How to find the Wilson-Fisher fixed point using ERG perturbatively

— a particle theorist's approach —

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Abstract

After reviewing generalized ERG differential equations, we introduce a recipe for calculating the critical exponents of the Wilson-Fisher fixed point (critical Ising in $D = 3$) perturbatively in loop expansions.

Plan

1. Generalized ERG differential equations
   — my attempt at deciphering sect. 11 of Wilson & Kogut

2. Non-perturbative formulation for the Wilson-Fisher fixed point

3. Loop expansions
   — calculate the critical exponents in loop expansions

Generalized ERG differential equations

1. Let $S[\phi]$ be the action of a real scalar field theory in D dimensional euclidean space.

2. We generate a one-parameter family of actions S_t equivalent to S:

$$\exp\left[S_t[\phi]\right] = \int [d\phi']\, \mathrm{e}^{S[\phi']}$$
$$\times \exp\left[\frac{-1}{2}\int_p A_t(p)^2 \left\{\phi(p) - Z_t(p)\phi'(p)\right\}\left\{\phi(-p) - Z_t(p)\phi'(-p)\right\}\right]$$

(a) $\phi(p) \sim Z_t(p)\phi'(p)$ is the **block spin**. In coordinate space

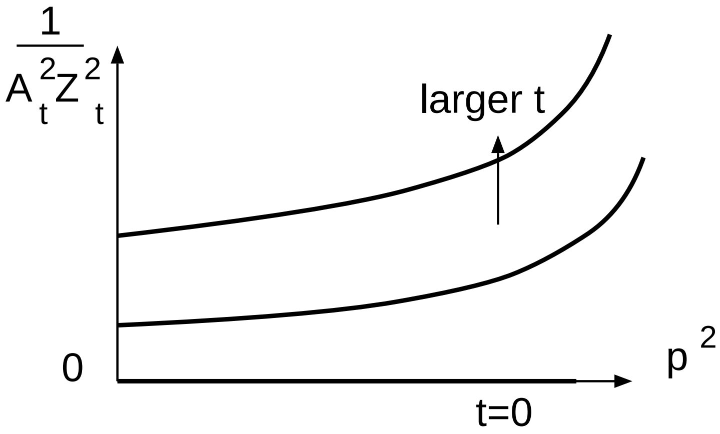
$$\int_p \mathrm{e}^{ipx} Z_t(p)\phi'(p) = \int d^D y\, \tilde{Z}_t(x-y)\tilde{\phi}'(y)$$

(b) $\frac{1}{A_t(p)}$ is the width of field diffusion:

$$\left| \frac{1}{Z_t(p)} \phi(p) - \phi'(p) \right| \sim \frac{1}{A_t(p) Z_t(p)}$$

i.e., $S_t[\phi]$ is obtained by an **incomplete integration** of $S[\phi']$.

(c) More integration for larger t, larger p^2.



$(\frac{1}{A_0} = 0,\ Z_0 = 1$ so that $S_0 = S.)$

3. Relation between S_t and S

(a) Define the generating functionals:

$$\begin{cases} \mathrm{e}^{W[J]} & \equiv \int [d\phi] \exp\left[S[\phi] + i \int_p J(p)\phi(-p) \right] \\ \mathrm{e}^{W_t[J]} & \equiv \int [d\phi] \exp\left[S_t[\phi] + i \int_p J(p)\phi(-p) \right] \end{cases}$$

(b) A simple gaussian integration gives

$$\mathrm{e}^{W_t[J]} = \exp\left[-\frac{1}{2} \int_p \frac{1}{A_t(p)^2} J(p) J(-p) + W\left[Z_t(p) J(p) \right] \right]$$

Hence,

$$\begin{cases} \langle\, \phi(p)\phi(-p)\, \rangle_{S_t} & = \frac{1}{A_t(p)^2} + Z_t(p)^2 \langle\, \phi(p)\phi(-p)\, \rangle_S \\ \langle\, \phi(p_1) \cdots \phi(p_{n>1})\, \rangle_{S_t}^{\mathrm{c}} & = \prod_{i=1}^n Z_t(p_i) \cdot \langle\, \phi(p_1) \cdots \phi(p_n)\, \rangle_S^c \end{cases}$$

