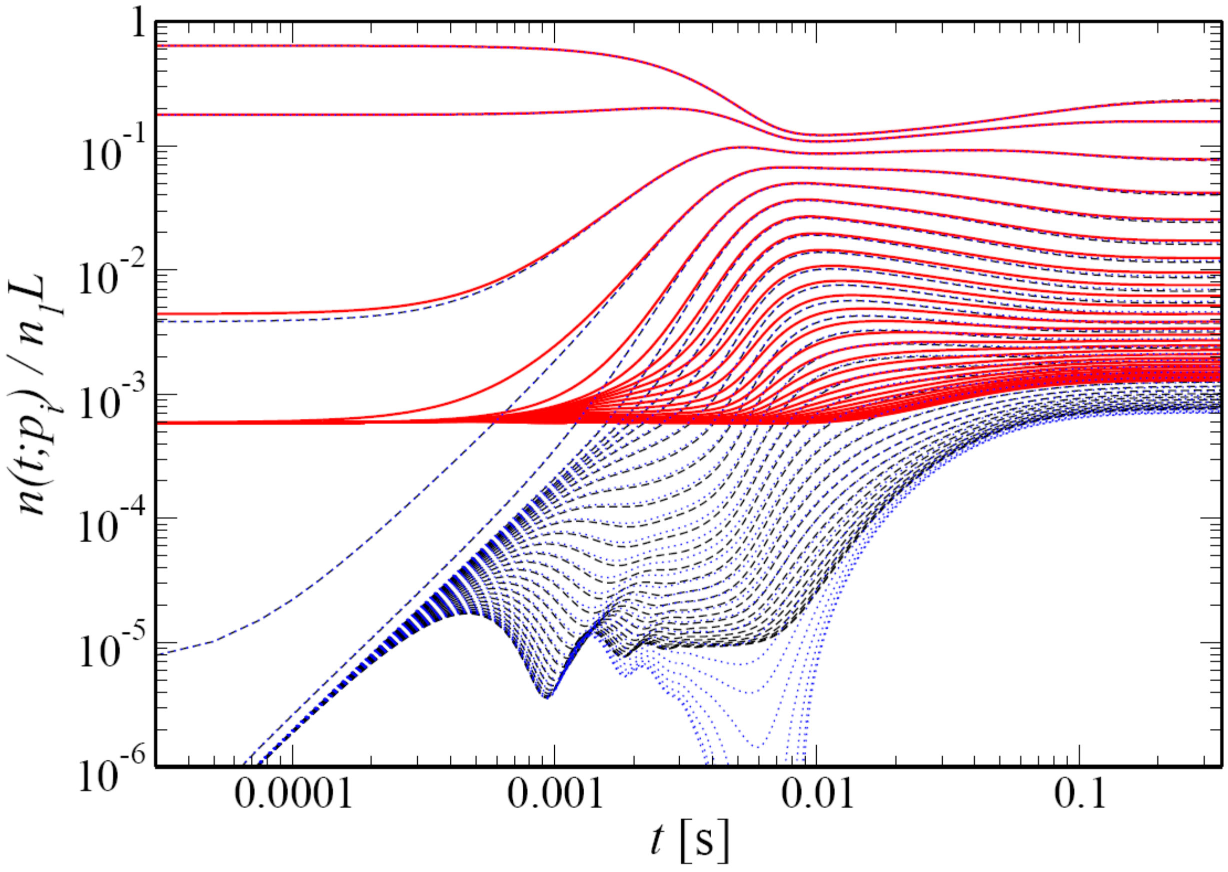
Occupation numbers according to quantum dynamic equations...



...vs. classical evolution for ‘quantum’ initial conditions.

