Stochastic Quantization with Colored Noise

Scale-controlled Gradient Flows

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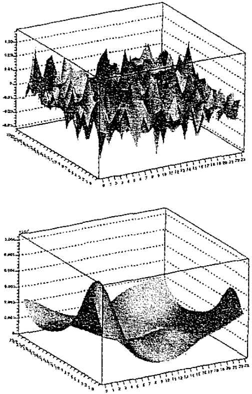
Motivation

UV-fluctuations on the lattice

- **Context:** QCD vacuum and topology in Yang-Mills theory
- **Problem:** short-distance fluctuations of order of the lattice spacing blur the underlying classical structure
- **Solution:** cooling methods



[Rothe, "Lattice Gauge Theories", 2005: Wantz, 2003]

Cooling methods

- **Wilson flow** [Lüscher, 2010]

$$\boxed{\frac{\partial V_{t_F}(x,\mu)}{\partial t_F} = -g_0^2[D_{x,\mu}S(V_{t_F})]V_{t_F}(x,\mu)}$$

- **Procedure:** Smoothen field configurations in a damping equation such as the Wilson flow or use action minimization methods [Garcia-Perez, Philipsen, Stamatescu, 1999]
- **Problem:** When does one stop cooling? Is there a characteristic scale above which observables are independent of short-distance fluctuations?

Wilson flow and cooling – a simple example

Massless scalar field [Bonati and D'Elia, 2014]

$$S_{\mathrm{E}} = \sum_{n} \sum_{\mu=1}^{d} \phi(n) \left[2\phi(n) - (\phi(n+\hat{\mu}) + \phi(n-\hat{\mu})) \right]$$

⇒ Gradient flow

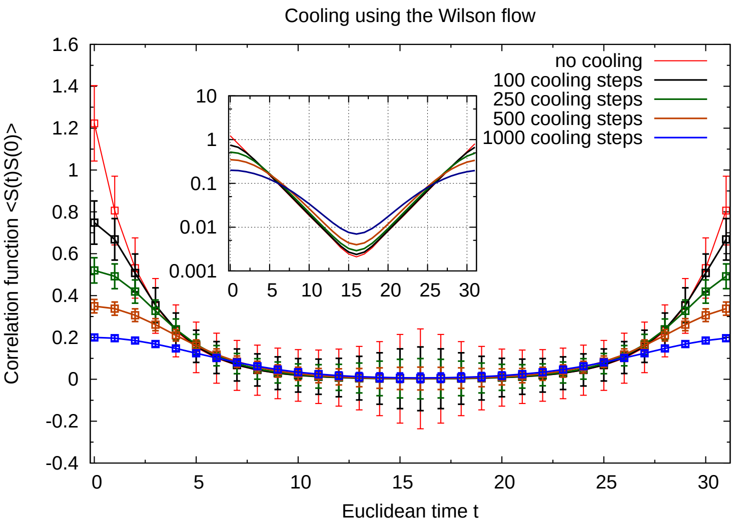
$$\frac{\partial \phi(n, t_{\mathrm{F}})}{\partial t_{\mathrm{F}}} = -\frac{\delta S}{\delta \phi(n, t_{\mathrm{F}})} = \sum_{\mu=1}^{d} [\phi(n+\hat{\mu}, t_{\mathrm{F}}) + \phi(n-\hat{\mu}, t_{\mathrm{F}})] - 2d\phi(n, t_{\mathrm{F}})$$

Limit of smoothly varying fields ($x_\mu = a n_\mu$)

$$\boxed{\frac{\partial \phi(x, t_{\mathrm{F}})}{\partial t_{\mathrm{F}}} = a^2 \Delta \phi(x, t_{\mathrm{F}})}$$