(c) Conversely,

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_S &= \frac{1}{Z_t(p)^2}\,\langle\, \phi(p)\phi(-p)\,\rangle_{S_t} - \frac{1}{A_t(p)^2 Z_t(p)^2} \\ \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_S^c &= \prod_{i=1}^n \frac{1}{Z_t(p_i)} \cdot \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_{S_t}^c \end{cases}$$

The original correlation functions can be constructed as long as Z_t and $A_t Z_t$ are non-vanishing. Hence,

S_t and S are equivalent.

4. The t dependence of the action is given by the ERG differential equation of Wilson [Wilson & Kogut '74, sect. 11]:

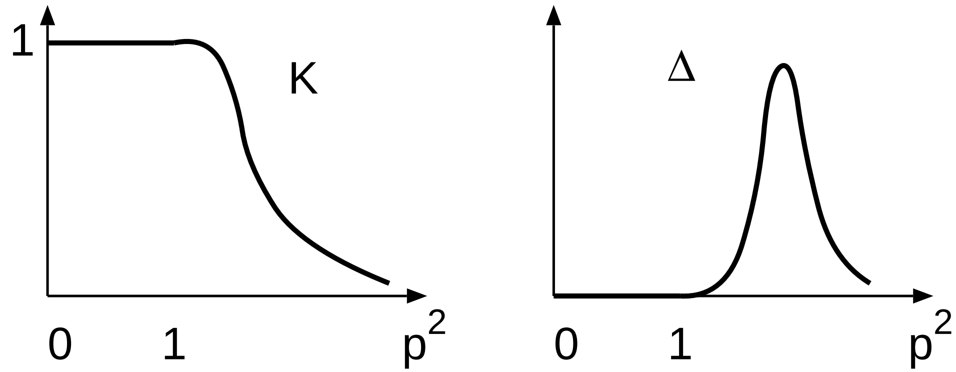
$$\begin{aligned}\partial_t S_t &= \int_p \left[F_t(p) \cdot \phi(p) \frac{\delta S_t}{\delta \phi(p)} \right. \\ &\quad \left. + G_t(p) \cdot \frac{1}{2} \left\{ \frac{\delta S_t}{\delta \phi(p)} \frac{\delta S_t}{\delta \phi(-p)} + \frac{\delta^2 S_t}{\delta \phi(p) \delta \phi(-p)} \right\} \right]\end{aligned}$$

where

$$\begin{cases} F_t(p) \equiv -\partial_t \ln Z_t(p) \\ G_t(p) \equiv -2 \frac{1}{A_t(p)^2} \partial_t \ln \left(A_t(p) Z_t(p) \right) \end{cases}$$

5. Choice of Z_t:

(a) Let $K(p)$ be an arbitrary cutoff function and $\Delta(p) \equiv -2p^2 dK(p)/dp^2$.

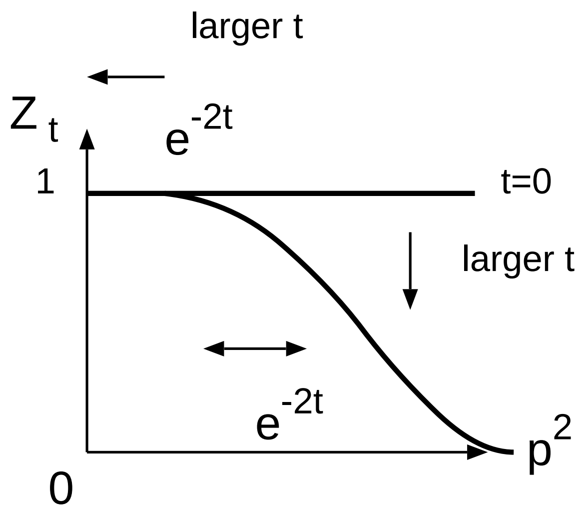


(b) We choose

$$\begin{cases} Z_t(p) = \frac{K(pe^t)}{K(p)} \exp\left\{\frac{1}{2}\int_0^t dt'\,\eta(t')\right\} \\ F_t(p) = \frac{\Delta(pe^t)}{K(pe^t)} - \frac{\eta(t)}{2} \end{cases}$$

where η is an anomalous dimension.