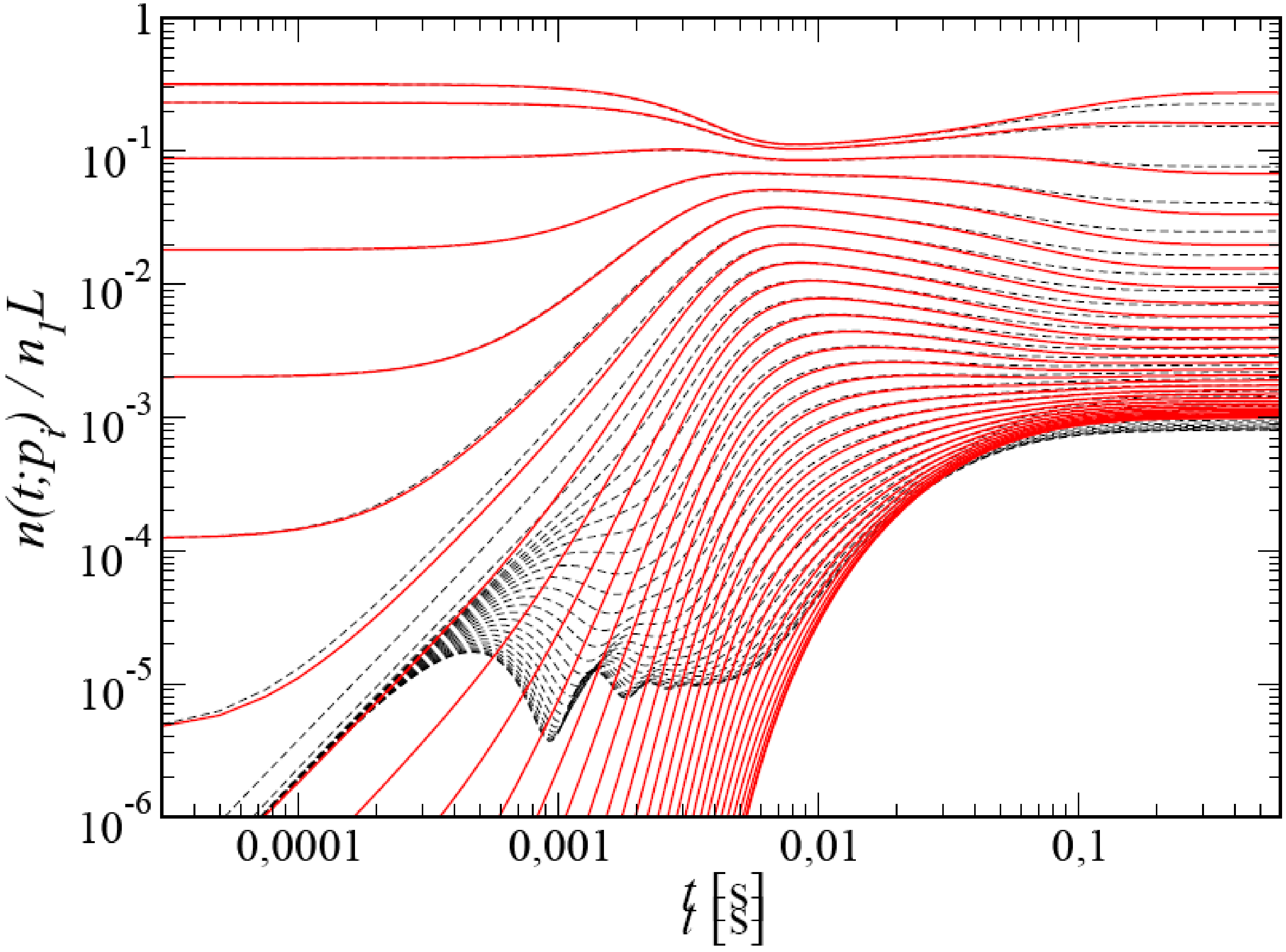
Classical vs. quantum evolution

Occupation numbers according to quantum dynamic equations...



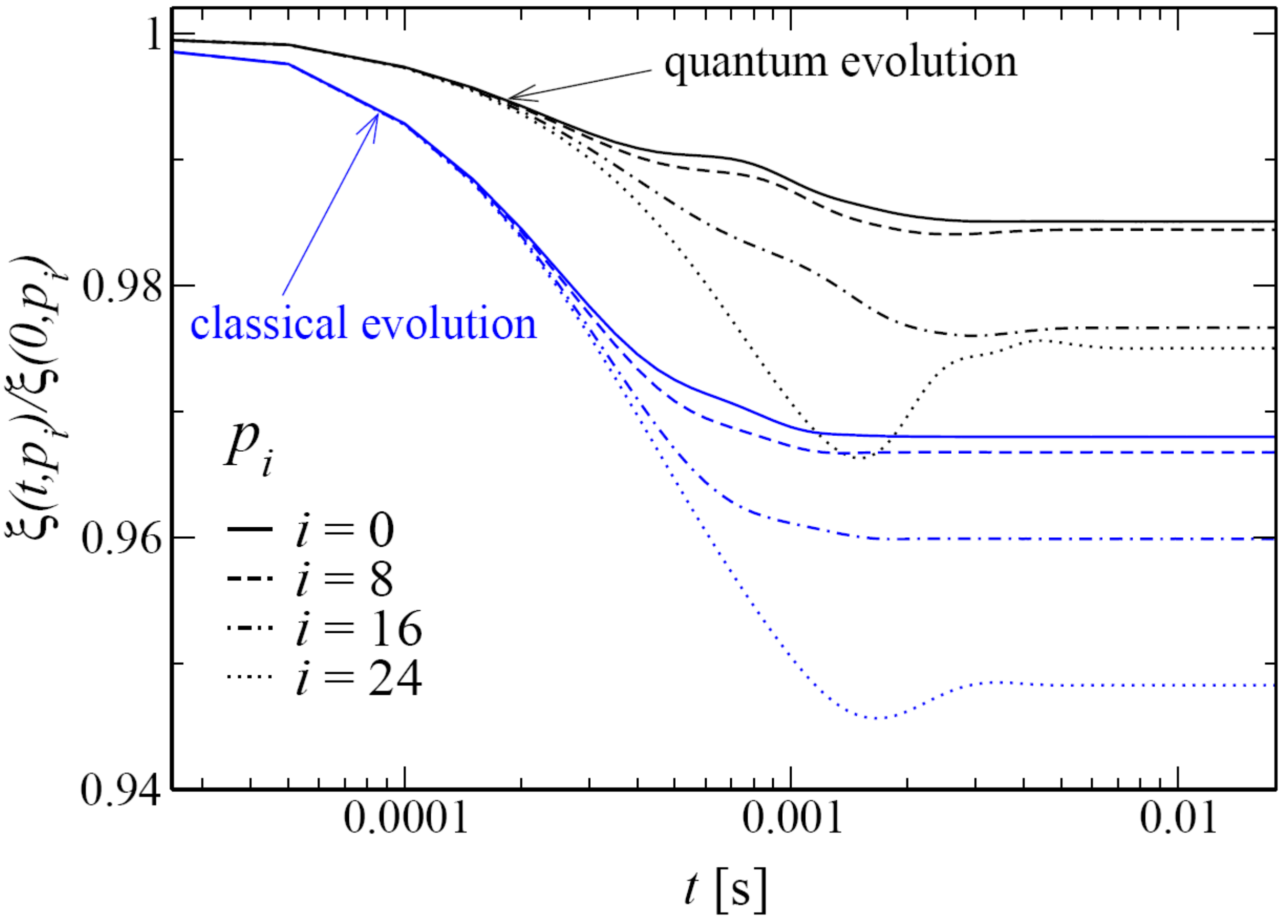
Compare to eq. **fluctuation-dissipation rel.**: $\overline{F^2(t,t';\mathrm{p})} \simeq \left(n(t,\mathrm{p})+\frac{1}{2}\right)^2 \overline{\rho^2(t,t';\mathrm{p})}$.