Heat diffusion equation

Wilson flow and cooling – a simple example



Methods and Setup

Stochastic quantization

- Analogy between a Euclidean quantum field theory and a classical statistical mechanical system in thermal equilibrium with a heat reservoir. [Parisi, Wu, 1981]
- **Stochastic process** – evolution of fields in fictitious time τ described by the **Langevin equation**

$$\boxed{\frac{\partial\phi(x,\tau)}{\partial\tau} = -\frac{\delta S_E}{\delta\phi(x,\tau)} + \eta(x,\tau)}$$

$$\langle\eta(x,\tau)\rangle_\eta = 0, \quad \langle\eta(x,\tau)\eta(y,\tau')\rangle_\eta = 2\delta^{(d)}(x-y)\delta(\tau-\tau')$$

- Quantum fluctuations encoded in Gaussian white noise
  $\Rightarrow$ **Aim:** attack UV fluctuations here!

Langevin vs. Wilson flow

Langevin

$$\frac{\partial\phi(x,\tau)}{\partial\tau} = -\frac{\delta S_E}{\delta\phi(x,\tau)} + \eta(x,\tau)$$

Generation of field configurations
(full fluctuation spectrum)

Wilson flow

$$\frac{\partial\varphi(x,t_{\mathrm{F}})}{\partial t_{\mathrm{F}}} = -\frac{\delta S_E}{\delta\varphi(x,t_{\mathrm{F}})}$$

Cooling of configurations
$\varphi(x, t_{\mathrm{F}} = 0) = \phi(x, \tau = \infty)$

Related cooling methods?
$\rightarrow$ modify noise spectrum in the Langevin equation

Colored noise

Access lattice momentum scale and control fluctuations by modifying the noise term using a UV-regulator

[Bern, Halpern, Sadun, Taubes, 1987]

$$\frac{\partial \phi(x,\tau)}{\partial \tau} = -\frac{\delta S}{\delta \phi(x,\tau)} + \int \mathrm{d}^d y \, R(x-y) \, \eta(y,\tau)$$

$$R(x-y) = \left(1 - \frac{\Delta_x}{\Lambda^2}\right)^{-1} \delta^{(d)}(x-y)$$

$$\frac{\partial}{\partial \tau} P[\phi,\tau] = \int \mathrm{d}^d x \frac{\delta}{\delta \phi(x)} \left(\frac{\delta S}{\delta \phi(x)} + \int \mathrm{d}^d y \, R^2_{xy} \frac{\delta}{\delta \phi(y)} \right) P[\phi,\tau]$$