(c) For $\eta = 0$, we obtain



This implies S_t has smaller field fluctuations for larger t and larger p^2, since

$$\langle\, \phi(p_1) \cdots \phi(p_n)\, \rangle^c_{S_t} = \prod_{i=1}^n Z_t(p_i) \cdot \langle\, \phi(p_1) \cdots \phi(p_n)\, \rangle^c_S$$

6. Three examples of A_t:

(a) **Wilson**

$$\begin{cases} \frac{1}{A_t(p)^2} &= \mathrm{e}^{2t} - \frac{K(p\mathrm{e}^t)^2}{K(p)^2}\,\mathrm{e}^{\int_0^t \eta} \quad \overset{t\to\infty}{\longrightarrow} \mathrm{e}^{2t} \\ G_t(p) &= 2\mathrm{e}^{2t}\left(\frac{\Delta(p\mathrm{e}^t)}{K(p\mathrm{e}^t)} + 1 - \frac{1}{2}\eta(t)\right) \end{cases}$$

This implies

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_{S_t} - \mathrm{e}^{2t} &= \mathrm{e}^{\int_0^t \eta}\,\frac{K(p\mathrm{e}^t)^2}{K(p)^2}\left(\langle\, \phi(p)\phi(-p)\,\rangle_S - 1\right) \\ \langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_{S_t} &= \mathrm{e}^{n\int_0^t \eta} \prod_{i=1}^{2n} \frac{K(p_i\mathrm{e}^t)}{K(p_i)} \cdot \langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_S \end{cases}$$

- This is the original choice of Wilson.
- $\eta(t)$ chosen to normalize the kinetic term:

$$-\frac{\partial}{\partial p^2}\frac{\delta^2 S_t}{\delta\phi(p)\delta\phi(-p)}\Bigg|_{\phi=0} \overset{p^2\to 0}{\longrightarrow} 1$$

(b) **Polchinski**

$$\begin{cases} \frac{1}{A_t(p)^2} &= \frac{K(pe^t)}{p^2+m^2}\left(1 - \frac{K(pe^t)}{K(p)}\right) \\ G_t(p) &= \frac{\Delta(pe^t)}{p^2+m^2} \end{cases}$$

This implies

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_{S_t} - \frac{1}{p^2+m^2}K(pe^t)\left(1 - K(pe^t)\right) \\ \qquad = \frac{K(pe^t)^2}{K(p)^2}\left(\langle\, \phi(p)\phi(-p)\,\rangle_S - \frac{1}{p^2+m^2}K(p)(1-K(p))\right) \\ \langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_{S_t} = \prod_{i=1}^{2n}\frac{K(p_i e^t)}{K(p_i)}\cdot\langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_S \end{cases}$$

- $\eta = 0$ and m^2 independent of t
- simplest for perturbative applications (Saturday's talk)

(c) **Polchinski** modified by non-vanishing $\eta(t)$ & t-dependent $m^2(t)$

$$
\begin{cases}
\frac{1}{A_t(p)^2} = K(p\mathrm{e}^t)^2 \left\{ \frac{1}{p^2+m^2(t)} \left(\frac{1}{K(p\mathrm{e}^t)} - 1 \right) - \frac{\mathrm{e}^{\int_0^t \eta}}{p^2+m^2(0)} \left(\frac{1}{K(p)} - 1 \right) \right\} \\
G_t(p) = \frac{1}{p^2+m^2(t)} \left\{ \Delta(p\mathrm{e}^t) - \left(\eta(t) + \frac{\frac{d}{dt} m^2(t)}{p^2+m^2(t)} \right) K(p\mathrm{e}^t) \left(1 - K(p\mathrm{e}^t)\right) \right\}
\end{cases}
$$