Far-from-equilibrium evolution



Strong coupling

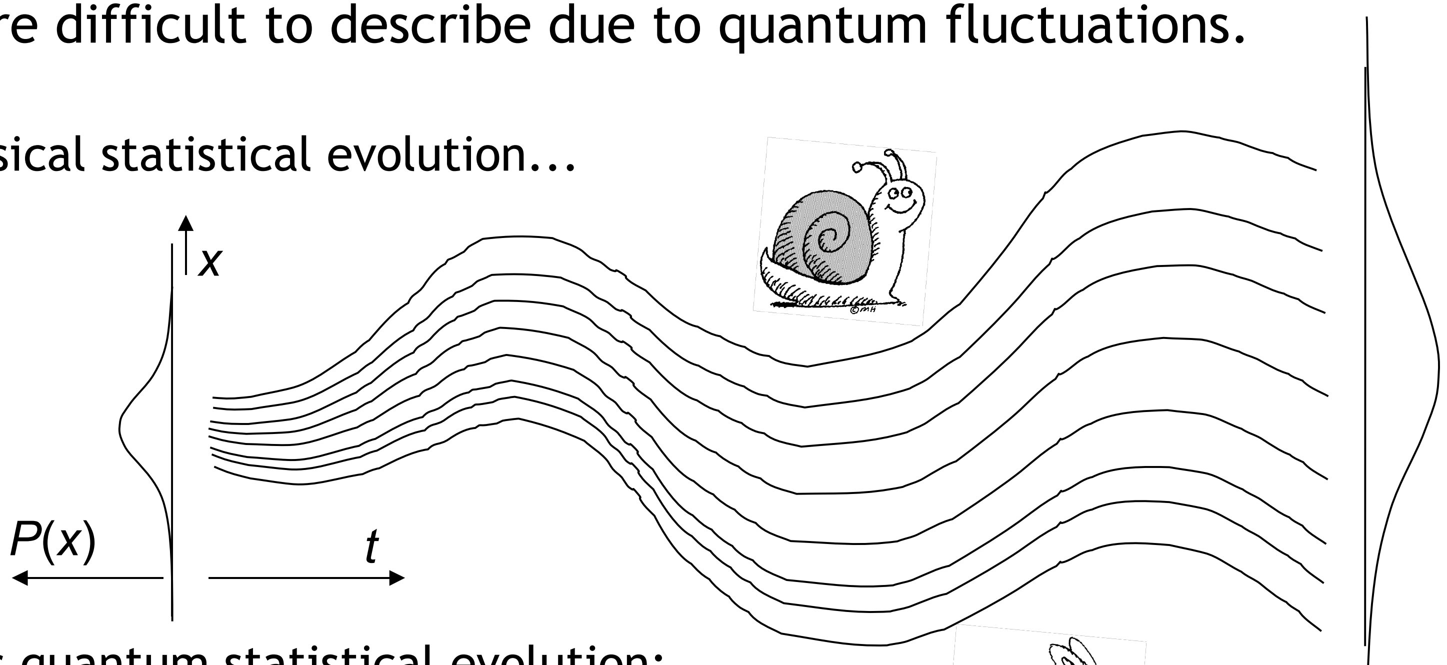
Time evolution of temporal correlations $\xi(t,p) = F(t,0;p)/\rho(t,0;p)$:



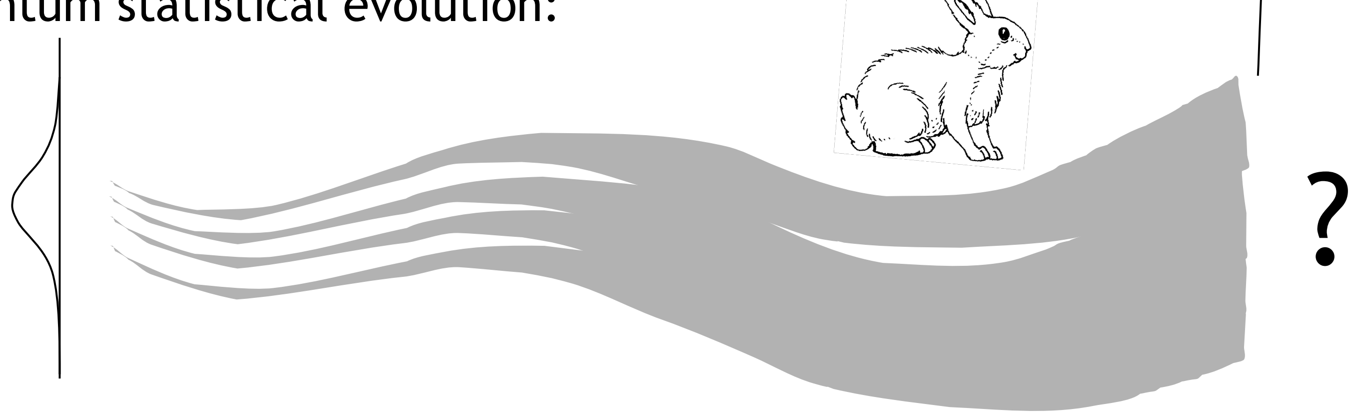
Evolving quantum fields...

...are difficult to describe due to quantum fluctuations.

Classical statistical evolution...