Fokker-Planck equation

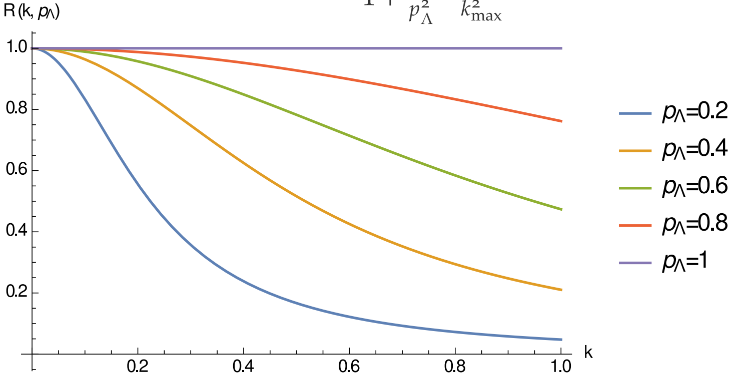
Colored noise - Lattice implementation

$$\hat{k}_\mu = \frac{2\pi}{N_\mu} n_\mu,$$

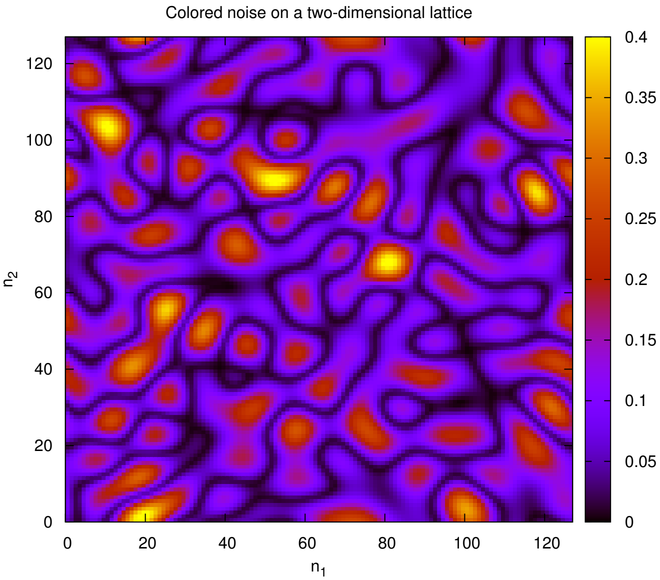
$$n_\mu = -N_\mu/2+1,\ldots,N_\mu/2$$

$$k^2 = 4\sum_{\mu=1}^{d} \sin^2\left(\frac{\hat{k}_\mu}{2}\right)$$

$$\tilde{R}(k,p_\Lambda) := \frac{1}{1+\frac{k^2}{p_\Lambda^2}\frac{k_{\max}^2-p_\Lambda^2}{k_{\max}^2}}$$



Colored noise - Lattice implementation



Aims and strategy

Aims

- Generate smoothened configurations in a lattice simulation
- Precision gain

Strategy

- Probe effects of colored noise
- Model: $O(1)$ Scalar field theory in d dimensions
- Try to find a relation to **flow time** and **physical scales**

Real scalar field theory on the lattice

Continuum

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial_\mu \phi_0(x))^2 + \frac{m_0^2}{2}\phi_0(x)^2 + \frac{g_0}{4!}\phi_0(x)^4\right]$$

Lattice discretization

$$\boxed{S = \sum_{x\in\Lambda}\left[-2\kappa\sum_{\mu=1}^{d}\phi(x)\phi(x+\hat{\mu}) + \phi(x)^2 + \lambda[\phi(x)^2 - 1]^2 - \lambda\right]}$$

- large range of applications in field theory
- simple model, appropriate for testing algorithms

Observables

- Magnetization (order parameter)

$$M := \frac{1}{\Omega} \sum_x \phi(x)$$

- Connected two-point correlation function

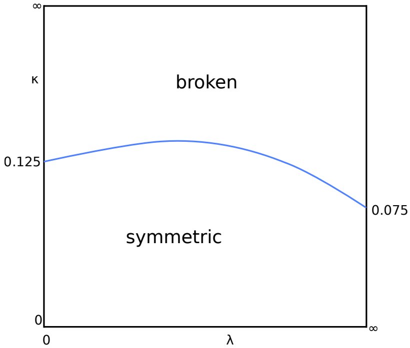
$$G_c(x, y) := \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle$$

- Connected susceptibility

$$\chi_2 = \sum_x G_c(x, 0) = \Omega \left(\langle M^2 \rangle - \langle M \rangle^2 \right)$$

- fourth order cumulant [Binder, 1981]

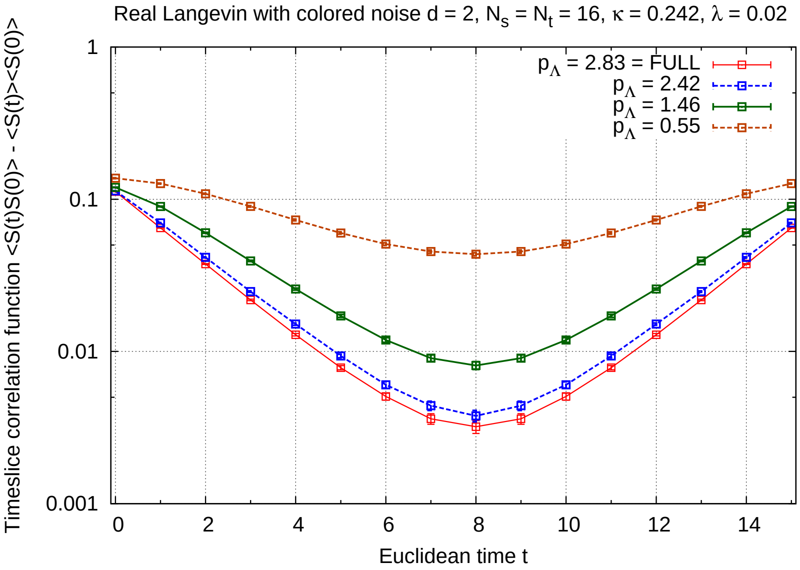
$$U_L = 1 - \frac{1}{3} \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2}$$