This implies

$$
\begin{cases}
\langle \phi(p)\phi(-p) \rangle_{S_t} - \frac{1}{p^2+m^2(t)} K(p\mathrm{e}^t)\left(1 - K(p\mathrm{e}^t)\right) \\
\qquad = \mathrm{e}^{\int_0^t \eta} \frac{K(p\mathrm{e}^t)^2}{K(p)^2} \left(\langle \phi(p)\phi(-p) \rangle_S - \frac{1}{p^2+m^2(0)} K(p)(1-K(p)) \right) \\
\langle \phi(p_1) \cdots \phi(p_{2n}) \rangle_{S_t} = \mathrm{e}^{n \int_0^t \eta} \prod_{i=1}^{2n} \frac{K(p_i \mathrm{e}^t)}{K(p_i)} \cdot \langle \phi(p_1) \cdots \phi(p_{2n}) \rangle_S
\end{cases}
$$

- $\eta(t)$ and $m^2(t)$ chosen so that

$$\frac{\delta^2 S_t}{\delta\phi(p)\delta\phi(-p)}\Big|_{\phi=0} = -m^2(t) - p^2 + \mathrm{O}\left(p^4 \mathrm{e}^{-2t}\right)$$

7. For the existence of a fixed point, it is essential to rescale momenta so that p is replaced by $p\mathrm{e}^t$.

8. With rescaling, (a) and (c) obtain the Wilson-Fisher fixed point for $D = 3$, but not (b); $\eta \neq 0$ is necessary.

Non-perturbative formulation

1. **Rescaling of space**

   (a) Rewrite $S_t[\phi]$ in terms of

   $$\phi'(p) \equiv \mathrm{e}^{-\frac{D+2}{2}t}\phi(p\mathrm{e}^{-t})$$

   (b) This converts

   $$S_t[\phi] = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int_{p_1,\cdots,p_{2n}} \phi(p_1)\cdots\phi(p_{2n})\, \mathcal{V}_{2n}(t;p_1,\cdots,p_{2n}) \times (2\pi)^D \delta^{(D)}(p_1+\cdots+p_{2n})$$

into

$$\underbrace{S'_t[\phi']}_{\equiv S_t[\phi]} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int_{p_1,\cdots,p_{2n}} \phi'(p_1)\cdots\phi'(p_{2n})\, \mathcal{V}'_{2n}(t; p_1, \cdots, p_{2n})$$
$$\times (2\pi)^D \delta^{(D)}(p_1 + \cdots + p_{2n})$$

where

$$\mathcal{V}'_{2n}(t; p_1, \cdots, p_{2n}) = \mathrm{e}^{y_{2n} t} \mathcal{V}_{2n}(t; p_1 \mathrm{e}^{-t}, \cdots, p_{2n} \mathrm{e}^{-t})$$

and

$$y_{2n} \equiv D - n(D-2)$$

(c) After rescaling we obtain

$$\partial_t S_t = \int_{p'} \left[\left\{ p'_\mu \frac{\partial \phi'(p')}{\partial p'_\mu} + \left(\frac{D+2}{2} + F'_t(p') \right) \phi'(p') \right\} \frac{\delta S_t}{\delta \phi'(p')} \right.$$
$$\left. + G'_t(p') \frac{1}{2} \left\{ \frac{\delta S_t}{\delta \phi'(p')} \frac{\delta S_t}{\delta \phi'(-p')} + \frac{\delta^2 S_t}{\delta \phi'(p') \delta \phi'(-p')} \right\} \right]$$

where

$$\begin{cases} F'_t(p') \equiv F_t(p' \mathrm{e}^{-t}) = \frac{\Delta(p')}{K(p')} - \frac{\eta(t)}{2} \\ G'_t(p') \equiv \mathrm{e}^{-2t} G_t(p' \mathrm{e}^{-t}) \end{cases}$$