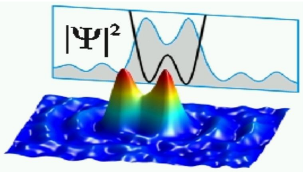


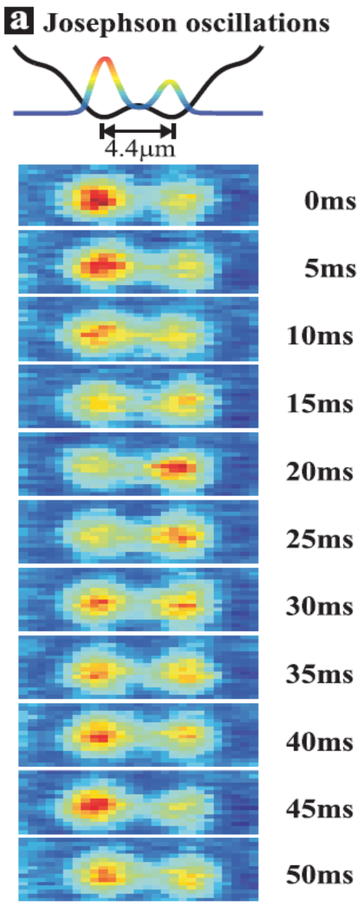
...vs quantum statistical evolution:



Squeezing & Entanglement

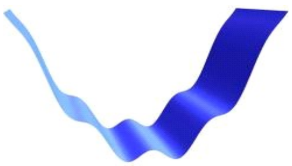
Josephson contacts

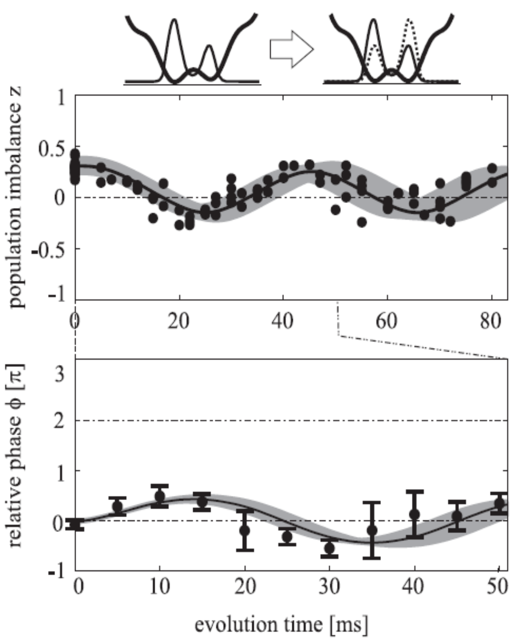




Experimenters can now

- ✔ Observe evolution in real time
- ✔ Model freely initial state
- ✔ Change boundary conditions
- ✔ Measure mean densities, phases, fluctuations
- ✔ Reduce atom numbers to a few hundreds & less





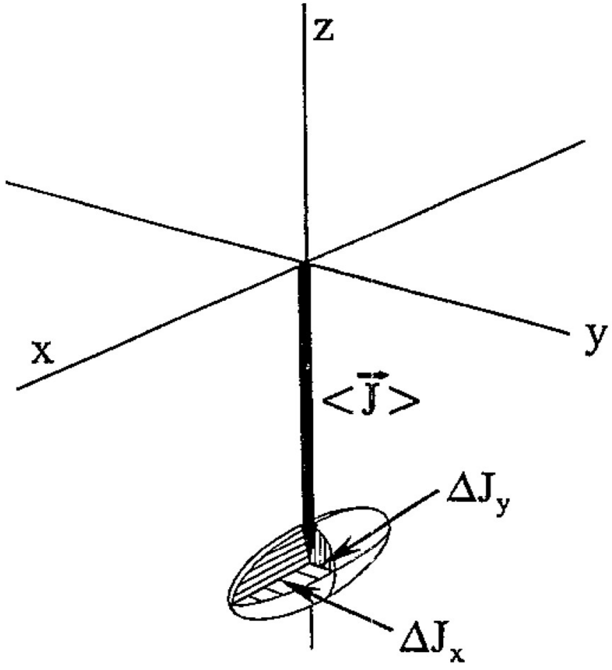
M. Oberthaler's labs (Heidelberg)

Spin squeezing

Schwinger representation of angular momentum:
(Starting point: 2 Fock modes a, b)

$$
\begin{aligned}
J_1 &= (a^\dagger b + b^\dagger a)/2, \\
J_2 &= (a^\dagger b - b^\dagger a)/2i, \\
J_3 &= (a^\dagger a - b^\dagger b)/2, \\
J &= (a^\dagger a + b^\dagger b)/2.
\end{aligned}
$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$



[D. Wineland et al., PRA 50, 67 (94)]

Spin squeezing & entanglement

[A. Sørensen et al., Nature 409, 63 (01)]

Sufficient (but not necessary) condition for non-separability of quantum state:

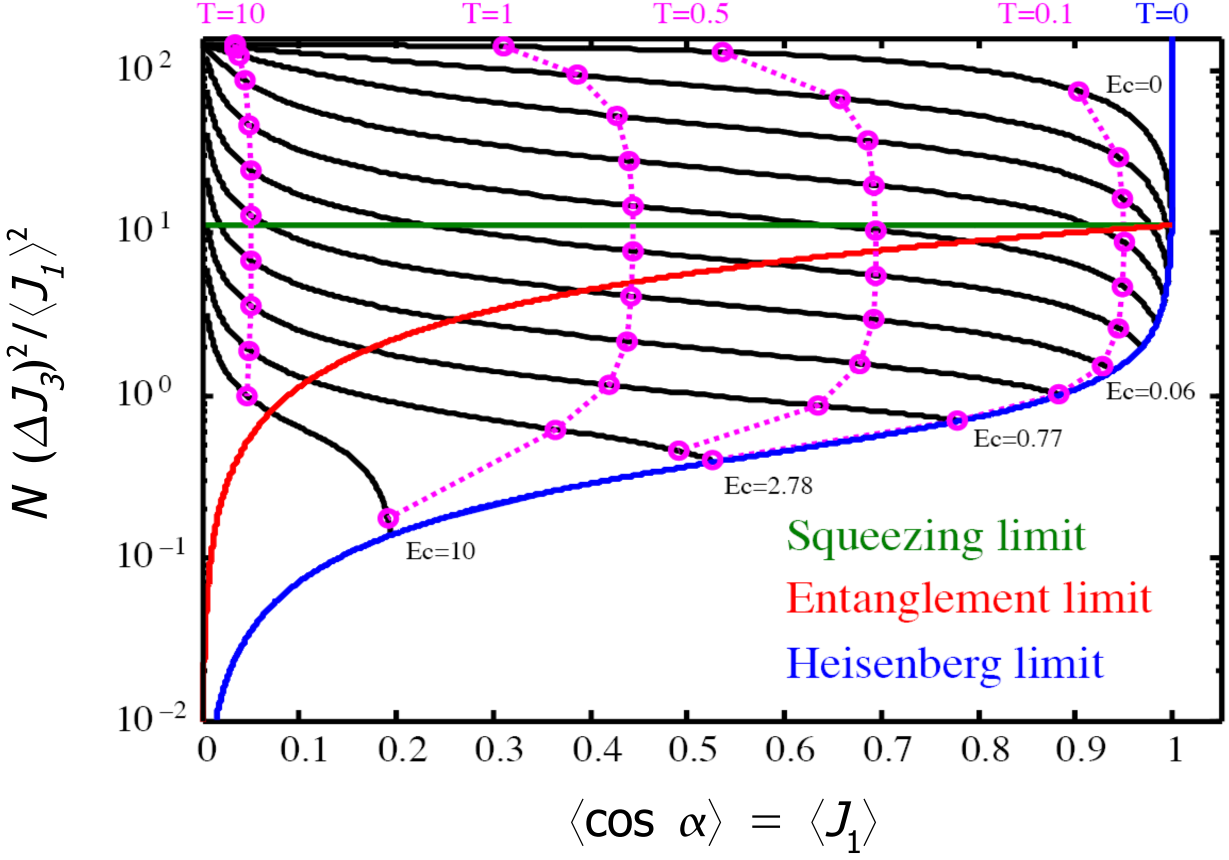
$$\xi^2 \equiv \frac{N(\Delta J_{\mathbf{n}_1})^2}{\langle J_{\mathbf{n}_2}\rangle^2 + \langle J_{\mathbf{n}_3}\rangle^2} < 1$$

Separable means that the density matrix can be written as

$$\rho = \sum_k P_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \ldots \otimes \rho_k^{(N)}$$

Spin squeezing & entanglement

[with C. Bodet, J. Esteve, M. Oberthaler]



Quantum Information inspired methods

Time dependent Density Matrix Renormalization Group (t-DMRG)
[e.g. Vidal PRL **93**, 040503 (04), Schollwöck & White cond-mat/0606018]

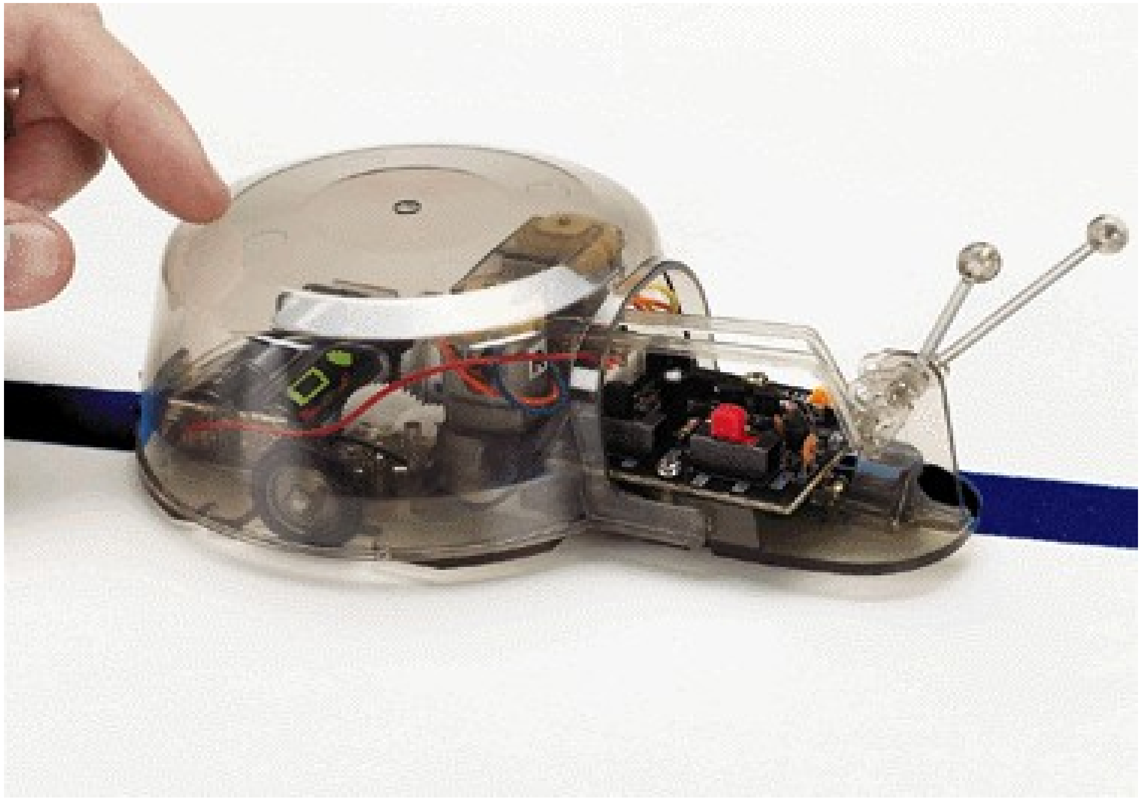
Central idea: Schmidt decomposition (here for n Spin-d sites)

$$|\Psi\rangle = \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} c_{i_1 \cdots i_n} |i_1\rangle \otimes \cdots \otimes |i_n\rangle,$$