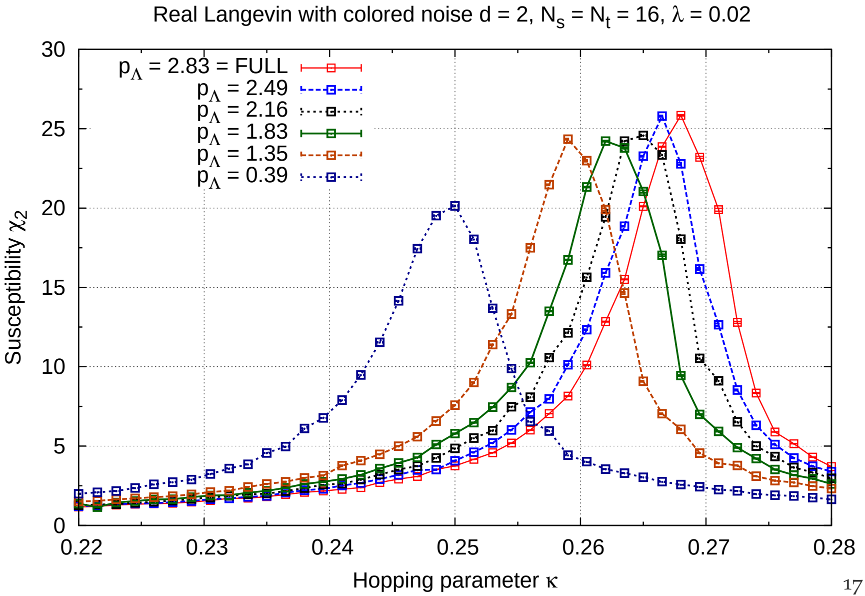
Phase diagram in d = 4

Numerical results

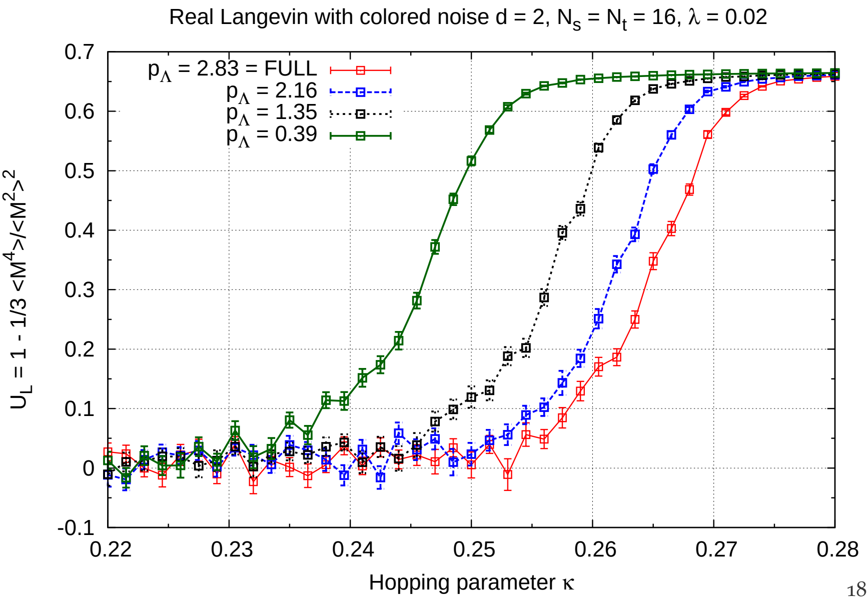
Time slice correlation function



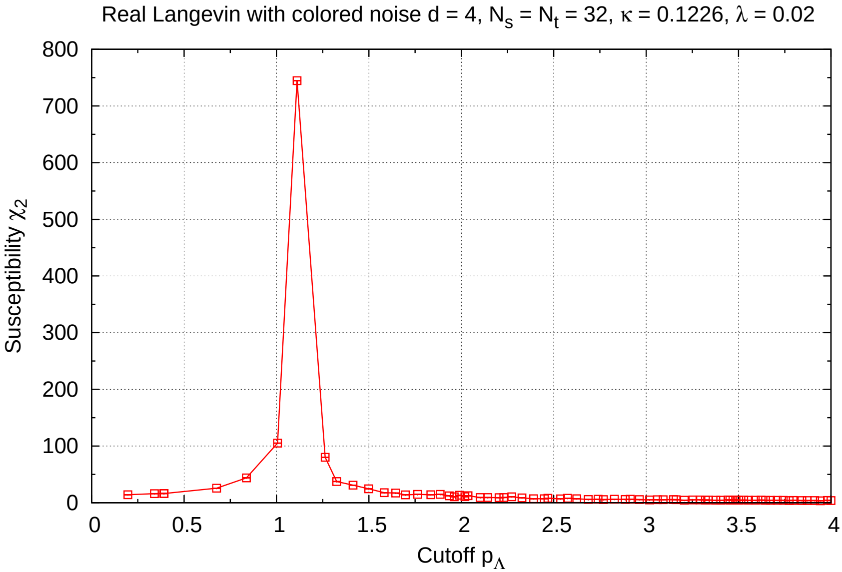
Susceptibility for fixed λ



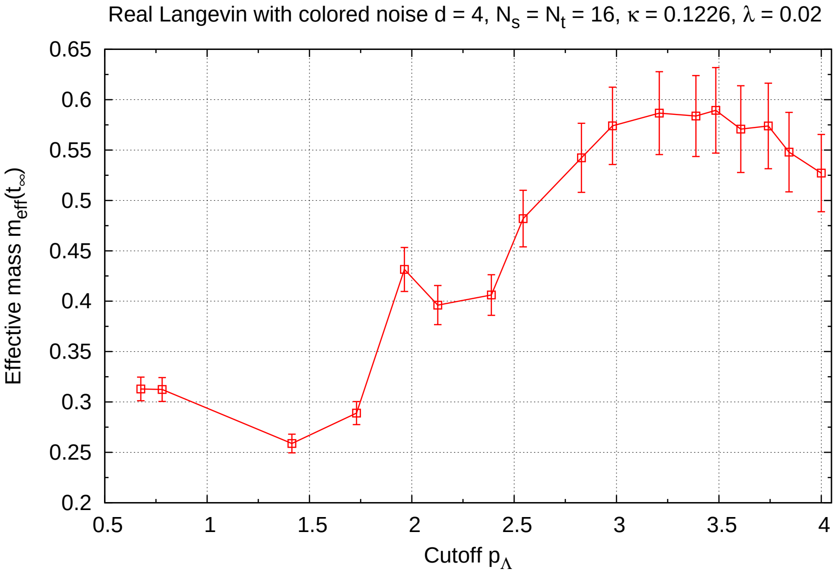
Binder cumulant for fixed λ



Susceptibility (fixed κ, λ)



Effective mass at large Euclidean times (fixed κ, λ)



Conclusions and Outlook

Conclusions

- application of colored noise causes "movement" from the symmetric to the broken phase
- plateau in mass: physics seems to be dominated by long-range modes
- CN approach consistent with Wilson flow $\Rightarrow$ match scale and flow time

Outlook

- Exploration of the broken phase: Tunneling and topological charge
- Cooling of kink – anti-kink configurations using CN
- Extension to pure gauge theory ⇒ comparison with existing results
- smoothened configurations are created from the beginning ⇒ optimization (?)

Thank you very much for your attention!