(d) **Example 1 (Wilson)**

$$G'_t(p) = 2\left(\frac{\Delta(p)}{K(p)} + 1 - \frac{1}{2}\eta(t)\right)$$

(e) **Example 2 (Polchinski)**

$$G'_t(p) = \frac{1}{p^2 + \mathrm{e}^{2t}m^2(t)}\left\{\Delta(p) - \left(\eta(t) + \frac{b_m(t)}{p^2 + \mathrm{e}^{2t}m^2(t)}\right)K(p)(1 - K(p))\right\}$$

where

$$\frac{d}{dt}\left(\mathrm{e}^{2t}m^2(t)\right) = 2\underbrace{\mathrm{e}^{2t}m^2(t)}_{\to m^2(t)} \;+\; \underbrace{\mathrm{e}^{2t}\frac{d}{dt}m^2(t)}_{\equiv b_m(t)}$$

We will omit the primes from ϕ' and F'_t, G'_t.

2. **Adopt the modified Polchinski.**

(a) ERG equation

$$\frac{\partial}{\partial t} S_t = \int_p \left\{ p_\mu \frac{\partial \phi(p)}{\partial p_\mu} + \left(\frac{D+2}{2} - \frac{\eta(t)}{2} + \frac{\Delta(p)}{K(p)} \right) \phi(p) \right\} \frac{\delta S_t}{\delta \phi(p)}$$
$$+ \int_p \frac{1}{p^2 + m^2(t)} \left\{ \Delta(p) - \left(\eta(t) + \frac{b_m(t)}{p^2 + m^2(t)} \right) K(p) \left(1 - K(p)\right) \right\}$$
$$\times \frac{1}{2} \left(\frac{\delta S_t}{\delta \phi(p)} \frac{\delta S_t}{\delta \phi(-p)} + \frac{\delta^2 S_t}{\delta \phi(p) \delta \phi(-p)} \right)$$

where

$$\frac{d}{dt} m^2(t) = 2 m^2(t) + b_m(t)$$

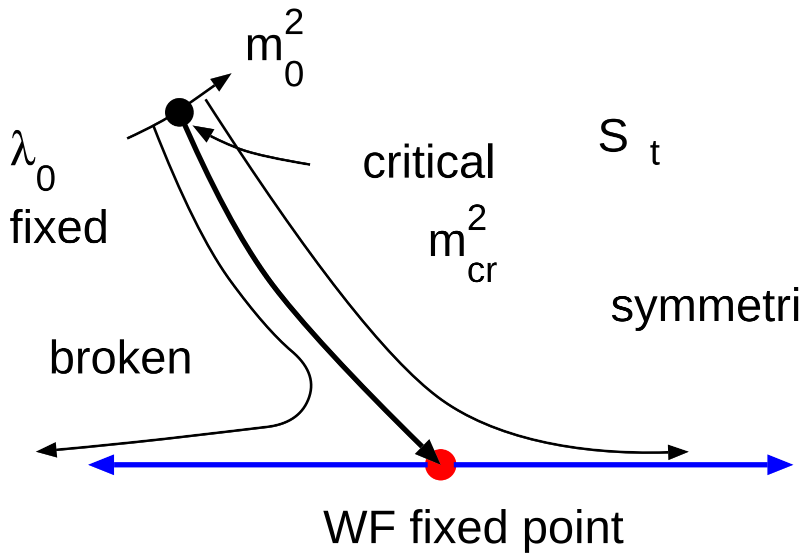
(b) $\mathcal{V}_2(t;0,0) = 0$ gives

$$-b_m(t) - \eta(t) m^2(t) = \frac{1}{2} \int_q \frac{1}{q^2 + m^2(t)}$$
$$\times \left\{ \Delta(q) - K(q)(1-K(q)) \left(\eta(t) + \frac{b_m(t)}{q^2 + m^2(t)} \right) \right\} \mathcal{V}_4(t;q,-q,0,0)$$