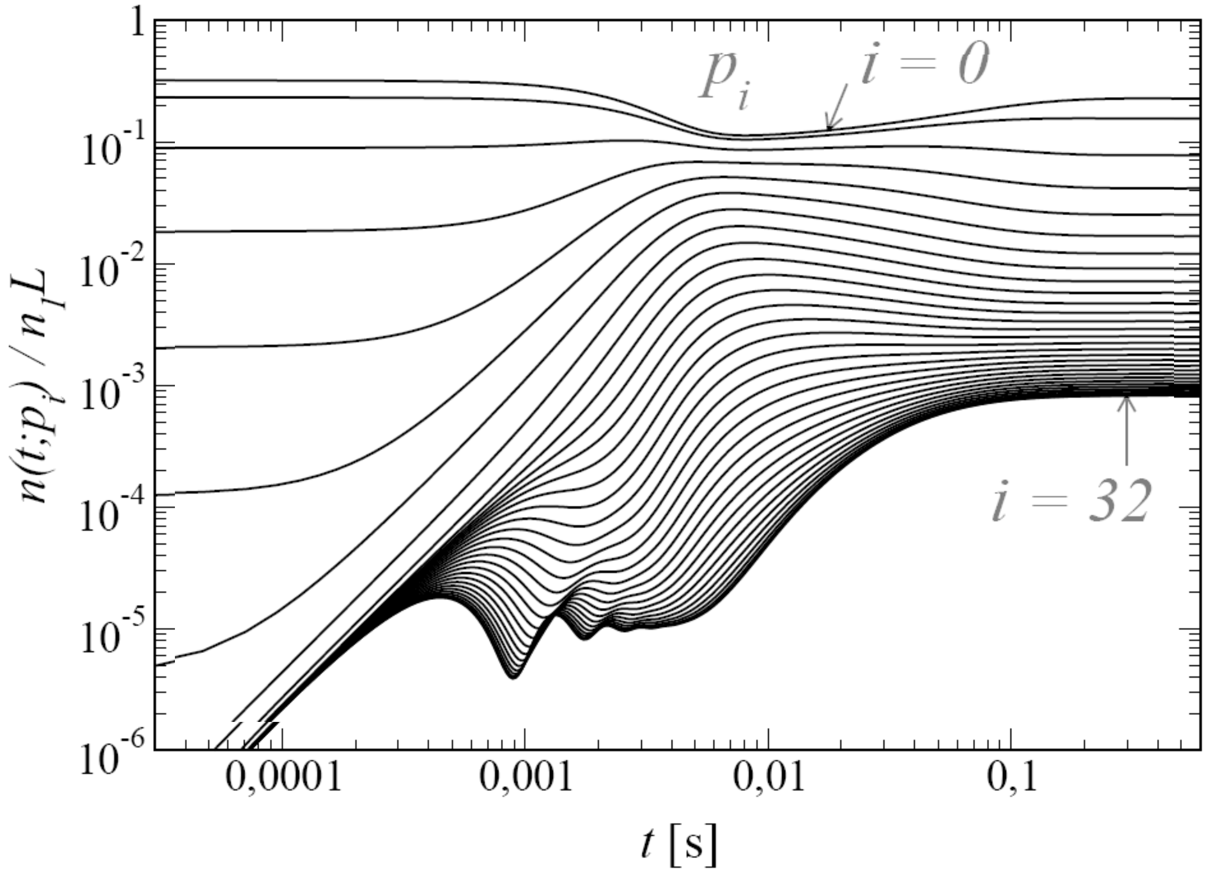
$$c_{i_1 i_2 \cdots i_n} = \sum_{\alpha_1,\ldots,\alpha_{n-1}} \Gamma^{[1]i_1}_{\alpha_1} \lambda^{[1]}_{\alpha_1} \Gamma^{[2]i_2}_{\alpha_1\alpha_2} \lambda^{[2]}_{\alpha_2} \Gamma^{[3]i_3}_{\alpha_2\alpha_3} \cdots \Gamma^{[n]i_n}_{\alpha_{n-1}}$$

Spectrum of eigenvalues $\lambda^{[l]}_{\alpha}$ is strongly peaked.

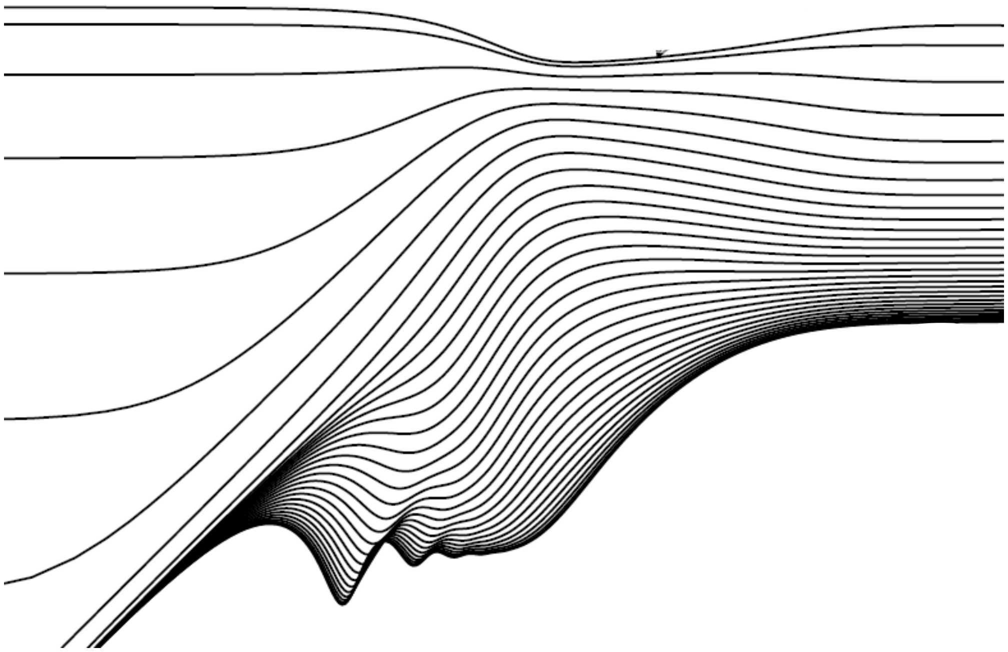
Classical Propagator



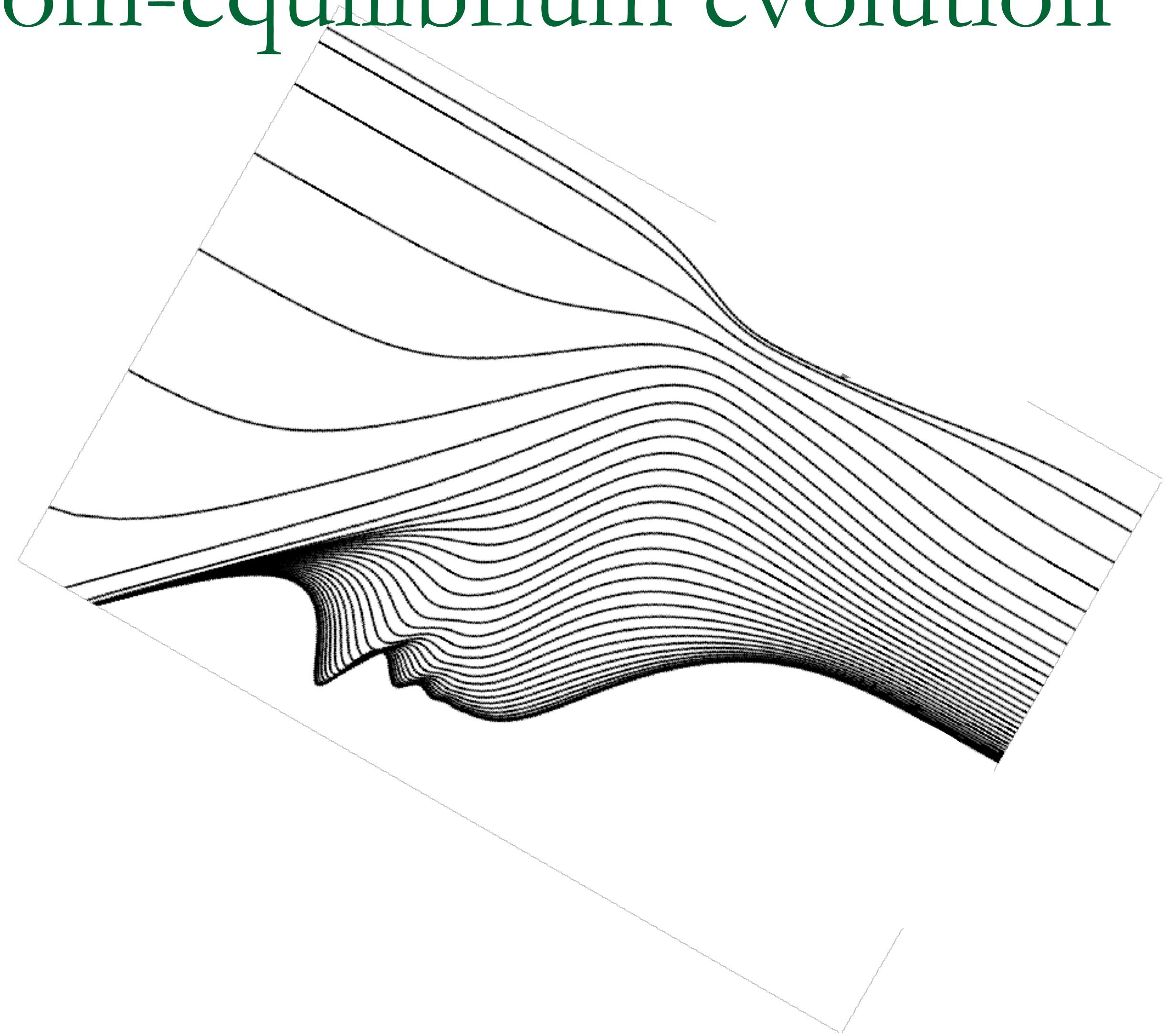
Far-from-equilibrium evolution



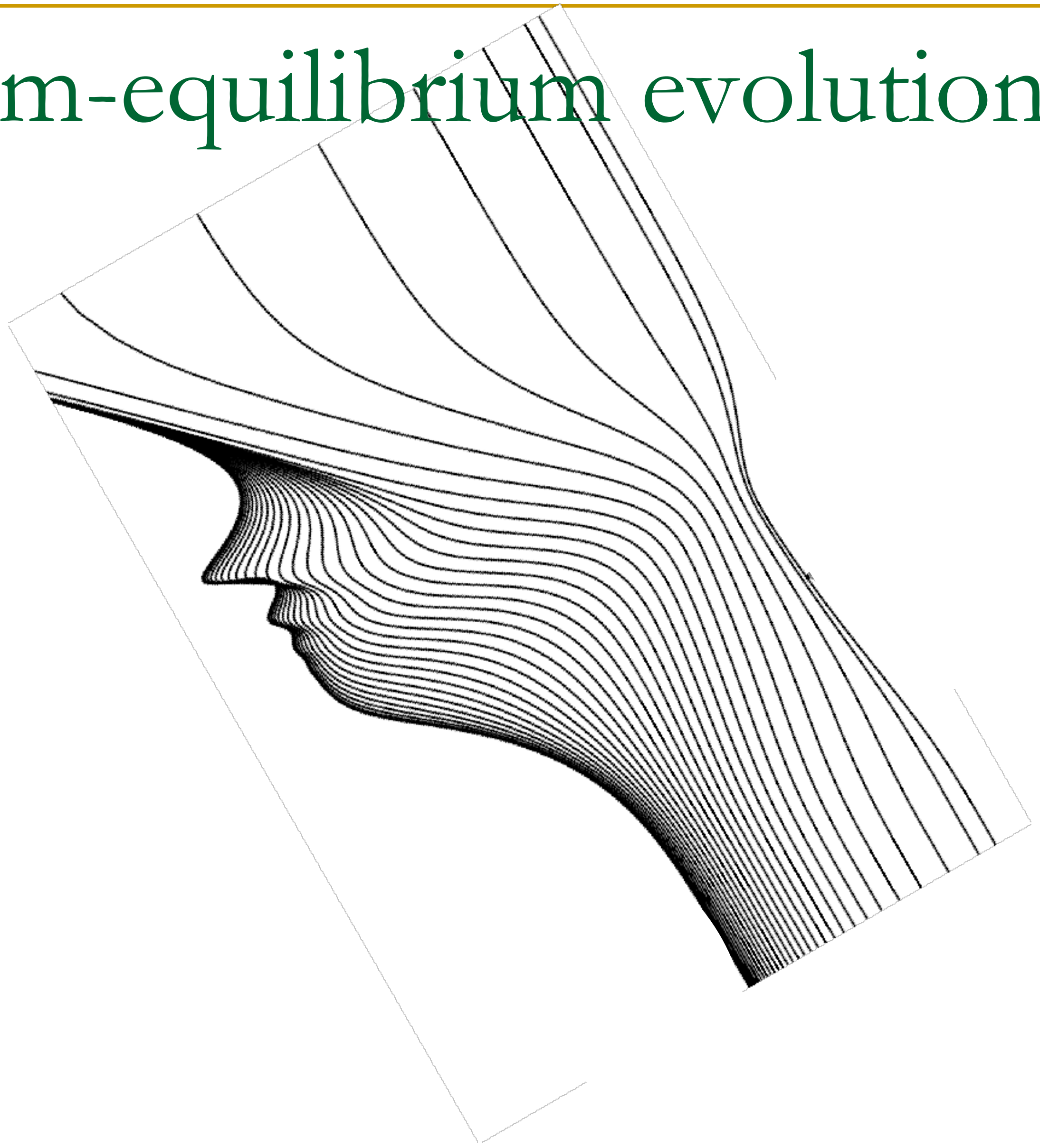
Far-from-equilibrium evolution