(c) $\frac{\partial}{\partial p^2} \mathcal{V}_2(t;p,-p) \Big|_{p=0} = 0$ gives

$$-\eta(t) = \frac{1}{2} \frac{\partial}{\partial p^2} \int_q \frac{1}{q^2 + m^2(t)}$$
$$\times \left\{ \Delta(q) - K(q)(1-K(q)) \left(\eta(t) + \frac{b_m(t)}{q^2 + m^2} \right) \right\} \mathcal{V}_4(t;q,-q,p,-p) \Big|_{p^2=0}$$

3. **Wilson-Fisher fixed point** $(D=3)$



(a) Choose the initial action

$$S_0 \equiv -\frac{1}{2}\int_p \phi(p)\phi(-p)\frac{p^2+m_0^2}{K(p)} - \frac{\lambda_0}{4!}\int_{p_1,p_2,p_3} \phi(p_1)\phi(p_2)\phi(p_3)\phi(-p_1-p_2-p_3)$$

(b) **Slightly off criticality**: for $m^2 \simeq m_{cr}^2(\lambda_0)$,

$$\left\{ \begin{array}{ccl} m^2(t) & \longrightarrow & m^{*2} + f(\lambda_0)\cdot(m_0^2 - m_c^2(\lambda_0))\cdot \mathrm{e}^{y_E t} \\ b_m(t) & \longrightarrow & -2m^{*2} + (y_E - 2)\left(m^2(t) - m^{*2}\right) \\ \gamma(t) & \longrightarrow & \gamma^* \end{array} \right.$$

2 critical exponents: y_E & γ^*

(c) Relace S_0 by the asymptotic condition:

$$S_t \stackrel{t\to-\infty}{\longrightarrow} S_{UV}^*, \qquad \text{where} \quad S_{UV}^* \equiv -\frac{1}{2}\int_p \phi(p)\phi(-p)\frac{p^2}{K(p)}$$

(d) More concretely, for $2n \geq 6$

$$\mathrm{e}^{-y_{2n}t}\,\mathcal{V}_{2n}(t; p_1\mathrm{e}^t, \cdots, p_{2n}\mathrm{e}^t) \stackrel{t\to-\infty}{\longrightarrow} 0 \quad \text{where} \quad y_{2n} \equiv 3 - n \leq 0$$

(e) Defining $\lambda(t) \equiv -\mathcal{V}_4(t; 0, 0, 0, 0)$, we obtain

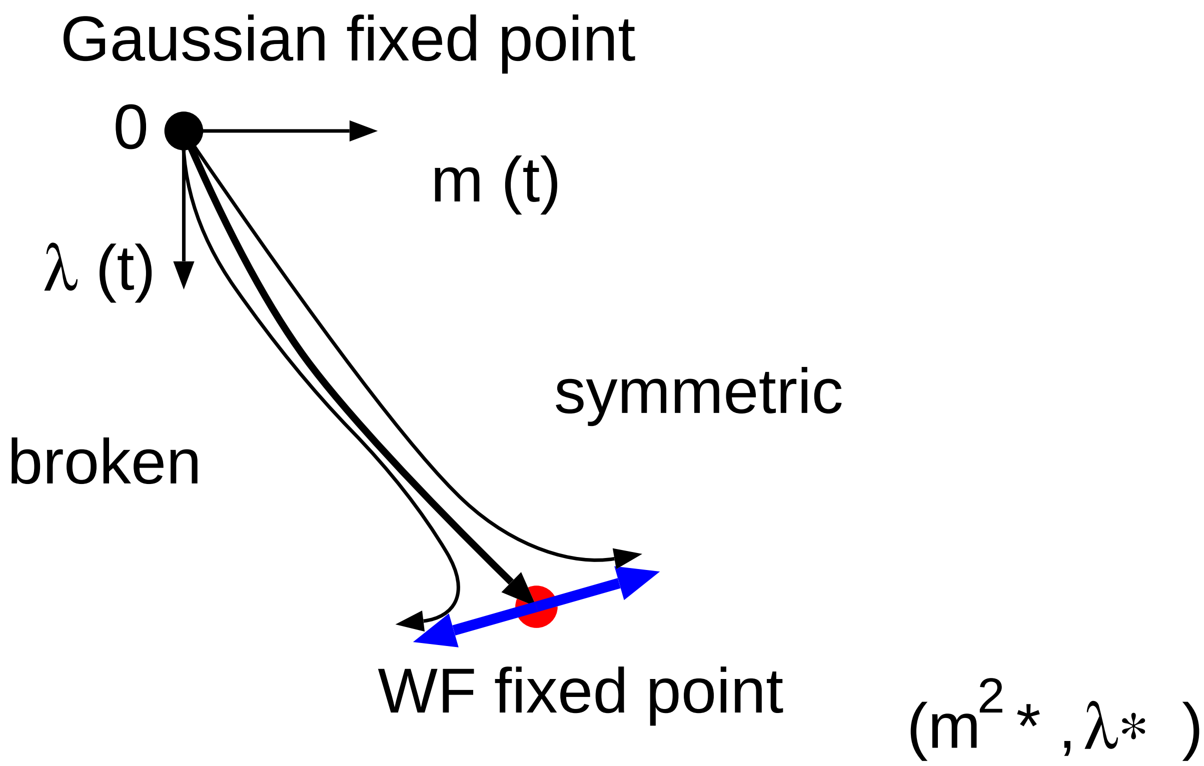
$$S_t = S(m^2(t), \lambda(t))$$

- $b_m(t)$ and $\eta(t)$ are functions of $m^2(t), \lambda(t)$.
- $\lambda(t)$ satisfies

$$\frac{d\lambda(t)}{dt} = \lambda(t) + \beta(t)$$

where

$$\begin{aligned} & -\beta(t) - 2\lambda(t) \cdot \eta(t) \\ & = \frac{1}{2} \int_q \frac{1}{q^2 + m^2(t)} \left\{ \Delta(q) - K(q)(1 - K(q)) \left(\eta(t) + \frac{b_m(t)}{q^2 + m^2(t)} \right) \right\} \\ & \qquad \times \mathcal{V}_6(t; q, -q, 0, 0, 0, 0) \end{aligned}$$


Gaussian fixed point
0
m (t)
λ (t)
symmetric
broken
WF fixed point
(m2 * , λ*)

Perturbative calculations

1. At 1-loop, we find

$$\begin{cases} b_m(m^2,\lambda) &= \frac{\lambda}{2}\int_q \frac{\Delta(q)}{q^2+m^2} \\ \beta(m^2,\lambda) &= -3\lambda^2 \int_q \frac{\Delta(q)(1-K(q))}{(q^2+m^2)^2} \\ \eta(m^2,\lambda) &= 0 \end{cases}$$

2. At 2-loop, we find

$$\eta(m^2,\lambda) = -\frac{\lambda^2}{2}\frac{\partial}{\partial p^2}\int_{q,r}\frac{\Delta(q)}{q^2+m^2}\frac{1-K(r)}{r^2+m^2}\frac{1-K(p+q+r)}{(p+q+r)^2+m^2}\Big|_{p^2=0}$$

3. These integrals depend on the choice of the cutoff function $K(q)$.

4. The critical exponents can formally be shown independent of the choice.

5. Choose

$$K(q) = \theta(1 - q^2)$$

to obtain

$$\begin{cases} y_E - 2 &= -\frac{1}{3} \simeq -0.333 \\ \eta &= -\frac{1}{54} + \frac{\ln 2}{27} \simeq 0.00715 \quad \text{much too small!} \end{cases}$$

6. How can one improve on this?