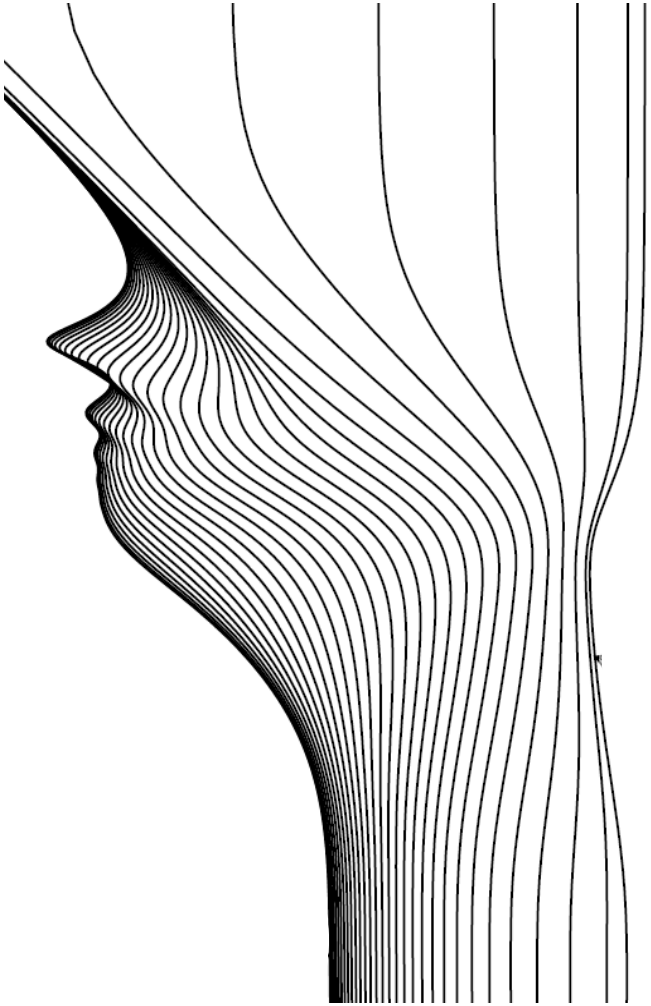
Far-from-equilibrium evolution



Far-from-equilibrium evolution



Far-from-equilibrium evolution



Santa Barbara.
Evening.
Stiff south-westerly wind.
Your hair doesn't care.